A hypergraph Turán problem with no stability

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Abstract

A fundamental barrier in extremal hypergraph theory is the presence of many near-extremal constructions with very different structure. Indeed, the classical constructions due to Kostochka imply that the notorious extremal problem for the tetrahedron exhibits this phenomenon assuming Turán's conjecture.

Our main result is to construct a finite family of triple systems $\mathcal{M}$, determine its Turán number, and prove that there are two near-extremal $\mathcal{M}$-free constructions that are far from each other in edit-distance. This is the first extremal result for a hypergraph family that fails to have a corresponding stability theorem.

1 Introduction

Let $r \geq 3$ and $\mathcal{F}$ be a family of r-uniform graphs (henceforth r-graphs). An r-graph $\mathcal{H}$ is $\mathcal{F}$-free if it contains no member of $\mathcal{F}$ as a subgraph. The Turán number $\mathrm{ex}(n, \mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The Turán density $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$. The study of $\mathrm{ex}(n, \mathcal{F})$ is perhaps the central topic in extremal graph and hypergraph theory.

Much is known about $\mathrm{ex}(n, \mathcal{F})$ when $r = 2$ and one of the most famous results in this regard is Turán's theorem, which states that for $\ell \geq 2$ the Turán number $\mathrm{ex}(n, K_{\ell+1})$ is uniquely achieved by $T(n, \ell)$ which is the ℓ-partite graph on n vertices with the maximum number of edges.

For $\ell > r \geq 3$, let K_ℓ^r be the complete r-graph on ℓ vertices. Extending Turán's theorem to hypergraphs (i.e. $r \geq 3$) is a major problem. Indeed, the problem of determining $\pi(K_\ell^r)$ was raised by Turán [27] and is still wide open. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geq 3$ and \$1000 for the determination of all $\pi(K_\ell^r)$ with $\ell > r \geq 3$.

Conjecture 1.1 (Turán [27]). $\pi(K_\ell^3) = 1 - \left(\frac{2}{\ell-1}\right)^2$.

The case $\ell = 4$ above, which states that $\pi(K_4^3) = 5/9$ has generated a lot of interest and activity over the years. Many constructions (e.g. see [2, 13, 7]) are known to achieve the value in Conjecture 1.1 for $\ell = 4$, and that is perhaps one of the reasons why it is so challenging. On the other hand, successively better upper bounds for $\pi(K_4^3)$ were obtained by de Caen [4], Giraud (see [3]), Chung and Lu [3], and Razborov [24]. The current record is $\pi(K_4^3) \leq 0.561666$, which was obtained by Razborov [24] using flag algebra machinery.

Many families $\mathcal{F}$ have the property that there is a unique $\mathcal{F}$-free graph (or hypergraph) $\mathcal{G}$ on n vertices achieving $\mathrm{ex}(n, \mathcal{F})$, and moreover, any $\mathcal{F}$-free graph (or hypergraph) $\mathcal{H}$ of size close to $\mathrm{ex}(n, \mathcal{F})$ can be transformed to $\mathcal{G}$ by deleting and adding very few edges. Such a property

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is called the stability of $\mathcal{F}$. The first stability theorem was proved independently by Erdős and Simonovits [26].

Theorem 1.2 (Erdős-Simonovits [26])**.** *Let $\ell \geq 2$. Then for every $\delta > 0$ there exists an $\epsilon > 0$ and n_0 such that the following statement holds for all $n \geq n_0$. Every $K_{\ell+1}$-free graph on n vertices with at least $(1-\epsilon)t(n,\ell)$ edges can be transformed to $T(n,\ell)$ by deleting and adding at most δn^2 edges.*

The stability phenomenon has been used to determine $\mathrm{ex}(n,\mathcal{F})$ exactly in many cases. It was first used by Simonovits in [26] to determine $\mathrm{ex}(n,\mathcal{F})$ exactly for all color-critical graphs and large n, and then by several authors (e.g. see [5, 6, 8, 9, 11, 12, 19, 20, 22]) to prove exact results for hypergraphs.

However, there are many Turán problems for hypergraphs that (perhaps) do not have the stability property. The example K_4^3 we mentioned before was shown to have exponentially many extremal constructions in the number of vertices (see Kostochka [13] and Brown [2]). We will prove (Proposition 1.7) that these constructions can be used to show that K_4^3 does not have the stability property (assuming Conjecture 1.1 is true). For K_ℓ^3 with $\ell \geq 5$, different near-extremal constructions were given by Sidorenko [25], and Keevash and the second author [10]. Although we do not provide the details, these also show that K_ℓ^3 does not have stability (assuming Conjecture 1.1 is true).

The absence of stability seems to be a fundamental barrier in determining the Turán numbers of some families. Indeed, the Turán numbers of the examples we presented above are not known, even asymptotically, and in fact, no Turán number of a family without the stability property has been determined.

This paper provides the first such example. We construct a family $\mathcal{M}$ of 3-graphs, prove that $\mathcal{M}$ does not have the stability property, and determine $\pi(\mathcal{M})$, and even $\mathrm{ex}(n,\mathcal{M})$ for infinitely many n (Theorems 1.6 and 1.10).

The present paper has a slightly similar flavor as [23] in the sense that we will define the extremal hypergraphs $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ first, and then define the forbidden family $\mathcal{M}$, which is a suitably chosen family based on $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$.

In order to state our results formally, we need some definitions.

Definition 1.3. *Let $r \geq 2$ and $\mathcal{H}$ be an r-graph. The transversal number of $\mathcal{H}$ is*

$$\tau(\mathcal{H}) := \min\left\{|S| : S \subset V(\mathcal{H}) \text{ such that } S \cap E \neq \emptyset \text{ for all } E \in \mathcal{H}\right\}.$$

We set $\tau(\mathcal{H}) = 0$ if $\mathcal{H}$ is an empty graph.

Let $\ell \geq r \geq 2$ and $\mathcal{K}_{\ell+1}^r$ be the collection of all r-graphs F on at most $(\ell+1)+(r-2)\binom{\ell+1}{2}$ vertices such that for some $(\ell+1)$-set S, which will be called the core of F, every pair $\{u,v\} \subset S$ is covered by an edge in F [1]. Let $V_1 \cup \cdots \cup V_\ell$ be a partition of $[n] := \{1,\ldots,n\}$ with each V_i of size either $\lfloor n/\ell \rfloor$ or $\lceil n/\ell \rceil$. The generalized Turán graph $T_r(n,\ell)$ is the collection of all r-subsets of $[n]$ that have at most one vertex in each V_i. Let $t_r(n,\ell) = |T_r(n,\ell)|$. It was shown by the second author [17] that $\mathrm{ex}(n,\mathcal{K}_{\ell+1}^r) = t_r(n,\ell)$ and $T_r(n,\ell)$ is the unique $\mathcal{K}_{\ell+1}^r$-free r-graph on n vertices with exactly $t_r(n,\ell)$ edges.

Suppose that $\mathcal{T}$ is an r-graph on s vertices and $t = (t_1,\ldots,t_s)$ with each t_i a positive integer. Then the blowup $\mathcal{T}(t)$ of $\mathcal{T}$ is obtained from $\mathcal{T}$ by replacing each vertex i by a set of size t_i, and replacing every edge in $\mathcal{T}$ by the corresponding complete r-partite r-graph.

An r-graph $\mathcal{S}$ is a star if all edges in $\mathcal{S}$ contain a fixed vertex v, which is called the center of $\mathcal{S}$.

[1] The original definition of $\mathcal{K}_{\ell+1}^r$ in [17] requires that F has at most $\binom{\ell+1}{2}$ edges. The new definition we used here will make our proofs simpler. Notice that $\mathcal{K}_{\ell+1}^r$ is a finite family in both definitions.

Definition 1.4. *Let $|A| = \lfloor n/3 \rfloor$ and $|B| = \lceil 2n/3 \rceil$ with $A \cap B = \emptyset$. Define*

$$\mathcal{G}_n^1 := \left\{ abb' : a \in A \text{ and } \{b, b'\} \subset B \right\}.$$

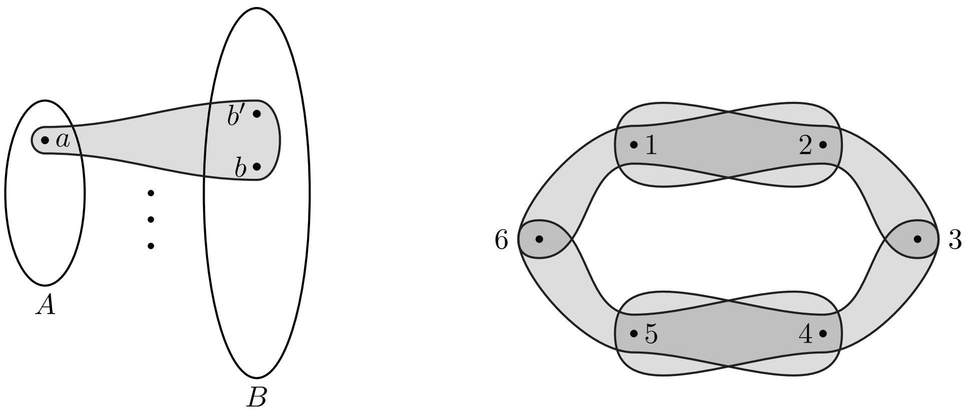
Let $\mathcal{G}_6^2$ be the 3-graph with vertex set $[6]$ whose complement is

$$\overline{\mathcal{G}_6^2} := \{123, 126, 345, 456\}.$$

For $n > 6$ let $\mathcal{G}_n^2$ be a 3-graph on n vertices which is a blowup of $\mathcal{G}_6^2$ with the maximum number of edges.

Remarks.

- Notice that $\mathcal{G}_n^1$ is a (unbalanced) blowup of a star.
- Simple calculations show that each part in $\mathcal{G}_n^2$ has size either $\lfloor n/6 \rfloor$ or $\lceil n/6 \rceil$.
- For $i = 1, 2$, let $g_i(n) = |\mathcal{G}_n^i|$. Then $\lim_{n\to\infty} g_i(n)/n^3 = 2/27$.



(a) The 3-graph $\mathcal{G}_n^1$.

(b) The complement of $\mathcal{G}_6^2$.

Figure 1: $\mathcal{G}^1$ and $\overline{\mathcal{G}_6^2}$.

Definition 1.5. *The family $\mathcal{M}$ is the union of the following three finite families.*

(a) M_1 is the set containing the complete 3-graph on five vertices with one edge removed, $M_1 = \left\{ K_5^{3-} \right\}$.

(b) M_2 is the collection of all 3-graphs in $\mathcal{K}_7^3$ with a core whose induced subgraph has transversal number at least two.

(c) M_3 is the collection of all 3-graphs $F \in \mathcal{K}_6^3$ such that both $F \not\subset \mathcal{G}_n^1$ and $F \not\subset \mathcal{G}_n^2$ for all positive integers n.

Our first result is about the Turán number of $\mathcal{M}$.

Theorem 1.6. *The inequality $ex(n, \mathcal{M}) \le 2n^3/27$ holds for all positive integers n. Moreover, equality holds whenever n is a multiple of six.*

For an r-graph $\mathcal{H}$ the shadow of $\mathcal{H}$ is

$$\partial \mathcal{H} := \left\{ A \in \binom{V(\mathcal{H})}{r-1} : \exists B \in \mathcal{H} \text{ such that } A \subset B \right\}.$$

Note that both $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ are $\mathcal{M}$-free and $g_1(n) \sim g_2(n) \sim 2n^3/27$. Moreover, it is easy to see that transforming $\mathcal{G}_n^1$ to $\mathcal{G}_n^2$ requires us to delete and add $\Omega(n^3)$ edges. Indeed, $\partial\mathcal{G}_n^1$ contains a clique on $\lfloor 2n/3 \rfloor$ vertices, whereas $\partial\mathcal{G}_n^2$ has clique number six. By Turán's theorem, one must thus delete strictly more that $(1-\pi(K_7))\binom{\lfloor 2n/3\rfloor}{2} = \Omega(n^2)$ edges from $\partial\mathcal{G}_n^2$ to obtain a copy of $\partial\mathcal{G}_n^2$. Since every edge in $\partial\mathcal{G}_n^1$ is covered by $\Omega(n)$ edges in $\mathcal{G}_n^1$, we need to remove at least $\Omega(n^3)$ edges from $\mathcal{G}_n^1$ before getting $\mathcal{G}_n^2$. So this proves that $\mathcal{M}$ does not have the stability property (in the sense of Theorem 1.2).

A family $\mathcal{F}$ is t-stable if there exist t near-extremal constructions, and every $\mathcal{F}$-free graph (or hypergraph) of size close to $\mathrm{ex}(n,\mathcal{F})$ is structurally close to one of these near-extremal constructions. The stability number of $\mathcal{F}$, denoted by $\xi(\mathcal{F})$, is the minimum integer t such that $\mathcal{F}$ is t-stable. If there is no such integer t, then we let $\xi(\mathcal{F}) = \infty$.

Although the concept of t-stable families was raised over a decade ago (see [18] and [21]), no example of t-stable families are known for any $t \geq 2$ before this work. However, if we assume that Turán's conjecture is true, then the following result shows that the stability number of K_4^3 is infinity.

Proposition 1.7. [2] *If Conjecture 1.1 is true, then $\xi(K_4^3) = \infty$.*

Our next result gives further detail about near-extremal $\mathcal{M}$-free constructions by showing that $\mathcal{M}$ is 2-stable with respect to $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$. More precisely, it shows that $\xi(\mathcal{M}) = 2$.

Definition 1.8. *Let $\mathcal{H}$ be a 3-graph. Then $\mathcal{H}$ is called semibipartite if $V(\mathcal{H})$ has a partition $A \cup B$ such that $|E \cap A| = 1$ and $|E \cap B| = 2$ for all $E \in \mathcal{H}$, and $\mathcal{H}$ is called $\mathcal{G}_6^2$-colorable if it is a subgraph of a blowup of $\mathcal{G}_6^2$.*

With some calculations one can get the following observation.

Observation 1.9. *Let $\mathcal{H}$ be a 3-graph on n-vertices. If $\mathcal{H}$ is semibipartite, then $|\mathcal{H}| \leq g_1(n)$. If $\mathcal{H}$ is $\mathcal{G}_6^2$-colorable, then $|\mathcal{H}| \leq g_2(n)$.*

Theorem 1.10 (2-stability)**.** *For every $\delta > 0$ there exists $\epsilon > 0$ and n_0 such that the following holds for all $n \geq n_0$. Every $\mathcal{M}$-free 3-graph on n vertices with at least $2n^3/27 - \epsilon n^3$ edges can be transformed to a 3-graph that is either semibipartite or $\mathcal{G}_6^2$-colorable by removing at most δn vertices. In other words, $\xi(\mathcal{M}) = 2$.*

Note that Theorem 1.10 is stronger than the requirement in the definition of 2-stability since removing at most δn vertices implies that the number of edges removed is at most δn^3 but not vice versa.

Let $\mathcal{H}$ be an r-graph on n vertices. The edge density of $\mathcal{H}$ is $d(\mathcal{H}) := |\mathcal{H}|/\binom{n}{r}$ and the shadow density of $\mathcal{H}$ is $d(\partial\mathcal{H}) := |\partial\mathcal{H}|/\binom{n}{r-1}$. The feasible region $\Omega(\mathcal{F})$ of $\mathcal{F}$ is the set of points $(x,y) \in [0,1]^2$ such that there exists a sequence of $\mathcal{F}$-free r-graphs $(\mathcal{H}_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty} v(\mathcal{H}_k) = \infty$, $\lim_{k\to\infty} d(\partial\mathcal{H}_k) = x$ and $\lim_{k\to\infty} d(\mathcal{H}_k) = y$. We introduced this notion recently in [15] to understand the extremal properties of $\mathcal{F}$-free hypergraphs beyond just the determination of $\pi(\mathcal{F})$ (it unifies and generalizes many classical problems). In particular, we proved that $\Omega(F)$ is completely determined by a left-continuous almost everywhere differentiable function $g(\mathcal{F}) : \mathrm{proj}\Omega(\mathcal{F}) \to [0,1]$, where

$$\mathrm{proj}\Omega(\mathcal{F}) = \{x : \exists y \in [0,1] \text{ such that } (x,y) \in \Omega(\mathcal{F})\},$$

and

$$g(\mathcal{F},x) = \max\{y : (x,y) \in \Omega(\mathcal{F})\}, \text{ for all } x \in \mathrm{proj}\Omega(\mathcal{F}).$$

Theorem 1.6 together with Theorem 1.10 yield the following result.

[2] We refer the reader to [14] for a proof.

Theorem 1.11. [3] *The set* proj$\Omega(\mathcal{M}) = [0,1]$, *and* $g(\mathcal{M}, x) \le 4/9$ *for all* $x \in [0,1]$. *Moreover,* $g(\mathcal{M}, x) = 4/9$ *iff* $x \in \{5/6, 8/9\}$.

In words, Theorem 1.11 says that $\mathcal{M}$-free 3-graphs can have any possible shadow density but the edge density is maximized for exactly two values of the shadow densities.

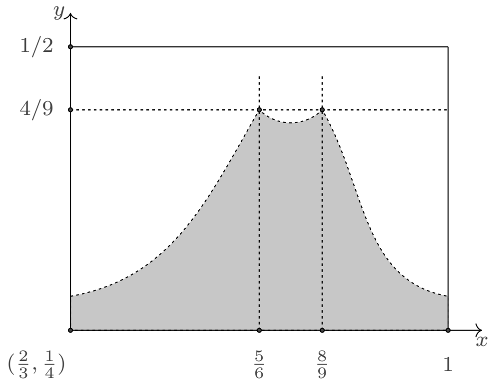


Figure 2: $g(\mathcal{M})$ has exactly two global maxima by Theorem 1.11.

This paper is organized as follows. In Section 2 we will present some preliminary definitions and lemmas for the proofs of Theorems 1.6 and 1.10. In Section 3 we will prove Theorem 1.6, and in Section 4 we will prove Theorem 1.10.

2 Preliminaries

For a graph G and two disjoint sets $A, B \subset V(G)$ denote by $G[A, B]$ the induced bipartite subgraph of G with two parts A and B.

Let $r \ge 2$ and $\mathcal{H}$ be an r-graph. For every $v \in V(\mathcal{H})$ the link $L_{\mathcal{H}}(v)$ of v in $\mathcal{H}$ is

$$L_{\mathcal{H}}(v) = \{A \in \partial\mathcal{H} \colon A \cup \{v\} \in \mathcal{H}\},$$

the degree of v in $\mathcal{H}$ is $d_{\mathcal{H}}(v) := |L_{\mathcal{H}}(v)|$, and the minimum degree of $\mathcal{H}$ is $\delta(\mathcal{H}) := \min\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$. For $S \subset V(\mathcal{H})$, the neighborhood[4] of S in $\mathcal{H}$ is

$$N_{\mathcal{H}}(S) := \{v \in V(\mathcal{H}) \setminus S : \exists E \in \mathcal{H} \text{ such that } \{v\} \cup S \subset E\}.$$

Two vertices $u, v \in V(\mathcal{H})$ are adjacent in $\mathcal{H}$ if $u \in N_{\mathcal{H}}(v)$. When it is clear from context we will omit the subscript $\mathcal{H}$ in the notations above.

Let $V(\mathcal{H}) = [n]$. For $x = (x_1, \ldots, x_n)$ define the weight polynomial of a hypergraph $\mathcal{H}$ as

$$p_{\mathcal{H}}(x) := \sum_{E \in \mathcal{H}} \prod_{i \in E} x_i.$$

The standard n-simplex is

$$\Delta^n := \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \ge 0 \text{ for all } i \in [n+1] \right\}.$$

The Lagrangian of $\mathcal{H}$ is

$$\lambda(\mathcal{H}) := \max\left\{ p_{\mathcal{H}}(x) : x \in \Delta^{n-1} \right\}.$$

[3] We refer the reader to [14] for a proof.

[4] Note that this is not a standard definition for the neighborhood. Some authors define the the neighborhood of an s-set S to be its $(r-s)$-uniform link.

Note that Δ^{n-1} is compact in $\mathbb{R}^n$ and $p_{\mathcal{H}}(x)$ is continuous, so $\lambda(\mathcal{H})$ is well-defined.

Recall that in Section 1 we defined the blowup of an r-graph $\mathcal{T}$. The next standard lemma gives a relationship between $\lambda(\mathcal{T})$ and the size of a blowup of $\mathcal{T}$.

Lemma 2.1. *Let $r \geq 2$ and $\mathcal{T}$ and $\mathcal{H}$ be two r-graphs. Suppose that $\mathcal{H}$ is a blowup of $\mathcal{T}$ with $v(\mathcal{H}) = n$. Then $|\mathcal{H}| \leq \lambda(\mathcal{T})n^r$.*

Proof. Suppose that $|V(\mathcal{T})| = s$ and $\mathcal{H} = \mathcal{T}(t)$ for some $t = (t_1, \dots, t_s)$. Then

$$|\mathcal{H}| = \sum_{E\in\mathcal{T}} \prod_{i\in E} t_i = n^r \sum_{E\in\mathcal{T}} \prod_{i\in E} \frac{t_i}{n} \leq \lambda(\mathcal{T})n^r,$$

where the last inequality follows from the definition of $\lambda(\mathcal{T})$ and $\sum_{i\in[s]} t_i = n$. ∎

Given another r-graph F we say $f : V(F) \to V(\mathcal{H})$ is a homomorphism if $f(E) \in \mathcal{H}$ for all $E \in F$, i.e., f preserves edges. We say that $\mathcal{H}$ is F-hom-free if there is no homomorphism from F to $\mathcal{H}$. In other words, $\mathcal{H}$ is F-hom-free if and only if all blowups of $\mathcal{H}$ are F-free. For a family $\mathcal{F}$ of r-graphs, $\mathcal{H}$ is $\mathcal{F}$-hom-free if it is F-hom-free for all $F \in \mathcal{F}$.

An r-graph F is 2-covered if every $\{u, v\} \subset V(F)$ is contained in some $E \in F$, and a family $\mathcal{F}$ is 2-covered if all $F \in \mathcal{F}$ are 2-covered. It is easy to see that if $\mathcal{F}$ is 2-covered, then $\mathcal{H}$ is $\mathcal{F}$-free if and only if it is $\mathcal{F}$-hom-free. Although $\mathcal{M}$ is not 2-covered, we still have a similar result.

Lemma 2.2. *A 3-graph $\mathcal{H}$ is $\mathcal{M}$-free if and only if it is $\mathcal{M}$-hom-free.*

Proof. The backward implication is clear. Now suppose conversely that $\mathcal{H}$ fails to be $\mathcal{M}$-hom-free, i.e., that there is a homomorphism $f\colon V(F) \to V(\mathcal{H})$ for some $F \in \mathcal{M}$. If $F \cong K_5^{3-}$, then f is injective due to the fact that K_5^{3-} is 2-covered. However, this implies that $K_5^{3-} \subset \mathcal{H}$, a contradiction. Therefore, $F \in M_2 \cup M_3$. Clearly the restriction of f to the core S of F is injective. So $f(F) \in \mathcal{K}_{|S|}^3 \cap \mathcal{M}$ and in view of $f(F) \subset \mathcal{H}$ it follows that $\mathcal{H}$ fails to be $\mathcal{M}$-free. ∎

3 Turán number of $\mathcal{M}$

In this section, we will prove Theorem 1.6. The first subsection contains some technical lemmas and calculations needed in the proof.

3.1 Lagrangian of some 3-graphs

Lemma 3.1. *Suppose that $\mathcal{T}$ is a 3-graph with at most four vertices. Then $\lambda(\mathcal{T}) \leq 1/16$.*

Proof. Without loss of generality we may assume that $v(\mathcal{T}) = 4$ and $|\mathcal{T}| = 4$, i.e., $\mathcal{T} \cong K_4^3$. It is easy to see that

$$p_{K_4^3}(x) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \leq 4(1/4)^3 = 1/16.$$

Therefore, $\lambda(\mathcal{T}) \leq 1/16$. ∎

Lemma 3.2. [5] *Suppose that $\mathcal{T}$ is a 3-graph on five vertices with at most eight edges. Then $\lambda(\mathcal{T}) < 0.067277$.*

Lemma 3.3. $\lambda(\mathcal{G}_6^2) \leq 2/27$.

[5] We refer the reader to [14] for a proof.

Proof. Notice that

$$p_{\mathcal{G}_6^2}(x_1,\ldots,x_6) = x_3x_6(x_1+x_2+x_4+x_5) + (x_1+x_2)(x_3+x_6)(x_4+x_5) + x_1x_2(x_4+x_5) + x_4x_5(x_1+x_2).$$

Setting $a = (x_3+x_6)/2$, $b = (x_1+x_2)/2$, $c = (x_4+x_5)/2$, $d = (b+c)/2$, and then it follows from the AM-GM inequality that

$$\begin{aligned} p_{\mathcal{G}_6^2}(x_1,\ldots,x_6) \le 2a^2(b+c) + 8abc + 2bc(b+c) &\le 4a^2d + 8ad^2 + 4d^3 \\ &= 2\left((a+d)\cdot(a+d)\cdot 2d\right) \\ &\le 2\left(\frac{(a+d)+(a+d)+2d}{3}\right)^3 = \frac{2}{27}. \end{aligned}$$

∎

Lemma 3.4. *Let $\mathcal{T}$ be a 2-covered 3-graph on $k \ge 7$ vertices. Suppose that $\tau(\mathcal{T}[S]) \le 1$ for all sets $S \subset V(\mathcal{T})$ with $|S| = 7$. Then $\mathcal{T}$ is a star.*

Remark. In fact, a weaker condition that $|S| = 6$ is sufficient for the proof of Lemma 3.4.

Proof. Suppose that $\mathcal{T}$ is not a star. Then for every vertex v in $\mathcal{T}$ there exists an edge E_v in $\mathcal{T}$ that does not contain v.

First notice that $\mathcal{T}$ cannot contain two disjoint edges. Therefore, $\mathcal{T}$ is intersecting. Suppose that $\mathcal{T}$ contains two edges $E_1 = \{u, v_1, v_2\}$ and $E_2 = \{u, w_1, w_2\}$, where $\{v_1, v_2\} \cap \{w_1, w_2\} = \emptyset$. Let $E_3 \in \mathcal{T}$ be an edge that does not contain u. Since $\mathcal{T}$ is intersecting, we may assume that $v_1, w_1 \in E_3$. Then, we have $|E_1 \cup E_2 \cup E_3| \le 6$, and $\tau(\{E_1, E_2, E_3\}) = 2$, a contradiction. Therefore, we may assume that the intersection of every two edges in $\mathcal{T}$ has size two. Let $E_1 = \{u, v, w_1\}$ and $E_2 = \{u, v, w_2\}$ be two edges in $\mathcal{T}$. By assumption there exists an edge $E_3 \in \mathcal{T}$ that does not contain u and, hence, we have $E_3 = \{v, w_1, w_2\}$. Similarly there exists $E_4 \in \mathcal{T}$ that does not contain v and, hence, we have $E_4 = \{u, w_1, w_2\}$. Then, we have $|E_1 \cup E_2 \cup E_3 \cup E_4| = 4$, and $\tau(\{E_1, E_2, E_3, E_4\}) = 2$, a contradiction. ∎

3.2 Proof of Theorem 1.6

In this section we complete the proof of Theorem 1.6.

For $v \in V(\mathcal{H})$ and $E \in \mathcal{H}$, $\mathcal{H} - v$ is obtained by removing v and all edges containing v from $\mathcal{H}$, and $\mathcal{H} - E$ is obtained by removing E from $\mathcal{H}$ and keeping $V(\mathcal{H})$ unchanged.

Definition 3.5 (Equivalence classes)**.** *Let $\mathcal{H}$ be an r-graph and u, v be two non-adjacent (i.e. no edge containing both) vertices in $\mathcal{H}$. Then u and v are equivalent if $L(u) = L(v)$, otherwise they are non-equivalent. If u and v are equivalent, then we write $u \sim v$. Let C_v denote the equivalence class of v.*

Algorithm 1 (Symmetrization without cleaning) Let $\mathcal{H}$ be an r-graph. We perform the following operation as long as there are two non-adjacent non-equivalent vertices in $\mathcal{H}$. Let u, v be two such vertices with $d(u) \ge d(v)$. Then we delete all vertices from C_v and duplicate u using $|C_v|$ vertices and still label these new vertices with labels in C_v. Another way to view this operation is that we remove all edges in $\mathcal{H}$ that have nonempty intersection with C_v and for every $E \in \mathcal{H}$ with $u \in E$ we add $E - \{u\} \cup \{v'\}$ for all $v' \in C_v$ into $\mathcal{H}$. We terminate the process when there is no non-adjacent non-equivalent pair.

Note that the number of equivalence classes in $\mathcal{H}$ strictly decreases after each step that can be performed, so Algorithm 1 always terminates. On the other hand, since symmetrization only deletes and duplicates vertices, by Lemma 2.2, Algorithm 1 preserves the $\mathcal{M}$-freeness of $\mathcal{H}$. The following lemma is immediate from the definition.

Lemma 3.6. *Let $\mathcal{H}_t$ be the 3-graph obtained from $\mathcal{H}$ by applying Algorithm 1, and let $T \subset V(\mathcal{H})$ such that T contains exactly one vertex from each equivalence class of $\mathcal{H}_t$. Then,*

(a) $|\mathcal{H}_t| \geq |\mathcal{H}|$.

(b) $\mathcal{H}_t[T]$ *is 2-covered and* $\mathcal{H}_t$ *is a blowup of* $\mathcal{H}_t[T]$.

Now we are ready to finish the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $\mathcal{H}$ be an $\mathcal{M}$-free 3-graph on n vertices. Apply Algorithm 1 to $\mathcal{H}$ and let $\mathcal{H}_t$ denote the resulting 3-graph. Let $T \subset V(\mathcal{H})$ such that T contains exactly one vertex from each equivalent class in $\mathcal{H}_t$, and let $\mathcal{T} = \mathcal{H}_t[T]$. By Lemma 3.6, in order to prove $|\mathcal{H}| \leq 2n^3/27$, it suffices to show $|\mathcal{H}_t| \leq 2n^3/27$. Since $\mathcal{H}_t$ is a blowup of $\mathcal{T}$, by Lemma 2.1, it suffices to show that $\lambda(\mathcal{T}) \leq 2/27$. Next, we will consider two cases depending on the size of T: either $|T| \geq 7$ or $|T| \leq 6$.

Case 1: $|T| \geq 7$.
Since $\mathcal{T}$ is 2-covered and it is M_2-free, $\tau(\mathcal{T}[S]) \leq 1$ for all $S \subset T$ with $|S| = 7$, and it follows from Lemma 3.4 that $\mathcal{T}$ is a star.

Let us calculate $\lambda(\mathcal{T})$. We may assume that $V(\mathcal{T}) = [s]$ for some integer s and 1 is the center of $\mathcal{T}$. Then,

$$p_{\mathcal{T}}(x) \leq x_1 \left(\sum_{\{i,j\} \subset [s] \setminus \{1\}} x_i x_j \right) \leq \frac{s-2}{2(s-1)} x_1 (1-x_1)^2 < \frac{1}{2} x_1 (1-x_1)^2 \leq \frac{2}{27},$$

which implies that $\lambda(\mathcal{T}) < 2/27$.

Case 2: $|T| \leq 6$.
If $|T| \leq 5$, then Lemmas 3.1 and 3.2 imply that $\lambda(\mathcal{T}) < 0.67277$. So we may assume that $|T| = 6$.

Lemma 3.6 implies that $\mathcal{T}$ is 2-covered, so $\mathcal{T} \in \mathcal{K}_6^3$. Since $\mathcal{H}_t$ does not contain any member in M_3 as a subgraph, either $\mathcal{T} \subset \mathcal{G}_n^1$ or $\mathcal{T} \subset \mathcal{G}_n^2$ for some $n \geq 6$. Due to the fact that $\mathcal{T}$ is 2-covered again, either $\mathcal{T}$ is a star or $\mathcal{T} \subset \mathcal{G}_6^2$. The former case has been handled by Case 1, so we may assume that $\mathcal{T} \subset \mathcal{G}_6^2$, and it follows from Lemma 3.3 that $\lambda(\mathcal{T}) \leq \lambda(\mathcal{G}_6^2) \leq 2/27$. ∎

4 Stability of $\mathcal{M}$

In this section we will prove Theorem 1.10. First we present an algorithm and some lemmas that will be used in the proof.

4.1 Symmetrization

Let $0 \leq \alpha \leq 1$ and $\mathcal{H}$ be a 3-graph. Then $\mathcal{H}$ is α-dense if $\delta(\mathcal{H}) \geq \alpha\binom{v(\mathcal{H})-1}{2}$. Let $(V, \prec_V)$ be a poset on V with relation $\prec_V$. For $S \subset V$ the induced poset of $(V, \prec_V)$ on S is denoted by $(S, \prec_V)$.

Algorithm 2 (Symmetrization and cleaning with threshold α).
Input: A 3-graph $\mathcal{H}$.
Operation:

- **Initial step**: If $\delta(\mathcal{H}) \geq \alpha\binom{v(\mathcal{H})-1}{2}$, then let $\mathcal{H}_0 = \mathcal{H}$ and $V_0 = V(\mathcal{H})$. Otherwise, we keep deleting vertices with the minimum degree one by one until the remaining 3-graph $\mathcal{H}_0$ is either empty or $\delta(\mathcal{H}_0) \geq \alpha\binom{v(\mathcal{H}_0)-1}{2}$. Let Z_0 be the set of deleted vertices during this process so that $V_0 := V(\mathcal{H}_0) = V(\mathcal{H}) - Z_0$.

Let $(V_0, \prec_{V_0})$ be the poset with V_0 itself an antichain, i.e., there is no relation between any two vertices in V_0.

Suppose we are at the i-th step for some $i \geq 1$. We terminate the algorithm if either

(a) $\mathcal{H}_{i-1} = \emptyset$ or

(b) $\delta(\mathcal{H}_{i-1}) \geq \alpha\binom{v(\mathcal{H}_{i-1})-1}{2}$ and there is no pair of non-adjacent non-equivalent vertices.

Otherwise, we iterate the following two operations.

Step 1 (Symmetrization): If $\mathcal{H}_{i-1}$ contains no pair of non-adjacent non-equivalent vertices, then let $\mathcal{G}_i = \mathcal{H}_{i-1}$ and go to Step 2. Otherwise, choose two non-adjacent non-equivalent vertices $u, v \in V(\mathcal{H}_{i-1})$ and assume that $d(u) \geq d(v)$. Delete all vertices in C_v and add $|C_v|$ new vertices into C_u by duplicating u and label these new vertices with labels in C_v, which is the same as what we did in Algorithm 1. Let $\mathcal{G}_i$ denote the resulting r-graph, and update the poset $(V_{i-1}, \prec_{V_{i-1}})$ by adding the following relations: $v' \prec u'$ for all $v' \in C_v$ and all $u' \in C_u$. This new poset is well-defined as one will see from the following operations that once two equivalence classes are merged they will never be split.

Step 2 (Cleaning): If $\delta(\mathcal{G}_i) \geq \alpha\binom{v(\mathcal{G}_i)-1}{2}$, then let $\mathcal{H}_i = \mathcal{G}_i$ and $(V_i, \prec_{V_i}) = (V_{i-1}, \prec_{V_{i-1}})$. Otherwise let $\mathcal{L} = \mathcal{G}_i$ and repeat Steps 2.1 and 2.2.

Step 2.1: Let $B = \{a \in V(\mathcal{L}) : d_{\mathcal{L}}(a) = \delta(\mathcal{L})\}$ and choose a minimal element $z \in (B, \prec_{V_{i-1}}$). Replace $\mathcal{L}$, V_{i-1}, and $(V_{i-1}, \prec_{V_{i-1}})$ by $\mathcal{L} - z$, $V_{i-1} \setminus \{z\}$, and $(V_{i-1} \setminus \{z\}, \prec_{V_{i-1}})$, respectively.

Step 2.2: If $\delta(\mathcal{L}) \geq \alpha\binom{v(\mathcal{L})-1}{2}$ or $\mathcal{L} = \emptyset$, then stop. Otherwise, go to Step 2.1.

Let $\mathcal{H}_i = \mathcal{L}$ and $(V_i, \prec_{V_i}) = (V_{i-1}, \prec_{V_{i-1}})$. Let Z_i denote the set of vertices removed by Step 2.1 so that $\mathcal{H}_i = \mathcal{G}_i - Z_i$.

Output: A 3-graph $\mathcal{H}_t$ for some t such that either $\mathcal{H}_t$ is empty or $\delta(\mathcal{H}_t) \geq \alpha\binom{v(\mathcal{H}_t)-1}{2}$ and there is no pair of non-adjacent non-equivalent vertices in $\mathcal{H}_t$.

Remark. The point of Step 2 is that the symmetrization step (Step 1) could potentially bring down the degree of some of the vertices in the hypergraph, making the pruning step (Step 2) necessary.

Let $\epsilon > 0$ be sufficiently small and n be sufficiently large. Let $\mathcal{H}$ be an $\mathcal{M}$-free 3-graph on n vertices with $|\mathcal{H}| \geq 2n^3/27 - \epsilon n^3$. Apply Algorithm 2 to $\mathcal{H}$ with threshold $\alpha = 4/9 - 3\epsilon^{1/2}$ and suppose that it stops at the t-th step. Let $\mathcal{H}_t$ denote the resulting 3-graph and $W = V(\mathcal{H}_t)$ and $\tilde{n} = |W|$. For $0 \leq i \leq t$ let $\widetilde{\mathcal{H}}_i = \mathcal{H}_i[W]$ and $\widetilde{\mathcal{G}}_i = \mathcal{G}_i[W]$. Note that $\widetilde{\mathcal{H}}_0 = \mathcal{H}[W]$ and $\widetilde{\mathcal{G}}_0 = \mathcal{G}[W]$, and we will omit the subscript 0 if there is no cause for confusion. Let $Z = \bigcup_{i=0}^{t} Z_i$ be the set of vertices in $\mathcal{H}$ that were removed by Algorithm 2. In the rest of the proof we will focus on $\widetilde{\mathcal{H}}_i$ and $\widetilde{\mathcal{G}}_i$. Notice from Algorithm 2 that $\mathcal{H}_i = \mathcal{G}_i - Z_i$ and $Z_i \subset V(\mathcal{H}) \setminus W$, therefore, $\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{G}}_i$ for all $1 \leq i \leq t$.

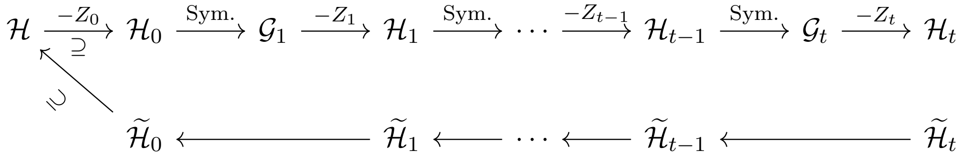


Figure 3: The first line contains the 3-graphs produced by Algorithm 2 and the second line contains the corresponding induced 3-graphs on W.

Lemma 4.1. *For every $i \in [t]$ either $\widetilde{\mathcal{H}}_{i-1} = \widetilde{\mathcal{H}}_i$ or there exist two nonempty equivalence classes $V_i \subset W$ and $U_i \subset W$ in $\widetilde{\mathcal{H}}_{i-1}$ such that $\widetilde{\mathcal{H}}_i$ is obtained from $\widetilde{\mathcal{H}}_{i-1}$ by deleting all vertices in V_i and adding $|V_i|$ new vertices by duplicating some vertex in U_i.*

Proof. Fix $1 \le i \le t$ and suppose that in forming $\mathcal{G}_i$ from $\mathcal{H}_{i-1}$ in Algorithm 2 we deleted all vertices in C_v and added $|C_v|$ new vertices by duplicating some $u \in C_u$, where C_v (resp. C_u) is the equivalence class of $v \in V(\mathcal{H}_{i-1})$ (resp. $u \in V(\mathcal{H}_{i-1})$) in $\mathcal{H}_{i-1}$. Let $\widehat{C}_u = C_v \cup C_u$ and notice that for every $i \le j \le t$ the set $\widehat{C}_u \cap V(\mathcal{G}_j)$ (resp. $\widehat{C}_u \cap V(\mathcal{H}_j)$) is an equivalence class in $\mathcal{G}_j$ (resp. $\mathcal{H}_j$).

If $C_v \cap W = \emptyset$, then $\widetilde{\mathcal{H}}_{i-1} = \widetilde{\mathcal{H}}_i$ and we are done. So we may assume that $C_v \cap W \neq \emptyset$.

First, we claim that $C_u \subset W$. Indeed, suppose that there exists $u' \in C_u \setminus W$. Then it means that u' was removed at the j-th step for some $i \le j \le t$. Since all $v' \in C_v$ satisfy $v' \prec_{V_k} u'$ and $d_{\mathcal{G}_k}(v') = d_{\mathcal{G}_k}(u')$ for all $i \le k \le j$, by definition of Algorithm 2 all vertices in C_v must be removed before u' was removed, which implies that $C_v \cap W = \emptyset$, a contradiction. Therefore, $C_u \subset W$.

Let $V_i = C_v \cap W$ and $U_i = C_u$ and note that neither of them is empty. Since C_v and C_u are equivalence classes in $\mathcal{H}_{i-1}$, V_i and U_i are equivalence classes in $\widetilde{\mathcal{H}}_{i-1}$. According to Algorithm 2, $\widetilde{\mathcal{H}}_i$ is obtained from $\widetilde{\mathcal{H}}_{i-1}$ by deleting all vertices in V_i and adding $|V_i|$ new vertices by duplicating some vertex in U_i. ∎

The following two lemmas show that the size of the set Z of vertices removed by Algorithm 2 is small, and the induced subgraph $\widetilde{\mathcal{H}}_i$ of $\mathcal{H}_i$ on W has a large minimum degree for $0 \le i \le t$. Their proofs can be found in [1].

Lemma 4.2. *We have $|Z| \le 3\epsilon^{1/2} n$, and hence $\tilde{n} \ge n - 3\epsilon^{1/2} n$.*

Lemma 4.3. *For all $0 \le i \le t$,*

$$\delta(\widetilde{\mathcal{H}}_i) > \left(4/9 - 10\epsilon^{1/2}\right) \binom{\tilde{n}-1}{2}$$

and, in particular,

$$|\widetilde{\mathcal{H}}_i| > \left(4/9 - 10\epsilon^{1/2}\right) \binom{\tilde{n}}{3}.$$

Notice that $\widetilde{\mathcal{H}}_t = \mathcal{H}_t$ and $\widetilde{\mathcal{H}}_0 = \mathcal{H}[W]$. In order to prove Theorem 1.10 it suffices to show that $\widetilde{\mathcal{H}}_0$ is either semibipartite or $\mathcal{G}_6^2$-colorable. We will proceed by backward induction on i. The following lemma establishes the base case of the induction.

Lemma 4.4. *Let $T \subset W$ such that T contains exactly one vertex in each equivalence class in $\mathcal{H}_t$ and $\mathcal{T} = \mathcal{H}_t[T]$. Then, either $\mathcal{T}$ is a star or $\mathcal{T} \subset \mathcal{G}_6^2$ and, in particular, $\mathcal{H}_t$ is either semibipartite or $\mathcal{G}_6^2$-colorable.*

Proof. First we claim that $|T| \ge 6$. Indeed, suppose that $|T| \le 5$. Then, Lemmas 3.1 and 3.2 imply that $\lambda(\mathcal{T}) < 0.067277$. It follows from Lemma 2.1 that $|\mathcal{H}_t| < 0.067277\tilde{n}^3 < (4/9 - 10\epsilon^{1/2})\binom{\tilde{n}}{3}$, which contradicts Lemma 4.3. Therefore, $|T| \ge 6$.

Suppose that $|T| \ge 7$. Since $\mathcal{T}$ is 2-covered and M_2-free, $\tau(\mathcal{T}[S]) \le 1$ for all $S \subset T$ with $|S| = 7$. So by Lemma 3.4, $\mathcal{T}$ is a star, and hence $\mathcal{H}_t$ is semibipartite.

Suppose that $|T| = 6$. Since $\mathcal{T} \in \mathcal{K}_6^3$ and $\mathcal{H}$ is M_3-free, either $\mathcal{T} \subset \mathcal{G}_m^1$ or $\mathcal{T} \subset \mathcal{G}_m^2$ for some integer $m \ge 6$. Moreover, due to the fact that $\mathcal{T}$ is 2-covered, either $\mathcal{T}$ is a star or $\mathcal{T} \subset \mathcal{G}_6^2$. In the former case, $\mathcal{H}_t$ is semibipartite, and in the latter case, $\mathcal{H}_t$ is $\mathcal{G}_6^2$-colorable. ∎

Next, we will consider two cases in the following two subsections depending on the structure of $\mathcal{H}_t$.

4.2 Semibipartite

In this section we will prove the following statement.

Lemma 4.5. *Suppose that $\mathcal{H}_t$ is semibipartite. Then $\widetilde{\mathcal{H}}_i$ is semibipartite for all $0 \le i \le t$. In particular, $\mathcal{H}[W] = \widetilde{\mathcal{H}}_0$ is semibipartite.*

We will use the following stability theorem due to Füredi, Pikhurko, and Simonovits [8] to prove Lemma 4.5.

Let $\mathbb{F}_{3,2}$ be the 3-graph with vertex set $[5]$ and edges set $\{123, 124, 125, 345\}$. Füredi, Pikhurko, and Simonovits [8] proved that if n is sufficiently large, then $\mathcal{G}_n^1$ is the unique $\mathbb{F}_{3,2}$-free 3-graph on n vertices with the maximum number of edges. Moreover, they proved the following strong stability result.

Theorem 4.6 (Füredi-Pikhurko-Simonovits [8])**.** *Let $\gamma \le 1/125$ be fixed and $n \ge n_0$. Let $\mathcal{H}$ be an $\mathbb{F}_{3,2}$-free 3-graph on n vertices with $\delta(\mathcal{H}) > (4/9 - \gamma)\binom{n}{2}$. Then $\mathcal{H}$ is semibipartite.*

Now we prove Lemma 4.5.

Proof of Lemma 4.5. The proof is by backward induction on i and the base case is $i = t$ as $\widetilde{\mathcal{H}}_t = \mathcal{H}_t$. Now suppose that $\widetilde{\mathcal{H}}_{i+1}$ is semibipartite with two parts A^{i+1} and B^{i+1} for some $0 \le i \le t-1$, and every edge in $\widetilde{\mathcal{H}}_{i+1}$ has exactly one vertex in A^{i+1}. We may assume that both A^{i+1} and B^{i+1} are union of some equivalence classes. Our goal is to show that $\widetilde{\mathcal{H}}_i$ is also semibipartite.

Recall that $\epsilon > 0$ is a sufficiently small constant and $\tilde{n}$ is a sufficiently large integer.

Denote by $\widehat{\mathcal{G}}$ the semibipartite 3-graph on W that consists of all triples that have exactly one vertex in A^{i+1}. Notice that $\widetilde{\mathcal{H}}_{i+1} \subset \widehat{\mathcal{G}}$ and $L_{\widetilde{\mathcal{H}}_{i+1}}(w) \subset L_{\widehat{\mathcal{G}}}(w)$ for all $w \in W$.

Claim 4.7. *We have $\left||A^{i+1}| - \tilde{n}/3\right| < 4\epsilon^{1/4}\tilde{n}$ and $\left||B^{i+1}| - 2\tilde{n}/3\right| < 4\epsilon^{1/4}\tilde{n}$.*

Proof of Claim 4.7. Let $\beta = |B^{i+1}|$. Since $\widetilde{\mathcal{H}}_{i+1}$ is semibipartite,

$$|\widetilde{\mathcal{H}}_{i+1}| \le (\tilde{n} - \beta)\binom{\beta}{2}.$$

On the other hand, by Lemma 4.3, $|\widetilde{\mathcal{H}}_{i+1}| \ge (4/9 - 10\epsilon^{1/2})\binom{\tilde{n}}{3}$. Therefore,

$$(4/9 - 10\epsilon^{1/2})\binom{\tilde{n}}{3} \le (\tilde{n} - \beta)\binom{\beta}{2},$$

which implies that $(2/3 - 4\epsilon^{1/4})\tilde{n} < \beta < (2/3 + 4\epsilon^{1/4})\tilde{n}$. ∎

For every vertex $w \in W$ let $M_w = L_{\widehat{\mathcal{G}}}(w) \setminus L_{\widetilde{\mathcal{H}}_{i+1}}(w)$. Members in M_w are called missing edges of $L_{\widetilde{\mathcal{H}}_{i+1}}(w)$.

Claim 4.8. *We have $|M_w| \le 10\epsilon^{1/4}\tilde{n}^2$ for all $w \in W$.*

Proof of Claim 4.8. If $w \in A^{i+1}$, then $L_{\widehat{\mathcal{G}}}(w)$ is a complete graph on B^{i+1}. If $w \in B^{i+1}$, then $L_{\widehat{\mathcal{G}}}(w)$ is a complete bipartite graph with two parts A^{i+1} and B^{i+1}. Claim 4.7 and Lemma 4.3 imply that for every $w \in A^{i+1}$ we have

$$|M_w| \le \binom{2\tilde{n}/3 + 4\epsilon^{1/4}\tilde{n}}{2} - (4/9 - 10\epsilon^{1/2})\binom{\tilde{n}}{2} \le 10\epsilon^{1/4}\tilde{n}^2,$$

and for every $w \in B^{i+1}$ we have

$$|M_w| \le (\tilde{n}/3 + 4\epsilon^{1/4}\tilde{n})(2\tilde{n}/3 + 4\epsilon^{1/4}\tilde{n}) - (4/9 - 10\epsilon^{1/2})\binom{\tilde{n}}{2} \le 10\epsilon^{1/4}\tilde{n}^2.$$

∎

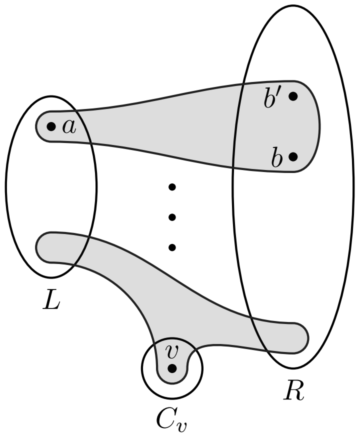


Figure 4: The 3-graph $\widetilde{\mathcal{H}}_{i+1}$ is obtained from $\widetilde{\mathcal{H}}_i$ by symmetrizing C_v to some equivalence class C_u that is contained in L or R.

Lemma 4.1 implies that either $\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{H}}_{i+1}$ or there exists two equivalence classes C_v and C_u in $\widetilde{\mathcal{H}}_i$ such that $\widetilde{\mathcal{H}}_{i+1}$ is obtained from $\widetilde{\mathcal{H}}_i$ by symmetrizing C_v to C_u (see Figure 4). In the former case, there is nothing to prove, so we may assume that we are in the latter case.

Let $L = A^{i+1} \setminus C_v$, $R = B^{i+1} \setminus C_v$ and $W' = W \setminus C_v$. Since $\widetilde{\mathcal{H}}_{i+1}$ is obtained from $\widetilde{\mathcal{H}}_i$ by symmetrizing C_v to C_u that is contained in either L or R, it follows that

$$\text{either} \quad (L \cup C_v, R) = \left(A^{i+1}, B^{i+1}\right) \quad \text{or} \quad (L, R \cup C_v) = \left(A^{i+1}, B^{i+1}\right), \tag{1}$$

and in particular, $L \neq \emptyset$ and $R \neq \emptyset$.

Since C_v is an equivalence class in $\widetilde{\mathcal{H}}_i$, $L_{\widetilde{\mathcal{H}}_i}(v') = L_{\widetilde{\mathcal{H}}_i}(v)$ for all $v' \in C_v$. Thus we may just focus on v. Notice that in forming $\widetilde{\mathcal{H}}_{i+1}$ from $\widetilde{\mathcal{H}}_i$ we only delete and add edges that have nonempty intersection with C_v, so $\widetilde{\mathcal{H}}_i[W'] = \widetilde{\mathcal{H}}_{i+1}[W']$. Since $\widetilde{\mathcal{H}}_{i+1}$ is semibipartite, it follows that $\widetilde{\mathcal{H}}_i[W']$ is semibipartite with two parts L and R.

Claim 4.9. *We have $|N_{\widetilde{\mathcal{H}}_i}(v) \cap R| \geq \left(1/3 - 5\epsilon^{1/4}\right)\tilde{n}$. In particular, $|R| \geq \left(1/3 - 5\epsilon^{1/4}\right)\tilde{n}$.*

Proof of Claim 4.9. By Lemma 4.3,

$$\binom{|N_{\widetilde{\mathcal{H}}_i}(v)|}{2} \geq d_{\widetilde{\mathcal{H}}_i}(v) \geq \left(4/9 - 10\epsilon^{1/2}\right)\binom{\tilde{n}-1}{2},$$

which implies that $|N_{\widetilde{\mathcal{H}}_i}(v)| \geq (2/3 - 15\epsilon^{1/2})\tilde{n}$. By Claim 4.7, $|L| \leq (1/3 + 4\epsilon^{1/4})\tilde{n}$, and hence

$$|N_{\widetilde{\mathcal{H}}_i}(v) \cap R| \geq \left(2/3 - 15\epsilon^{1/2}\right)\tilde{n} - \left(1/3 + 4\epsilon^{1/4}\right)\tilde{n} > \left(1/3 - 5\epsilon^{1/4}\right)\tilde{n}.$$

∎

Claim 4.10. *For every vertex $w \in W'$ we have $|N_{\widetilde{\mathcal{H}}_i} \cap R| \geq |R| - \tilde{n}/100$.*

Proof of Claim 4.10. Notice that $L_{\widetilde{\mathcal{H}}_{i+1}}(w)[W'] = L_{\widetilde{\mathcal{H}}_i}(w)[W']$ for every $w \in W'$. Therefore, for every $w \in L$ we have $|L_{\widehat{\mathcal{G}}}(w)[R] \setminus L_{\widetilde{\mathcal{H}}_i}(w)[R]| \leq |M_w| \leq 10\epsilon^{1/4}\tilde{n}^2$. By Claim 4.9, $|R| \geq \left(1/3 - 5\epsilon^{1/4}\right)\tilde{n}$. So the number of vertices in R that have degree 0 in $L_{\widetilde{\mathcal{H}}_i}(w)[R]$ is at most $2 \times 10\epsilon^{1/4}\tilde{n}^2/|R| < 80\epsilon^{1/4}\tilde{n} < \tilde{n}/100$.

Now fix $u \in R$. If $|L| \geq \tilde{n}/100$, then a similar argument as above applied to graphs $L_{\widehat{\mathcal{G}}}(u)[L, R]$ and $L_{\widetilde{\mathcal{H}}_i}(u)[L, R]$ yields the number of vertices in R that have degree 0 in $L_{\widetilde{\mathcal{H}}_i}(u)[L, R]$ is at most $2 \times 10\epsilon^{1/4}\tilde{n}^2/|L| \leq 2000\epsilon^{1/4}\tilde{n} < \tilde{n}/100$.

So we may assume that $|L| < \tilde{n}/100$. Due to Claim 4.7 and (1), we must have $C_v \cup L = A^{i+1}$ since otherwise we would have $|L| = |A^{i+1}| \geq (1/6 - 4\epsilon^{1/4})\tilde{n} > \tilde{n}/100$, a contradiction. In particular, $|C_v| \leq |A^{i+1}| \leq (1/3 + 4\epsilon^{1/4})\tilde{n}$ and $|R| = |B^{i+1}|$. Notice that $L_{\widetilde{\mathcal{H}}_i}(u)$ is a 3-partite graph with three parts L, R, and C_v (note that C_v is an equivalence class, so no pair in C_v is covered). Let x denote the number of vertices in R that have degree 0 in $L_{\widetilde{\mathcal{H}}_i}(u)$, and note that for a vertex $u' \in R$ with degree 0 in $L_{\widetilde{\mathcal{H}}_i}(u)$ every vertex $u'' \in L \cup C_v$ forms a pair $\{u', u''\}$ that is not contained in $L_{\widetilde{\mathcal{H}}_i}(u)$. Then due to $d_{\widetilde{\mathcal{H}}_i}(u) \geq (4/9 - 10\epsilon^{1/2})\binom{\tilde{n}-1}{2}$, we have

$$(4/9 - 10\epsilon^{1/2})\binom{\tilde{n}-1}{2} + x(|L| + |C_v|) \leq d_{\widetilde{\mathcal{H}}_i}(u) + x(|L| + |C_v|) \leq |L||C_v| + |R|(|L| + |C_v|),$$

which implies that $x \leq \tilde{n}/100$. ∎

We may assume that $\widetilde{\mathcal{H}}_i$ contains a copy of $\mathbb{F}_{3,2}$, since otherwise by Theorem 4.6 we are done. Let $S \subset W$ be a set of size 5 such that $\mathbb{F}_{3,2} \subset \widetilde{\mathcal{H}}_i[S]$. Observe that $S \cap C_v \neq \emptyset$ and due to the fact that $\mathbb{F}_{3,2}$ is 2-covered, we actually have $|S \cap C_v| = 1$. We may assume that $\{v\} = S \cap C_v$. Let $\{w_1, w_2, w_3, w_4\} = S \setminus \{v\}$. Define $R' = R \cap N_{\widetilde{\mathcal{H}}_i}(v) \cap \left(\bigcap_{j\in[4]} N_{\widetilde{\mathcal{H}}_i}(w_j)\right)$. Then Claims 4.9 and 4.10 imply that $|R'| \geq \left(1/3 - 5\epsilon^{1/4}\right)\tilde{n} - 4 \times \tilde{n}/100 > \tilde{n}/6$. Fix a vertex $u \in L$ (it is possible that $u \in \{w_1, w_2, w_3, w_4\}$). By Claim 4.8, $|L_{\widehat{\mathcal{G}}}(u)[R'] \setminus L_{\widetilde{\mathcal{H}}_i}(u)[R']| \leq |M_u| \leq 10\epsilon^{1/4}\tilde{n}^2$. So there exists an edge $w_5w_6 \in L_{\widetilde{\mathcal{H}}_i}(u)[R']$. Let $E \subset \widetilde{\mathcal{H}}_i$ be a set of edges of size at most 10 that covers all pairs in $\{v, w_1, w_2, w_3, w_4\} \times \{w_5, w_6\}$, and let $F = \widetilde{\mathcal{H}}_i[\{v, w_1, w_2, w_3, w_4\}] \cup \{uw_5w_6\} \cup E$. Then it is easy to see that F is a member in M_2 (since $\mathbb{F}_{3,2} \subset \widetilde{\mathcal{H}}_i[\{v, w_1, w_2, w_3, w_4\}]$ has transversal number at least two), a contradiction. ∎

4.3 $\mathcal{G}_6^2$-colorable

In this section we will prove the following statement.

Lemma 4.11. *Suppose that $\mathcal{H}_t$ is $\mathcal{G}_6^2$-colorable. Then $\widetilde{\mathcal{H}}_i$ is $\mathcal{G}_6^2$-colorable for all $0 \leq i \leq t$. In particular, $\mathcal{H}[W] = \widetilde{\mathcal{H}}_0$ is $\mathcal{G}_6^2$-colorable.*

The following lemma, which will be used in the proof of Lemma 4.11, can be easily proved using a probabilistic argument. Its proof can be found in [16].

Consider a 3-graph with $V(\mathcal{G}) = [m]$ and pairwise disjoint sets $V_1, \ldots, V_m$. The blowup $\mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$ is obtained from $\mathcal{G}$ by replacing each vertex $j \in [m]$ with the set V_j and each edge $\{j_1, j_2, j_3\} \in \mathcal{G}$ with the complete 3-partite 3-graph with vertex classes V_{j_1}, V_{j_2}, and V_{j_3}. For a 3-graph $\mathcal{H}$ we say that a partition $V(\mathcal{H}) = \bigcup_{j\in[m]} V_j$ is a $\mathcal{G}$-coloring of $\mathcal{H}$ if $\mathcal{H} \subseteq \mathcal{G}[V_1, \ldots, V_m]$.

Lemma 4.12 ([16])**.** *Fix a real $\eta \in (0,1)$ and integers $m, n \geq 1$. Let $\mathcal{G}$ be a 3-graph with vertex set $[m]$ and let $\mathcal{H}$ be a further 3-graph with $v(\mathcal{H}) = n$. Consider a vertex partition $V(\mathcal{H}) = \bigcup_{i\in[m]} V_i$ and the associated blowup $\widehat{\mathcal{G}} = \mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$. If two sets $T \subseteq [m]$ and $S \subseteq V$ (we allow S to contain vertices from V_i for $i \in T$) have the properties*

(a) $|V_j| \geq (|S| + 1)|T|\eta^{1/3}n + |S|$ for all $j \in T$,

(b) $|\mathcal{H}[V_{j_1}, V_{j_2}, V_{j_3}]| \geq |\widehat{\mathcal{G}}[V_{j_1}, V_{j_2}, V_{j_3}]| - \eta n^3$ for all $\{j_1, j_2, j_3\} \in \binom{T}{3}$, and

(c) $|L_{\mathcal{H}}(v)[V_{j_1}, V_{j_2}]| \geq |L_{\widehat{\mathcal{G}}}(v)[V_{j_1}, V_{j_2}]| - \eta n^3$ for all $v \in S$ and $\{j_1, j_2\} \in \binom{T}{2}$.

then there exists a selection of vertices $u_j \in V_j \setminus S$ for all $j \in [T]$ such that $U = \{u_j : j \in T\}$ satisfies $\widehat{\mathcal{G}}[U] \subseteq \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] \subseteq L_{\mathcal{H}}(v)[U]$ for all $v \in S$. In particular, if $\mathcal{H} \subseteq \widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}[U] = \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] = L_{\mathcal{H}}(v)[U]$ for all $v \in S$.

Now we prove Lemma 4.11.

Lemma 4.11. Similar to Lemma 4.5, the proof of Lemma 4.11 is also by backward induction on i, and the base case is $i = t$ as $\widetilde{\mathcal{H}}_t = \mathcal{H}_t$. Now suppose that $\widetilde{\mathcal{H}}_{i+1}$ is $\mathcal{G}_6^2$-colorable for some $0 \le i \le t-1$, and we want to show that $\widetilde{\mathcal{H}}_i$ is also $\mathcal{G}_6^2$-colorable.

Since $\widetilde{\mathcal{H}}_{i+1}$ is $\mathcal{G}_6^2$-colorable, let

$$\mathcal{P} = \{V_1^{i+1}, \ldots, V_6^{i+1}\}.$$

be the set of six parts in $\widetilde{\mathcal{H}}_{i+1}$ such that there is no edge between $V_1^{i+1}V_2^{i+1}V_3^{i+1}$, $V_1^{i+1}V_2^{i+1}V_6^{i+1}$, $V_3^{i+1}V_4^{i+1}V_5^{i+1}$, and $V_4^{i+1}V_5^{i+1}V_6^{i+1}$ (and every edge in $\widetilde{\mathcal{H}}_{i+1}$ hits at most one vertex in V_j^{i+1} for every $j \in [6]$). We may assume that each set V_j^{i+1} is a union of some equivalence classes.

Let

$$y = (y_1, \ldots, y_6) = \left(|V_1^{i+1}|/\tilde{n}, \ldots, |V_6^{i+1}|/\tilde{n}\right),$$

and notice that a similar argument as in the proof of Lemma 2.1 yields

$$|\widetilde{\mathcal{H}}_{i+1}| \le p_{\mathcal{G}_6^2}(y)\tilde{n}^3. \tag{2}$$

First we give a lower bound and an upper bound for the size of every set in $\mathcal{P}$.

Claim 4.13. *We have $||A| - \tilde{n}/6| < 20\epsilon^{1/4}\tilde{n}$ for every set $A \in \mathcal{P}$.*

Proof of Claim 4.13. Let $\eta = 4\epsilon^{1/2}$ and note that by assumption $\eta > 0$ is sufficiently small and $\tilde{n}$ is sufficiently large. First, it follows from (2) and Lemma 4.3 that

$$p_{\mathcal{G}_6^2}(y_1, \ldots, y_6) \ge \left(4/9 - 10\epsilon^{1/2}\right)\binom{\tilde{n}}{3}/\tilde{n}^3 \ge 2/27 - \eta.$$

On the other hand, let $a = (y_3 + y_6)/2$, $b = (y_1 + y_2)/2$, $c = (y_4 + y_5)/2$, $d = (b + c)/2$ and recall from the proof of Lemma 3.3 that

$$\begin{aligned} p_{\mathcal{G}_6^2}(y_1, \ldots, y_6) &= y_3y_6(y_1 + y_2 + y_4 + y_5) \\ &\quad + (y_1 + y_2)(y_3 + y_6)(y_4 + y_5) + y_1y_2(y_4 + y_5) + y_4y_5(y_1 + y_2) \\ &\le 2a^2(b + c) + 8abc + 2bc(b + c) \le 4a^2d + 8ad^2 + 4d^3 = 2\left((a + d) \cdot (a + d) \cdot 2d\right). \end{aligned}$$

Therefore,

$$(a + d) \cdot (a + d) \cdot 2d \ge 1/27 - \eta/2, \tag{3}$$

and

$$4d(a^2 - y_3y_6) \le \eta, \quad 4d(d^2 - bc) \le \eta, \quad 2c(b^2 - y_1y_2) \le \eta, \quad 2b(c^2 - y_4y_5) \le \eta. \tag{4}$$

Now (3) and $2a + 4d = 1$ yield

$$\begin{aligned} \eta/2 \ge 1/27 - (a + d)^2 \cdot 2d = 1/27 - (1 + 2a)^2(1 - 2a)/32 &= (a - 1/6)^2(a/4 + 5/24) \\ &\ge (a - 1/6)^2/8, \end{aligned}$$

whence $|a - 1/6| \le 2\eta^{1/2}$. By $2|a - 1/6| = 4|d - 1/6|$ this implies $|d - 1/6| \le \eta^{1/2}$. Since η is sufficiently small, it follows that $a, d \ge 1/8$. So the first inequality in (4) leads to $(y_3 - y_6) \le 8\eta$, whence $|y_3 - y_6| \le 3\eta^{1/2}$. By the triangle inequality we obtain

$$2|y_3 - 1/6| \le |y_3 - y_6| + |y_3 + y_6 - 1/3| \le 3\eta^{1/2} + 2|a - 1/6| \le 7\eta^{1/2},$$

which shows $|y_3 - 1/6| \le 4\eta^{1/2}$. Similarly, $|y_6 - 1/6| \le 4\eta^{1/2}$. Applying the same reasoning to the other estimates in (4) we obtain first $|b - 1/6|, |c - 1/6| \le 3\eta^{1/2}$ and then $|y_i - 1/6| \le 5\eta^{1/2}$ for every $i \in \{1, 2, 4, 5\}$. ∎

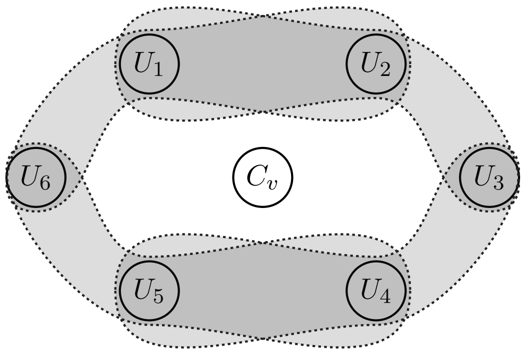


Figure 5: $\widetilde{\mathcal{H}}_{i+1}$ is obtained from $\widetilde{\mathcal{H}}_i$ by symmetrizing C_v to some equivalence class C_u that is contained in U_j for some $j \in [6]$. Dashed lines indicate that there is no edge between these parts in $\widetilde{\mathcal{H}}_i$.

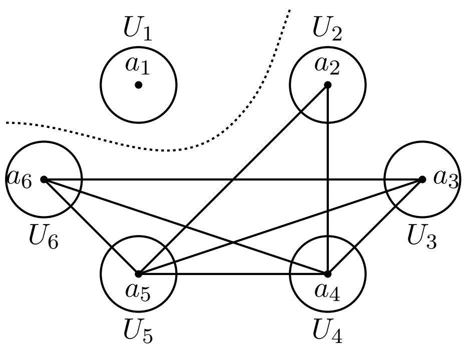
Lemma 4.1 implies that either $\widetilde{\mathcal{H}}_i = \widetilde{\mathcal{H}}_{i+1}$ or there exists two equivalence classes C_v and C_u in $\widetilde{\mathcal{H}}_i$ such that $\widetilde{\mathcal{H}}_i$ is obtained from $\widetilde{\mathcal{H}}_{i+1}$ by symmetrizing C_v to C_u (see Figure 5). In the former case, there is nothing to prove, so we may assume that we are in the latter case. Notice that C_v and C_u are contained in the same member in $\mathcal{P}$, and in particular, Claim 4.13 implies that $|C_v| \le (1/6 + 10\epsilon^{1/4})\tilde{n}$. In the rest of the proof we will focus on the structure of $\widetilde{\mathcal{H}}_i$. Let $U_j = V_j^{i+1} \setminus C_v$ for $j \in [6]$, $W' = W \setminus C_v$, and

$$\mathcal{P}' = \{U_1, \ldots, U_6\}.$$

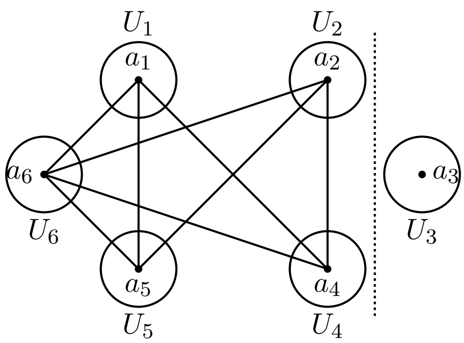
Notice that there exists $j_0 \in [6]$ such that $U_{j_0} \cup C_v = V_{j_0}^{i+1}$, and $U_j = V_j^{i+1}$ holds for all $j \in [6] \setminus \{j_0\}$. In particular, no set in $\mathcal{P}'$ is the empty set.

First we will prove several claims about sets in $\mathcal{P}'$. Since U_1 is a representative for sets in $\{U_1, U_2, U_4, U_5\}$ and U_3 is a representative for sets in $\{U_3, U_6\}$, we shall only prove the statements for U_1 and U_3, and by symmetry, the statements hold for all sets in $\mathcal{P}'$.

Denote by $\widehat{\mathcal{G}}$ the blowup $\mathcal{G}_6^2[U_1, \ldots, U_6]$ of $\mathcal{G}_6^2$, and notice that $\widetilde{\mathcal{H}}_i[W'] \subset \widehat{\mathcal{G}}$. For $j \in [6]$ fix a vertex $a_j \in U_j$, let $\widetilde{G}_j = L_{\widehat{\mathcal{G}}}(a_j)$, $G_j = \widetilde{G}_j[\{a_1, \ldots, a_6\} \setminus \{a_j\}]$, and notice that $\widetilde{G}_j$ is a graph on $W' \setminus U_j$ and is a blowup of G_j (see Figure 6).



(a) The graph G_1 is the 5-vertex graph above, and $\widetilde{G}_1$ is a blowup of G_1.



(b) The graph G_3 is the 5-vertex graph above, and $\widetilde{G}_3$ is a blowup of G_3.

Figure 6: Graphs G_1 and G_3.

For every $w \in W$, let $L(w) = L_{\widetilde{\mathcal{H}}_i}(w)$ and $N(w) = N_{\widetilde{\mathcal{H}}_i}(w)$. Since $\widetilde{\mathcal{H}}_{i+1}$ is $\mathcal{G}_6^2$-colorable and $\widetilde{\mathcal{H}}_i[W'] = \widetilde{\mathcal{H}}_{i+1}[W']$, it follows that $L(w)[W'] \subset \widetilde{G}_j$ for all $j \in [6]$ and $w \in U_j$. For every $j \in [6]$

and every $w \in U_j$ let

$$M(w) = \left\{ w_1w_2 \in \widetilde{G}_j : w_1w_2 \notin L(w)[W'] \right\},$$

and call members in $M(w)$ missing edges of $L(w)[W']$.

Claim 4.14. *We have $|M(w)| \le 30\epsilon^{1/4}\tilde{n}^2$ for every $w \in W$.*

Proof of Claim 4.14. We shall only prove the case $w \in U_1$, since the arguments for other cases are similar. Fix a vertex $w \in U_1$. Let $\widehat{G}_1$ be the blowup of G_1 obtained by replacing each vertex in $V(G_1)$ with the set in $\mathcal{P}$ that contains it. Since $\widetilde{\mathcal{H}}_{i+1}$ is $\mathcal{G}_6^2$-colorable, $L_{\widetilde{\mathcal{H}}_{i+1}}(w) \subset \widehat{G}_1$. On the other hand, since $L_{\widetilde{\mathcal{H}}_i}(w)[W'] = L_{\widetilde{\mathcal{H}}_{i+1}}(w)[W']$, it follows from Lemma 4.3 and Claim 4.13 that

$$\begin{aligned} |M(w)| = |\widetilde{G}_1 \setminus L_{\widetilde{\mathcal{H}}_i}(w)[W']| &\le |\widehat{G}_1 \setminus L_{\widetilde{\mathcal{H}}_{i+1}}(w)| \\ &= |\widehat{G}_1| - |L_{\widetilde{\mathcal{H}}_{i+1}}(w)| \\ &< 8\left(1/6 + 10\epsilon^{1/4}\right)^2 \tilde{n}^2 - \left(4/9 - 10\epsilon^{1/2}\right)\binom{\tilde{n}-1}{2} \\ &< 30\epsilon^{1/4}\tilde{n}^2. \end{aligned}$$

∎

By Lemma 4.3 and Claim 4.14, $\widetilde{\mathcal{H}}_i$ and $\widehat{\mathcal{G}}$ satisfy the following statements, which will be useful later when we applying Lemma 4.12.

(a) $|\widetilde{\mathcal{H}}_i[A_1, A_2, A_3]| \ge |\widehat{\mathcal{G}}[A_1, A_2, A_3]| - 2\epsilon^{1/2}n^3$ for every triple $\{A_1, A_2, A_3\} \subset \mathcal{P}'$, and

(b) $|L_{\widetilde{\mathcal{H}}_i}(u)[A_1, A_2]| \ge |L_{\widehat{\mathcal{G}}}(u)[A_1, A_2]| - 30\epsilon^{1/4}n^3$ for every $u \in W'$ and every pair $\{A_1, A_2\} \subset \mathcal{P}'$ satisfying $u \notin A_1 \cup A_2$.

Claim 4.15. *Let $j \in [6]$ and $w \in U_j$. Then $|N(w) \cap (W' \setminus U_j)| > |W' \setminus U_j| - 400\epsilon^{1/4}\tilde{n}$.*

Proof of Claim 4.15. We shall only prove the case that $P = U_1$, since the arguments for other cases are similar. Let $w \in U_1$ and $W'' = W' \setminus U_1$. Since C_v is contained in exactly one set in $\mathcal{P}$, it follows from Claim 4.13 that all but at most one set in $\mathcal{P}'$ have size at least $\left(1/6 - 10\epsilon^{1/4}\right)\tilde{n}$. On the other hand, since $\delta(G_1) \ge 2$ and $\widetilde{G}_1$ is a blowup of G_1, we obtain

$$\delta(\widetilde{G}_1) > \left(1/6 - 10\epsilon^{1/4}\right)\tilde{n}.$$

So it follows from Claim 4.14 that the number of vertices in W'' with degree 0 in $L(w)[W']$ is at most

$$\frac{2|M_U(w)|}{\delta(\widetilde{G}_1)} < \frac{60\epsilon^{1/4}\tilde{n}^2}{(1/6 - 10\epsilon^{1/4})\tilde{n}} < 400\epsilon^{1/4}\tilde{n}.$$

∎

Recall that $\widetilde{\mathcal{H}}_{i+1}$ is obtained from $\widetilde{\mathcal{H}}_i$ by symmetrizing C_v to C_u, where C_v and C_u are equivalence classes of v and u in $\widetilde{\mathcal{H}}_i$, respectively. Let P_u denote the member in $\mathcal{P}'$ that contains u and notice that $P_u \cup C_v$ is a member in $\mathcal{P}$. So Claim 4.13 implies that $|P_u \cup C_v| \le (1/6 + 10\epsilon^{1/4})\tilde{n}$.

Claim 4.16. *Suppose that $|C_v| > \tilde{n}/12$. Then every vertex in $W' \setminus P_u$ is adjacent to all vertices in C_v in $\widetilde{\mathcal{H}}_i$.*

Proof of Claim 4.16. We shall only prove the case $P_u = U_2$, since the arguments for other cases are similar. First it follows from $|P_u \cup C_v| \le (1/6+10\epsilon^{1/4})\tilde{n}$ that $|P_u| < (1/6+10\epsilon^{1/4})\tilde{n} - \tilde{n}/12 < \tilde{n}/10$. Let $w \in W' \setminus P_u$, and suppose that w is not adjacent to any vertex in C_v. We shall only prove that case $w \in U_1$, since the arguments for other cases are similar.

Since $\widetilde{\mathcal{H}}_i[W'] = \widetilde{\mathcal{H}}_{i+1}[W']$ and $\widetilde{\mathcal{H}}_{i+1}$ is $\mathcal{G}_6^2$-colorable, $L_{\widetilde{\mathcal{H}}_i}(w)[W'] \subset \widetilde{G}_1$. On the other hand, since $N_{\widetilde{\mathcal{H}}_i}(w) \cap C_v = \emptyset$, we actually have $L_{\widetilde{\mathcal{H}}_i}(w) \subset \widetilde{G}_1$. It follows from the definition of $\widetilde{G}_1$, Claim 4.13, and $|U_2| = |P_u| < \tilde{n}/10$ that

$$|L_{\widetilde{\mathcal{H}}_i}(w)| \le |\widetilde{G}_1| < 6\left(1/6 + 10\epsilon^{\frac{1}{4}}\right)^2 \tilde{n}^2 + 2 \times \frac{\tilde{n}}{10}\left(1/6 + 10\epsilon^{\frac{1}{4}}\right)\tilde{n} < \left(2/9 - 10\epsilon^{1/2}\right)\tilde{n}^2,$$

which contradicts Lemma 4.3.

Therefore, w is adjacent to some vertex in C_v (in $\widetilde{\mathcal{H}}_i$). Since C_v is an equivalence class in $\widetilde{\mathcal{H}}_i$, w is adjacent to all vertices in C_v (in $\widetilde{\mathcal{H}}_i$). ∎

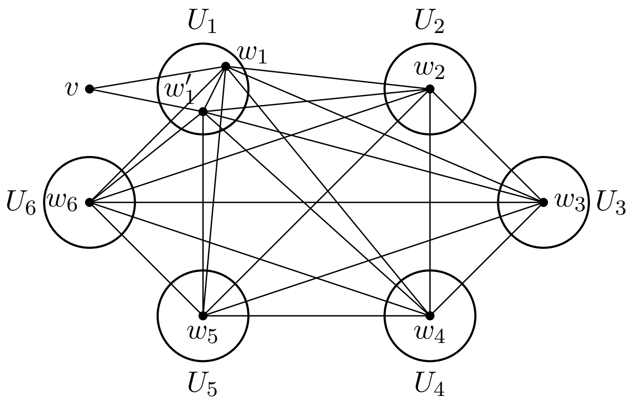


Figure 7: The 3-graph $F = \widetilde{\mathcal{H}}_i[\{w_1, w_2, \ldots, w_6\}] \cup \widetilde{\mathcal{H}}_i[\{w_1', w_2, \ldots, w_6\}] \cup \{vw_1w_1'\}$ is a member in M_2 with core $\{w_1, w_1', w_2, \ldots, w_6\}$. In particular, $\tau(\{w_1w_3w_4, w_1'w_5w_6\}) > 1$.

Claim 4.17. *We have $L(v)[A] = \emptyset$ for every set $A \in \mathcal{P}'$.*

Proof of Claim 4.17. Suppose to the contrary that there exists an edge $w_1w_1' \in L_{\widetilde{\mathcal{H}}_i}(v)[A]$ for some $A \in \mathcal{P}'$. We shall only prove the case $A = U_1$, since the arguments for other cases are similar. It follows from Claim 4.15 that

$$|N(w_1) \cap N(w_1') \cap (W' \setminus U_1)| > |W' \setminus U_1| - 800\epsilon^{1/4}\tilde{n}. \tag{5}$$

Suppose that $|W' \setminus U_1| > 11\tilde{n}/15$. Then by Claim 4.13, $|U_j| \ge 11\tilde{n}/15 - 4(1/6 + 20\epsilon^{1/4})\tilde{n} > \tilde{n}/20$ for every $j \in [2, 6]$. Applying Lemma 4.12 with $S = \{w_1, w_1'\}$, $T = [2, 6]$, and $\eta = 30\epsilon^{1/4}$ we obtain $w_j \in U_j$ for $j \in [2, 6]$ (see Figure 7) such that the induced subgraphs of $\widetilde{\mathcal{H}}_i$ on sets $\{w_1, w_2, \ldots, w_6\}$ and $\{w_1', w_2, \ldots, w_6\}$ are isomorphic to $\mathcal{G}_6^2$. Let $F = \widetilde{\mathcal{H}}_i[\{w_1, w_2, \ldots, w_6\}] \cup \widetilde{\mathcal{H}}_i[\{w_1', w_2, \ldots, w_6\}] \cup \{vw_1w_1'\}$. Then it is easy to see that $F \in M_2$ with core $\{w_1, w_1', w_2, \ldots, w_6\}$ (see Figure 7), a contradiction.

Suppose that $|W' \setminus U_1| \le 11\tilde{n}/15 \le 5(1/6 - 10\epsilon^{1/4})\tilde{n}$. Then by Claim 4.13, $|C_v| \ge \tilde{n} - (1/6 + 10\epsilon^{1/4})\tilde{n} - 11\tilde{n}/15 > \tilde{n}/12$ and $P_u \ne U_1$. We shall only prove that case $P_u = U_2$, since the arguments for other cases are similar. Applying Lemma 4.12 with $S = \{w_1, w_1'\}$, $T = [3, 6]$, and $\eta = 30\epsilon^{1/4}$ we obtain $w_j \in U_j$ for $j \in [3, 6]$ (see Figure 8) such that the induced subgraphs of $\widetilde{\mathcal{H}}_i$ and $\widehat{\mathcal{G}}$ on the sets $\{w_1, w_3, \ldots, w_6\}$ and $\{w_1', w_3, \ldots, w_6\}$ are isomorphic (and they are all 2-covered), respectively. For $j \in [3, 6]$ let $e_j \in \widetilde{\mathcal{H}}_i$ be an edge containing v and w_j (by Claim 4.16, v is adjacent to w_j, so such e_j exists). Define $F = \widetilde{\mathcal{H}}_i[\{w_1, w_3, \ldots, w_6\}] \cup \widetilde{\mathcal{H}}_i[\{w_1', w_3, \ldots, w_6\}] \cup \{vw_1w_1'\} \cup \{e_j \colon j \in [3, 6]\}$. Then it is easy to see that $F \in M_2$ with core $\{v, w_1, w_1', w_3, \ldots, w_6\}$ (see Figure 8), a contradiction. ∎

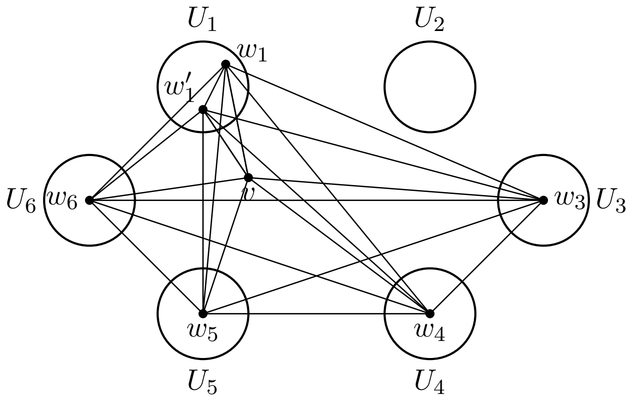


Figure 8: The 3-graph $F = \widetilde{\mathcal{H}}_i[\{w_1, w_3, \ldots, w_6\}] \cup \widetilde{\mathcal{H}}_i[\{w'_1, w_3, \ldots, w_6\}] \cup \{vw_1w'_1\} \cup \{e_j : j \in [3,6]\}$ is a member in M_2 with core $\{v, w_1, w'_1, w_3, \ldots, w_6\}$. In particular, $\tau(\{w_1w_3w_4, w'_1w_5w_6\}) > 1$.

Claim 4.18. *There is at most one set $A \in \mathcal{P}'$ such that $|N(v) \cap A| < \tilde{n}/48$.*

Proof of Claim 4.18. Let $U'_j = N(v) \cap U_j$ for $j \in [6]$. By Claim 4.17, $L(v)$ is a 6-partite graph (not necessarily complete) with the set of parts $\mathcal{P}'' := \{U'_1, U'_2, U'_3, U'_4, V'_1, V'_2\}$. Suppose to the contrary that there are at least two sets in $\mathcal{P}''$ that have size at most $\tilde{n}/48$. Then, by Claim 4.13,

$$|L(v)| \le 6\left(1/6 + 10\epsilon^{1/4}\right)^2 \tilde{n}^2 + (\tilde{n}/48)^2 + 8 \times \tilde{n}/48 \times \left(1/6 + 10\epsilon^{1/4}\right)\tilde{n} < \left(2/9 - 10\epsilon^{1/2}\right)\tilde{n}^2,$$

which contradicts Lemma 4.3. ∎

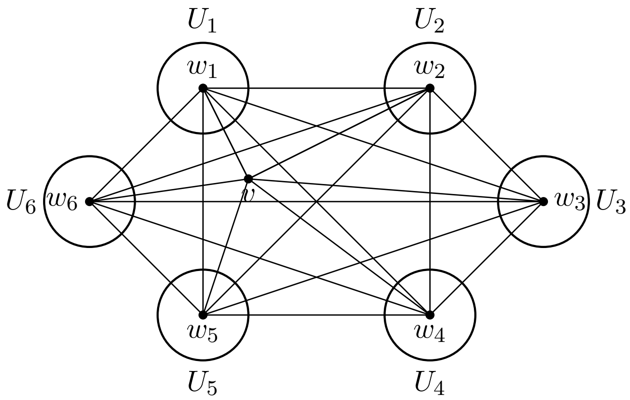


Figure 9: The 3-graph $F = \widetilde{\mathcal{H}}_i[\{w_1, \ldots, w_6\}] \cup \{e_j : j \in [6]\}$ is a member in M_2 with core $\{v, w_1, \ldots, w_6\}$. In particular, $\tau(\{w_1w_3w_4, w_2w_5w_6\}) > 1$.

Claim 4.19. *There exists a set $A \in \mathcal{P}'$ such that $N(v) \cap A = \emptyset$.*

Proof of Claim 4.19. Suppose to the contrary that every set $A \in \mathcal{P}'$ satisfies $A \cap N(v) \neq \emptyset$. By Claim 4.18, there are at least five sets $A' \in \mathcal{P}'$ with $|A' \cap N(v)| \ge \tilde{n}/48$. We shall only prove the case that every set $A' \in \mathcal{P}' \setminus \{U_1\}$ satisfies $|A' \cap N(v)| \ge \tilde{n}/48$, since the arguments for other cases are similar.

Fix a vertex $w_1 \in N(v) \cap U_1$. Let $U'_j = U_j \cap N(v)$ for $i \in [2,6]$. By assumption, $|U'_j| \ge \tilde{n}/48$ for $j \in [2,6]$. So applying Lemma 4.12 with $T = \{w_1\}$, $S = [2,6]$, and $\eta = 30\epsilon^{1/4}$ we obtain

$w_j \in U'_j$ for $j \in [2,6]$ (see Figure 9) such that the induced subgraph of $\widetilde{\mathcal{H}}_i$ on $\{w_1, \ldots, w_6\}$ is isomorphic to $\mathcal{G}_6^2$. For $j \in [6]$ let $e_j \in \widetilde{\mathcal{H}}_i$ be an edge containing v and w_j. Define $F = \widetilde{\mathcal{H}}_i[\{w_1, \ldots, w_6\}] \cup \{e_j : j \in [6]\}$. Then it is easy to see that F is a member in M_2 with core $\{v, w_1, \ldots, w_6\}$ (see Figure 9), a contradiction. ∎

Our next step is to show that $\widetilde{\mathcal{H}}_i$ is $\mathcal{G}_6^2$-colorable with the sets of parts $\widetilde{\mathcal{P}}$, where $\widetilde{\mathcal{P}}$ is obtained from $\mathcal{P}'$ by replacing A with $A \cup C_v$ and the set A is guaranteed by Claim 4.19. We shall only prove the case $A = U_1$, since the arguments for other cases are similar.

Let

$$B_v = \left\{ww' \in L(v) : ww' \notin \widetilde{G}_1\right\}, \quad \text{and} \quad M_v = \left\{ww' \in \widetilde{G}_1 : ww' \notin L(v)\right\}.$$

Members in B_v are called bad edges of $L(v)$ and members in M_v are called missing edges of $L(v)$.

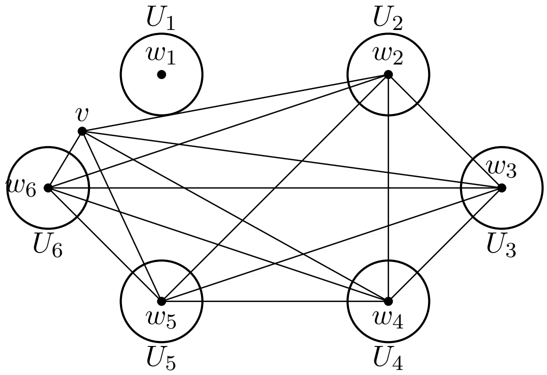


Figure 10: The 3-graph $F = \widetilde{\mathcal{H}}_i[\{w_1, \ldots, w_6\}] \cup \{e_j : j \in \{4,5,6\}\} \cup \{vw_2w_3\}$ is a member in M_3 with core $\{v, w_1, \ldots, w_5\}$.

Claim 4.20. *We have $|B_v| < 300\epsilon^{1/12}\tilde{n}^2$.*

Proof of Claim 4.20. Suppose to the contrary that $|B_v| \geq 300\epsilon^{1/12}\tilde{n}^2$. Notice that every edge in B_v must have one vertex in U_2 and the other vertex in $U_3 \cup U_6$. By symmetry and the Pigeonhole principle, we may assume that at least $|B_v|/2$ edges in B_v have one vertex in U_2 and the other vertex in U_3. Then Claim 4.13 and an easy averaging argument show that there exists a vertex $w_2 \in U_2$ such that

$$|N_{B_v}(w_2) \cap U_3| \geq \frac{|B_v|/2}{|U_2|} > \frac{300\epsilon^{1/12}\tilde{n}^2/2}{\tilde{n}/5} > 600\epsilon^{1/12}\tilde{n}.$$

Let $U'_3 = N_{B_v}(w_2) \cap U_3$, and $U'_j = N(v) \cap U_j$ for $j \in \{4,5,6\}$. Since $|U'_3| \geq 600\epsilon^{1/12}\tilde{n}$ and $|U'_j| \geq \tilde{n}/13$ for $j \in \{4,5,6\}$, applying Lemma 4.12 with $T = \{w_2\}$, $S = \{1,3,4,5,6\}$, and $\eta = 30\epsilon^{1/4}$ we obtain $w_1 \in U_1$ and $w_j \in U'_j$ for $j \in \{3,4,5,6\}$ (see Figure 10) such that the induced subgraph of $\widetilde{\mathcal{H}}_i$ on $\{w_1, \ldots, w_6\}$ is a copy of $\mathcal{G}_6^2$. For $j \in \{4,5,6\}$ let $e_j \in \widetilde{\mathcal{H}}_i$ be an edge containing v and w_j. Let $F = \widetilde{\mathcal{H}}_i[\{w_1, \ldots, w_6\}] \cup \{e_j : j \in \{4,5,6\}\} \cup \{vw_2w_3\}$. It is easy to see that F is a member in $\mathcal{K}_6^3$ with core $\{v, w_2, \ldots, w_6\}$ (see Figure 10). So, by assumption, either $F \subset \mathcal{G}_m^1$ or $F \subset \mathcal{G}_m^2$ for any integer m. It is easy to see that the former case cannot hold since the induced subgraph of F on the set $\{w_1, \ldots, w_6\}$ is a copy of $\mathcal{G}_6^2$ and $\mathcal{G}_6^2 \not\subset \mathcal{G}_m^1$ for any integer m. So, $F \subset \mathcal{G}_n^2$ for some integer m. In other words, there exists a map $\psi \colon V(F) \to V(\mathcal{G}_6^2)$ such that $\psi(e) \in \mathcal{G}_6^2$ for all $e \in F$. Notice that both $\{w_1, \ldots, w_6\}$ and $\{v, w_2, \ldots, w_6\}$ are 2-covered in F, so the restrictions of ψ on $\{w_1, \ldots, w_6\}$ and $\{v, w_2, \ldots, w_6\}$ are both injective (similar to the proof of Lemma 2.2), and moreover, $\psi(v) = \psi(w_1)$. Let $w = \psi(v) = \psi(w_1)$. Notice that the induced subgraph of $L_F(w_1)$ on $\{w_2, \dots, w_3\}$ has size 8 and $w_2w_3 \in L_F(v) \setminus L_F(w_1)$. Since ψ preserves edges, the degree of w in $\mathcal{G}_6^2$ should be at least $8 + 1 = 9$. However, this contradicts the fact that the maximum degree of $\mathcal{G}_6^2$ is 8. ∎

A consequence of Claim 4.20 is that the size of M_v satisfies

$$\begin{aligned} |M_v| = |\widetilde{G}_1 \setminus L(v)| &= |\widetilde{G}_1| - |\widetilde{G}_1 \cap L(v)| \\ &= |\widetilde{G}_1| - (|L(v)| - |B_v|) \\ &< 8\left(1/6 + 10\epsilon^{1/4}\right)^2 \tilde{n}^2 - \left(\left(2/9 - 10\epsilon^{1/2}\right)\tilde{n}^2 - |B_v|\right) < 400\epsilon^{1/12}\tilde{n}^2. \end{aligned}$$

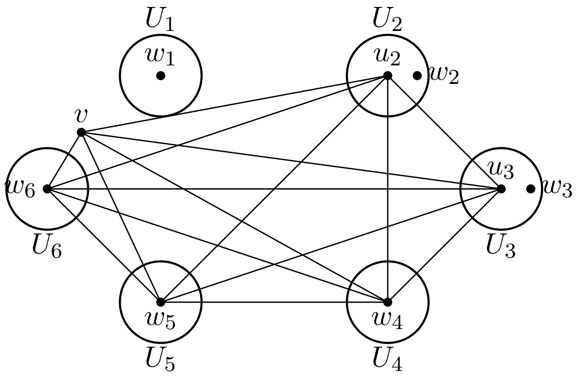


Figure 11: The 3-graph $F = \widetilde{\mathcal{H}}_i[\{v, u_2, u_3, w_1, \dots, w_6\}] \cup \{vu_2u_3\} \cup \{e_{u_3w_4}\}$ is a member in M_3 with core $\{v, u_2, u_3, w_4, w_5, w_6\}$.

Claim 4.21. *We have $B_v = \emptyset$. In other words, $L_{\widetilde{\mathcal{H}}_i}(v) \subset \widetilde{G}_1$.*

Proof of Claim 4.21. Suppose to the contrary that there exists an edge $u_2u_3 \in B_v$ and by symmetry we may assume that $u_2 \in U_2$ and $u_3 \in U_3$. For $j \in \{4, 5, 6\}$ let $U_j' = U_j \cap N(v) \cap N(u_1) \cap N(u_2)$ and notice that due to $|M_v| \le 400\epsilon^{1/12}\tilde{n}^2$ and Claim 4.13 we have $|U_j'| \ge |U_j|/2 > \tilde{n}/20$. Applying Lemma 4.12 with $T = \{u_2, u_3\}$, $S = [6]$, and $\eta = 400\epsilon^{1/36}$ we obtain $w_j \in U_j'$ for $j \in [6]$ (see Figure 11) such that

(a) $\widetilde{\mathcal{H}}_i[\{w_1, \dots, w_6\}] \cong \mathcal{G}_6^2$,

(b) $L_{\widetilde{\mathcal{H}}_i}(v)[\{w_2, \dots, w_6\}] = L_{\widehat{\mathcal{G}}}(w_1)[\{w_2, \dots, w_6\}]$,

(c) $L_{\widetilde{\mathcal{H}}_i}(u_2)[\{w_1, w_3, \dots, w_6\}] = L_{\widehat{\mathcal{G}}}(u_2)[\{w_1, w_3, \dots, w_6\}]$, and

(d) $L_{\widetilde{\mathcal{H}}_i}(u_3)[\{w_1, w_2, w_4, w_5, w_6\}] = L_{\widehat{\mathcal{G}}}(u_3)[\{w_1, w_2, w_4, w_5, w_6\}]$.

Let $e_{u_3w_4} \in \widetilde{\mathcal{H}}_i$ be an edge containing u_3 and w_4. Let $F = \widetilde{\mathcal{H}}_i[\{v, u_2, u_3, w_1, \dots, w_6\}] \cup \{vu_2u_3\} \cup \{e_{u_3w_4}\}$. Then it is easy to see that F is a member in $\mathcal{K}_6^3$ with core $\{v, u_2, u_3, w_4, w_5, w_6\}$ (see Figure 11). Similar to the proof of Claim 4.20, $F \subset \mathcal{G}_m^2$ for some integer m. In other words, there exists a map $\psi\colon V(F) \to V(\mathcal{G}_6^2)$ such that $\psi(e) \in \mathcal{G}_6^2$ for all $e \in F$. Notice that both $\{w_1, \dots, w_6\}$ and $\{v, u_2, u_3, w_4, w_5, w_6\}$ are 2-covered in F, so the restrictions of ψ on sets $\{w_1, \dots, w_6\}$ and $\{v, u_2, u_3, w_4, w_5, w_6\}$ are both injective (similar to the proof of Lemma 2.2), and moreover, $\psi(v) = \psi(w_1)$ (due to (b), v is adjacent to all vertices in $\{w_2, \dots, w_6\}$, so $\psi(v)$ is distinct from $\{\psi(w_2), \dots, \psi(w_6)\}$), $\psi(u_2) = \psi(w_2)$ (due to (c) and a similar reason), and $\psi(u_3) = \psi(w_3)$ (due to (d) and a similar reason). Let $w = \psi(v) = \psi(w_1)$. Notice that the induced subgraph of $L_F(w_1)$ on $\{w_2, \dots, w_6\}$ has size 8 and $u_2u_3 \in L_F(v) \setminus L_F(w_1)$. Since ψ preserves edges, the degree of w in $\mathcal{G}_6^2$ should be at least $8 + 1 = 9$. However, this contradicts the fact that the maximum degree of $\mathcal{G}_6^2$ is 8. ∎

Define

$$V_j^i = \begin{cases} U_1 \cup C_v, & \text{if } j = 1, \\ U_j, & \text{otherwise.} \end{cases}$$

By Claim 4.21, $L(v) \subset \widetilde{G}_1$. Therefore, $\widetilde{\mathcal{H}}_i$ is $\mathcal{G}_6^2$-colorable with set of parts $\{V_1^i, \ldots, V_6^i\}$. This completes the proof of Lemma 4.11. ∎

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