Constructions in Ramsey theory

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Abstract

We provide several constructions for problems in Ramsey theory. First, we prove a superexponential lower bound for the classical 4-uniform Ramsey number $r(5, n)$, and the same for the iterated $(k - 4)$-fold logarithm of the $k$-uniform version $r_k(k + 1, n)$. This is the first improvement of the original exponential lower bound for $r_4(5, n)$ implicit in work of Erdős and Hajnal from 1972 and also improves the current best known bounds for larger $k$ due to the authors.

Second, we prove an upper bound for the hypergraph Erdős-Rogers function $f_{k+1,k+2}(N)$ that is an iterated $(k - 13)$-fold logarithm in $N$. This improves the previous upper bounds that were only logarithmic and addresses a question of Dudek and the first author that was reiterated by Conlon, Fox and Sudakov. Third, we generalize the results of Erdős and Hajnal about the 3-uniform Ramsey number of $K_4$ minus an edge versus a clique to $k$-uniform hypergraphs.

1 Introduction

A $k$-uniform hypergraph $H$ ($k$-graph for short) with vertex set $V$ is a collection of $k$-element subsets of $V$. We write $K^k_n$ for the complete $k$-uniform hypergraph on an $n$-element vertex set. Given $k$-graphs $F, G$, the Ramsey number $r(F, G)$ is the minimum $N$ such that every red/blue coloring of the edges of $K^k_N$ results in a monochromatic red copy of $F$ or a monochromatic blue copy of $G$.

In this paper, we study several problems in hypergraph Ramsey theory. We describe each problem in detail in its relevant section. Here we provide a brief summary. In Section 2, we give new lower bounds on the classical Ramsey number $r(K^k_{k+1}, K^k_n)$, improving the previous best known bounds obtained by the authors [18]. In particular, we give the first superexponential lower bound for $r(K^4_5, K^4_n)$ since the problem was first explicitly stated by Erdős and Hajnal [12] in 1972. In Section 3, we establish a new upper bound for the hypergraph Erdős-Rogers function $f_{k+1,k+2}^k(N)$ that is an iterated logarithm function in $N$. More precisely, we construct $k$-graphs on $N$ vertices, with no copy of $K^k_{k+2}$, yet every set of $n$ vertices contains a copy of $K^k_{k+1}$ where $n$ is the $(k - 13)$-fold iterated logarithm of $N$. This addresses questions posed by Dudek and the first author [8] as well as by Conlon, Fox, and Sudakov [7] and significantly improves the previous best known bound in [8] of $n = O((\log N)^{1/(k-1)})$. In Section 4 we study the Ramsey numbers for $k$-half-graphs versus...
cliques, generalizing the results of Erdős and Hajnal [12] about the 3-uniform Ramsey number of $K_4$ minus an edge versus a clique. The upper bound is a straightforward extension of the method in [12], while the constructions are new.

All logarithms are base 2 unless otherwise stated. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 A new lower bound for $r_k(k+1,n)$

In order to avoid the excessive use of superscripts, we use the simpler notation $r(K_k^s, K_k^n) = r_k(s,n)$. Estimating the Ramsey number $r_k(s,n)$ is a classical problem in extremal combinatorics and has been extensively studied [13, 14, 16]. Here we study the off-diagonal Ramsey number, that is, $r_k(s,n)$ with $k,s$ fixed and $n$ tending to infinity. It is known that for fixed $s \geq k+1$, $r_2(s,n)$ grows polynomially in $n$ [1, 2, 3] and $r_3(s,n)$ grows exponentially in a power of $n$ [6]. In 1972, Erdős and Hajnal [12] raised the question of determining the correct tower growth rate for $r_k(s,n)$. We define the tower function $\text{twr}_k(x)$ by

$$\text{twr}_1(x) = x \quad \text{and} \quad \text{twr}_{i+1} = 2^{\text{twr}_i(x)}.$$ 

By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting (see [17]), it follows that $r_k(s,n) \geq \text{twr}_{k-1}(\Omega(n))$, for $k \geq 4$ and for all $s \geq 2^{k-1} - k + 3$. However they conjectured the following.

**Conjecture 2.1. (Erdős-Hajnal [12])** For $s \geq k + 1 \geq 5$ fixed, $r_k(s,n) \geq \text{twr}_{k-1}(\Omega(n))$.

In [5], Conlon, Fox, and Sudakov modified the Erdős-Hajnal stepping-up lemma to show that Conjecture 2.1 holds for all $s \geq \lceil 5k/2 \rceil - 3$. Recently the authors nearly proved the conjecture by establishing the following.

**Theorem 2.2 ([18]).** There is a positive constant $c > 0$ such that the following holds. For $k \geq 4$ and $n > 3k$, we have

1. $r_k(k+3,n) \geq \text{twr}_{k-1}(cn)$,
2. $r_k(k+2,n) \geq \text{twr}_{k-1}(c\log^2 n)$,
3. $r_k(k+1,n) \geq \text{twr}_{k-2}(cn^2)$.

Implicit in work of Erdős and Hajnal [12] is the bound $r_4(5,n) > 2^{cn}$ for some absolute positive constant $c$. While the authors [18] recently improved this to $2^{cn^2}$ above, there has been no superexponential lower bound given for this basic problem. Here we provide such a lower bound.

**Theorem 2.3.** There is an absolute constant $c > 0$ such that

$$r_4(5,n) > 2^{n^{c\log \log n}},$$

and more generally for $k > 4$,

$$r_k(k+1,n) > \text{twr}_{k-2}(n^{c\log \log n}).$$
One of the building blocks we will use in our construction is the following lower bound of Conlon, Fox, and Sudakov [6]: there is an absolute positive constant $c > 0$ such that

\[ r_3(4, t) > 2^{ct \log t}. \tag{1} \]

Our lower bound for $r_4(5, n)$ is proved via the following theorem.

**Theorem 2.4.** For $n$ sufficiently large, we have

\[ r_4(5, n) > 2^{r_3(4, \lfloor \log n \rfloor /2) - 1}. \]

**Proof.** The idea is to apply a variant of the Erdős-Hajnal stepping up lemma (see [17]). Set $t = \lfloor \log n / 2 \rfloor$. Let $\phi$ be a red/blue coloring of the edges of the complete 3-uniform hypergraph on the vertex set $\{0, 1, \ldots, r_3(4, t) - 2\}$ without a red $K^3_4$ and without a blue $K^3_4$. We use $\phi$ to define a red/blue coloring $\chi$ of the edges of the complete 4-uniform hypergraph $K^4_N$ on the vertex set $V = \{0, 1, \ldots, N - 1\}$ with $N = 2^{r_3(4, t) - 1}$, as follows.

For any $a \in V$, write $a = \sum_{i=0}^{r_3(4, t) - 2} a(i)2^i$ with $a(i) \in \{0, 1\}$ for each $i$. For $a \neq b$, let $\delta(a, b)$ denote the largest $i$ for which $a(i) \neq b(i)$. Notice that we have the following stepping-up properties (again see [17]):

**Property A:** For every triple $a < b < c$, $\delta(a, b) \neq \delta(b, c)$.

**Property B:** For $a_1 < \cdots < a_r$, $\delta(a_1, a_r) = \max_{1 \leq j \leq r-1} \delta(a_j, a_{j+1})$.

Given any 4-tuple $a_1 < \cdots < a_4$ of $V$, consider the integers $\delta_i = \delta(a_i, a_{i+1}), 1 \leq i \leq 3$. Say that $\delta_1, \delta_2, \delta_3$ forms a monotone sequence if $\delta_1 < \delta_2 < \delta_3$ or $\delta_1 > \delta_2 > \delta_3$. Now, define $\chi$ as follows:

\[ \chi(a_1, a_2, a_3, a_4) = \begin{cases} \phi(\delta_1, \delta_2, \delta_3) & \text{if } \delta_1, \delta_2, \delta_3 \text{ is monotone} \\ \text{blue} & \text{if } \delta_1, \delta_2, \delta_3 \text{ is not monotone} \end{cases} \]

Hence we have the following property which can be easily verified using Properties A and B (see [17]):

**Property C:** For $a_1 < \cdots < a_r$, set $\delta_j = \delta(a_j, a_{j+1})$ and suppose that $\delta_1, \ldots, \delta_{r-1}$ form a monotone sequence. If $\chi$ colors every 4-tuple in $\{a_1, \ldots, a_r\}$ red (blue), then $\phi$ colors every triple in $\{\delta_1, \ldots, \delta_{r-1}\}$ red (blue).

For sake of contradiction, suppose that the coloring $\chi$ produces a red $K^4_5$ on vertices $a_1 < \cdots < a_5$, and let $\delta_i = \delta(a_i, a_{i+1}), 1 \leq i \leq 4$. Then $\delta_1, \ldots, \delta_4$ form a monotone sequence and, by Property C, $\phi$ colors every triple in $\{\delta_1, \ldots, \delta_4\}$ red which is a contradiction. Therefore, there is no red $K^4_5$ in coloring $\chi$.

Next we show that there is no blue $K^4_4$ in coloring $\chi$. Our argument is reminiscent of the standard argument for the bound $r_2(n, n) < 4^n$, though it must be adapted to this setting. For sake of
contradiction, suppose we have vertices $a_1, \ldots, a_n \in V$ such that $a_1 < \cdots < a_n$ and $\chi$ colors every 4-tuple in the set $\{a_1, \ldots, a_n\}$ blue. Let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq n - 1$. We greedily construct a set $D_h = \{\delta_1, \ldots, \delta_h\} \subset \{\delta_1, \ldots, \delta_{n-1}\}$ and a set $S_h \subset \{a_1, \ldots, a_n\}$ such that the following holds.

1. We have $\delta_1 > \cdots > \delta_h$.
2. For each $\delta_{ij} = \delta(a_{ij}, a_{ij+1}) \in D_h = \{\delta_1, \ldots, \delta_h\}$, consider the set of vertices
   $$A = \{a_{ij+1}, a_{ij+1} + 1, \ldots, a_h, a_{h+1}\} \cup S_h.$$  

   Then either every element in $A$ is greater than $a_{ij}$ or every element in $A$ is less than $a_{ij+1}$. In the former case we will label $\delta_{ij}$ white, in the latter case we label it black.
3. The indices of the vertices in $S_h$ are consecutive, that is, $S_h = \{a_r, a_{r+1}, \ldots, a_s, a_s\}$ for $1 \leq r < s \leq n$.

We start with the $D_0 = \emptyset$ and $S_0 = \{a_1, \ldots, a_n\}$. Having obtained $D_h = \{\delta_1, \ldots, \delta_h\}$ and $S_h = \{a_r, \ldots, a_s\}$, $1 \leq r < s \leq n$, we construct $D_{h+1}$ and $S_{h+1}$ as follows. Let $\delta_{ih+1} = \delta(a_{ih}, a_{ih+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \ldots, \delta_{s-1}\}$, and set $D_{h+1} = D_h \cup \delta_{ih+1}$. The uniqueness of $\delta_{ih+1}$ follows from Properties A and B. If $|\{a_r, a_{r+1}, \ldots, a_t\}| \geq |S_h|/2$, then we set $S_{h+1} = \{a_r, a_{r+1}, \ldots, a_t\}$. Otherwise by the pigeonhole principle, we have $|\{a_{\ell+1}, a_{\ell+2}, \ldots, a_s\}| \geq |S_h|/2$ and we set $S_{h+1} = \{a_{\ell+1}, a_{\ell+2}, \ldots, a_s\}$.

Since $|S_0| = n$, $t = \left\lceil \frac{\log n}{2} \right\rceil$ and $|S_{h+1}| \geq |S_h|/2$ for $h \geq 0$, we can construct $D_{2t} = \{\delta_{i_1}, \ldots, \delta_{i_{2t}}\}$ with the desired properties. By the pigeonhole principle, at least $t$ elements in $D_{2t}$ have the same label, say white. The other case will follow by a symmetric argument. We remove all black labeled elements in $D_{2t}$, and let $\{\delta_{j_1}, \ldots, \delta_{j_t}\}$ be the resulting set. Now consider the vertices $a_{j_1}, a_{j_2}, \ldots, a_{j_t} \in V$. By construction and by Property B, we have $a_{j_1} < a_{j_2} < \cdots < a_{j_t}$ and $\delta(a_{j_1}, a_{j_2}) = \delta_{i_1}, \delta(a_{j_2}, a_{j_3}) = \delta_{i_2}, \ldots, \delta(a_{j_{t-1}}, a_{j_t}) = \delta_{i_{t-1}}$. Therefore we have a monotone sequence

$$\delta(a_{j_1}, a_{j_2}) > \delta(a_{j_2}, a_{j_3}) > \cdots > \delta(a_{j_{t-1}}, a_{j_t}).$$

By Property C, $\phi$ colors every triple from this set blue which is a contradiction. Therefore there is no red $K_5^3$ and no blue $K_n^4$ in coloring $\chi$.

Applying the lower bound in (1), we obtain that

$$r_4(5, n) \geq 2^{n\log n} - 1 > 2^{n \log n \log \log n} = 2^{n \log \log n}$$

for some absolute positive constant $c$ and this establishes the first part of Theorem 2.3.

We next prove Theorem 2.3 for $k \geq 5$. Independently, Conlon, Fox and Sudakov [4] gave a different proof of Theorem 2.2 part 1. Their approach was to begin with a known 4-uniform construction that yields $r_4(7, n) > 2^{2^{2n}}$ and then use a variant of the stepping up lemma to give tower-type lower bounds for larger $k$. Unfortunately, this variant of the stepping up lemma does not work if one begins instead with a lower bound for $r_4(5, n)$ which is our case. However, a further variant of the approach does work, and this is what we do below.
Lemma 2.5. For \( k \geq 5 \) and \( n \) sufficiently large, we have

\[
r_k(k+1, n) \geq 2^{r_{k-1}(k, \lfloor n/6 \rfloor)-1}.
\]

Proof. Again we apply a variant of the stepping-up lemma. Let \( \phi \) be a red/blue coloring of the edges of the complete \((k - 1)\)-uniform hypergraph on the vertex set \( \{0, 1, \ldots, r_{k-1}(k, \lfloor n/6 \rfloor) - 2\} \) without a red \( K_k^{k-1} \) and without a blue \( K_{\lfloor n/6 \rfloor}^{k-1} \). We use \( \phi \) to define a red/blue coloring \( \chi \) of the edges of the complete \( k \)-uniform hypergraph \( K_N^k \) on the vertex set \( V = \{0, 1, \ldots, N - 1\} \) with \( N = 2^{r_{k-1}(k, \lfloor n/6 \rfloor)-1} \), as follows.

Just as above, for any \( a \in V \), write \( a = \sum_{i=0}^{r_{k-1}(k, \lfloor n/6 \rfloor)-2} a(i)2^i \) with \( a(i) \in \{0, 1\} \) for each \( i \). For \( a \neq b \), let \( \delta(a, b) \) denote the largest \( i \) for which \( a(i) \neq b(i) \). Hence Properties A and B hold.

Given any \( k \)-tuple \( a_1 < a_2 < \ldots < a_k \) of \( V \), consider the integers \( \delta_i = \delta(a_i, a_{i+1}), 1 \leq i \leq k - 1 \). We say that \( \delta_i \) is a local minimum if \( \delta_{i-1} > \delta_i < \delta_{i+1} \), a local maximum if \( \delta_{i-1} < \delta_i > \delta_{i+1} \), and a local extremum if it is either a local minimum or a local maximum. We say that \( \delta_i \) is locally monotone if \( \delta_{i-1} < \delta_i < \delta_{i+1} \) or \( \delta_{i-1} > \delta_i > \delta_{i+1} \). Since \( \delta_{i-1} \neq \delta_i \) for every \( i \), every nonmonotone sequence \( \delta_1, \ldots, \delta_k-1 \) has a local extremum. If \( \delta_1, \ldots, \delta_k-1 \) form a monotone sequence, then let \( \chi(a_1, a_2, \ldots, a_k) = \phi(\delta_1, \delta_2, \ldots, \delta_{k-1}) \). Otherwise, if \( \delta_1, \ldots, \delta_k-1 \) is not monotone, then let \( \chi(a_1, a_2, \ldots, a_k) \) be red if and only if \( \delta_2 \) is a local maximum and \( \delta_3 \) is a local minimum. Hence the following generalization of Property C holds.

Property D: For \( a_1 < \cdots < a_r \), set \( \delta_j = \delta(a_j, a_{j+1}) \) and suppose that \( \delta_1, \ldots, \delta_{r-1} \) form a monotone sequence. If \( \chi \) colors every \( k \)-tuple in \( \{a_1, \ldots, a_r\} \) red (blue), then \( \phi \) colors every \((k - 1)\)-tuple in \( \{\delta_1, \ldots, \delta_{r-1}\} \) red (blue).

For sake of contradiction, suppose that the coloring \( \chi \) produces a red \( K_k^k \) on vertices \( a_1 < \cdots < a_{k+1} \), and let \( \delta_i = \delta(a_i, a_{i+1}), 1 \leq i \leq k \). We have two cases.

Case 1. Suppose \( \delta_1, \ldots, \delta_{k-1} \) is monotone. Then if \( \delta_2, \ldots, \delta_k \) is also a monotone sequence, \( \phi \) colors every \((k - 1)\)-tuple in \( \{\delta_1, \ldots, \delta_k\} \) red by Property D, which is a contradiction. Otherwise, \( \delta_{k-1} \) is the only local extremum and \( \chi(a_2, \ldots, a_{k+1}) \) is blue, which is again a contradiction.

Case 2. Suppose \( \delta_1, \ldots, \delta_{k-1} \) is not monotone. Then we know that \( \delta_2 \) is a local maximum and \( \delta_3 \) is a local minimum. However this implies that \( \chi(a_2, \ldots, a_{k+1}) \) is blue, which is a contradiction. Hence there is no red \( K_k^k \) in coloring \( \chi \).

Next we show that there is no blue \( K_n^k \) in coloring \( \chi \). For sake of contradiction, suppose we have vertices \( a_1, \ldots, a_n \in V \) such that \( a_1 < \cdots < a_n \) and \( \chi \) colors every \( k \)-tuple blue, and let \( \delta_i = \delta(a_i, a_{i+1}) \) for \( 1 \leq i \leq n-1 \). By Property D, there is no integer \( r \) such that \( \delta_r, \delta_{r+1}, \ldots, \delta_{r+[n/6]} \) is monotone, since this implies that \( \phi \) colors every \((k - 1)\)-tuple in the set \( \{\delta_r, \delta_{r+1}, \ldots, \delta_{r+[n/6]}\} \) blue which is a contradiction. Therefore the sequence \( \delta_1, \ldots, \delta_{n-1} \) contains at least four local extrema. Let \( \delta_j \) be the first local maximum, and let \( \delta_j \) be the next local extremum, which must be a local minimum. Recall that \( \delta_j = \delta(a_{j-1}, a_{j+1}) \) and \( \delta_j = \delta(a_{j-1}, a_{j+1}) \). Consider the \( k \) vertices

\[
(a_{j-1}, a_{j+1}, a_{j+2}, \ldots, a_{j+k-3})
\]
and the sequence
\[
\delta(a_{j_1-1}, a_{j_1}), \delta(a_{j_1}, a_{j_2}), \delta(a_{j_2}, a_{j_2+1}), \ldots, \delta(a_{j_2+k-4}, a_{j_2+k-3}).
\]

By Property B we have \(\delta(a_{j_1}, a_{j_2}) = \delta_{j_1}\), and therefore \(\delta(a_{j_1}, a_{j_2})\) is a local maximum and \(\delta(a_{j_2}, a_{j_2+1})\) is a local minimum. Therefore \(\chi(a_{j_1-1}, a_{j_1}, a_{j_2}, a_{j_2+1}, \ldots, a_{j_2+k-3})\) is red and we have our contradiction. Hence there is no blue \(K^k_n\) in coloring \(\chi\). 

By combining Theorem 2.4 with Lemma 2.5, we establish Theorem 2.3.

3 The Erdős-Rogers function for hypergraphs

An \(s\)-independent set in a \(k\)-graph \(H\) is a vertex subset that contains no copy of \(K^k_s\). So if \(s = k\), then it is just an independent set. Let \(\alpha_s(H)\) denote the size of the largest \(s\)-independent set in \(H\).

**Definition 3.1.** For \(k \leq s < t < N\), the Erdős-Rogers function \(f^k_{s,t}(N)\) is the minimum of \(\alpha_s(H)\) taken over all \(K^k_t\)-free \(k\)-graphs \(H\) of order \(N\).

To prove the lower bound \(f^k_{s,t}(N) \geq n\) one must show that every \(K^k_t\)-free \(k\)-graph of order \(N\) contains an \(s\)-independent set with \(n\) vertices. On the other hand, to prove the upper bound \(f^k_{s,t}(N) < n\), one must construct a \(K^k_t\)-free \(k\)-graph \(H\) of order \(N\) with \(\alpha_s(H) < n\).

The problem of determining \(f^k_{s,t}(n)\) extends that of finding Ramsey numbers. Formally,

\[
r_k(s, n) = \min\{N : f^k_{s,t}(N) \geq n\}.
\]

For \(k = 2\) the above function was first considered by Erdős and Rogers [15] only for \(t = s + 1\), which might be viewed as the most restrictive case. Since then the function has been studied by several researchers culminating in the work of Wolfovitz [20] and Dudek, Retter and Rödl [9] who proved the upper bound that follows (the lower bound is due to Dudek and the first author [8]): for every \(s \geq 3\) there are positive constants \(c_1\) and \(c_2(s)\) such that

\[
c_1 \left( \frac{N \log N}{\log \log N} \right)^{1/2} < f^2_{s,s+1}(N) < c_2(\log N)^{4s^2} N^{1/2}.
\]

The problem of estimating the Erdős-Rogers function for \(k > 2\) appears to be much harder. Let us denote

\[
g(k, N) = f^k_{k+1, k+2}(N)
\]

so that the above result (for \(s = 3\)) becomes \(g(2, N) = N^{1/2+o(1)}\). Dudek and the first author [8] proved that \((\log N)^{1/4+o(1)} < g(3, N) < O(\log N)\) and more generally that there are positive constants \(c_1\) and \(c_2\) with

\[
c_1(\log(k-2))^{1/4} N^{1/4} < g(k, N) < c_2(\log N)^{1/(k-2)}
\]
where \( \log(i) \) is the log function iterated \( i \) times. The exponent 1/4 was improved to 1/3 by Conlon, Fox, Sudakov [7]. Both sets of authors asked whether the upper bound could be improved (presumably to an iterated log function). Here we prove this where the number of iterations is \( k - O(1) \). It remains an open problem to determine the correct number of iterations (which may well be \( k - 2 \)).

**Theorem 3.2.** Fix \( k \geq 14 \). Then \( g(k, N) < O(\log(k-13) N) \).

**Proof.** We will proceed by induction on \( k \). The base case of \( k = 14 \) follows from the upper bound in (2). For the inductive step, let \( k > 14 \) and assume that the result holds for \( k - 1 \). We will show that

\[
g(k, 2^N) < k \cdot g(k-1, N),
\]

and this recurrence clearly implies the theorem. Indeed, it easily implies the upper bound

\[
g(k, N) < 2^k k! \log(k-13) N
\]

by induction on \( k \), as \( g(k+1, N) \) is at most

\[
g(k+1, 2^{[\log N]}) < (k+1) g(k, [\log N])
\]

\[
< 2^k (k+1)! \log(k-13) [\log N]
\]

\[
< 2^{k+1} (k+1)! \log(k-12) N.
\]

Our strategy is to apply a variant of the stepping-up lemma. Let us begin with a \( K_{k+1}^{k-1} \)-free \((k-1)\)-graph \( H' \) on \( N \) vertices for which \( \alpha_k(H') = g(k-1, N) \). Note that this exists by definition of \( g(k-1, N) \). We will use \( H' \) to produce a \( K_k^{k+2} \)-free \( k \)-graph \( H \) on \( 2^N \) vertices with \( \alpha_{k+1}(H) < k\alpha_k(H') = kg(k-1, N) \).

Let \( V(H') = \{0,1,\ldots,N-1\} \) and \( V(H) = \{0,1,\ldots,2^N-1\} \). For any \( a \in V(H) \), write \( a = \sum_{i=0}^{N-1} a(i) 2^i \) with \( a(i) \in \{0,1\} \) for each \( i \). For \( a \neq b \), let \( \delta(a,b) \) denote the largest \( i \) for which \( a(i) \neq b(i) \). Therefore Properties A and B in the previous section hold.

Given any set of \( s \) vertices \( a_1 < a_2 < \ldots < a_s \) of \( V(H) \), consider the integers \( \delta_i = \delta(a_i, a_{i+1}) \), \( 1 \leq i \leq s-1 \). For \( e = (a_1, \ldots, a_s) \), let \( m(e) \) denote the number of local extrema in the sequence \( \delta_1, \ldots, \delta_{s-1} \). In the case \( s = k \), we define the edges of \( H \) as follows. If \( \delta_1, \ldots, \delta_{k-1} \) form a monotone sequence, then let \( (a_1, a_2, \ldots, a_k) \in E(H) \) if and only if \( (\delta_1, \delta_2, \ldots, \delta_{k-1}) \in E(H') \). Otherwise if \( \delta_1, \ldots, \delta_{k-1} \) is not monotone, then \( (a_1, a_2, \ldots, a_k) \in E(H) \) if and only if \( m(e) \in \{k-4, k-3\} \).

In other words, given that \( \delta_1, \ldots, \delta_{k-1} \) is not monotone, \( (a_1, a_2, \ldots, a_k) \in E(H) \) if and only if \( \delta_1, \ldots, \delta_{k-1} \) has at most one locally monotone element. Note that we have the following variant of Property D.

**Property E:** For \( a_1 < \cdots < a_r \), set \( \delta_j = \delta(a_j, a_{j+1}) \) and suppose that \( \delta_1, \ldots, \delta_{r-1} \) form a monotone sequence. If every \( k \)-tuple in \( \{a_1, \ldots, a_r\} \) is in \( E(H) \) (in \( E(H) \)), then every \( (k-1) \)-tuple in \( \{\delta_1, \ldots, \delta_{r-1}\} \) is in \( E(H') \) (in \( E(H') \)).

We are to show that \( H \) contains no \((k+2)\)-clique and \( \alpha_{k+1}(H) < k\alpha_k(H') \). First let us establish the following lemma.
Lemma 3.3. Given \( e = (a_1, \ldots, a_7) \) with \( a_1 < \cdots < a_7 \), let \( \delta_i = \delta(a_i, a_{i+1}) \) for \( 1 \leq i \leq 6 \). If \( m(e) = 4 \), then there is an \( a_i \) such that \( 2 \leq i \leq 6 \) and \( m(e - a_i) = 2 \).

Proof. Suppose first that \( \delta_2 \) is a local minimum, so \( \delta_1 > \delta_2 < \delta_3 > \cdots \). Then we have \( m(e - a_1) = 2 \). Indeed, since \( \delta_4 \) is a local minimum, Property B implies \( \delta(a_3, a_5) = \delta_3 \). If \( \delta_5 > \delta_3 \), then we have \( \delta_2 < \delta(a_3, a_5) < \delta_5 \) and therefore \( m(e - a_4) = 2 \). If \( \delta_5 < \delta_3 \), then we have \( \delta(a_3, a_5) > \delta_5 > \delta_6 \) which again implies that \( m(e - a_4) = 2 \).

Now suppose that \( \delta_2 \) is a local maximum, so \( \delta_1 < \delta_2 > \delta_3 > \cdots \). Then we have \( m(e - a_3) = 2 \). Indeed, by Property B we have \( \delta(a_2, a_4) = \delta_2 \). If \( \delta_4 < \delta_2 \), then we have \( \delta(a_2, a_4) > \delta_4 > \delta_5 \) which implies \( m(e - a_3) = 2 \). If \( \delta_4 > \delta_2 \), then we have \( \delta_1 < \delta(a_2, a_4) < \delta_4 \) which again implies \( m(e - a_3) = 2 \).

For sake of contradiction, suppose there are \( k + 2 \) vertices \( a_1 < \cdots < a_{k+2} \) that induce a \( K_{k+2}^k \) in \( H \). Define \( \delta_i = \delta(a_i, a_{i+1}) \) for all \( 1 \leq i \leq k + 1 \). Given the sequence \( \delta_1, \delta_2, \ldots, \delta_{k+1} \), let us consider the number of locally monotone elements in \( D = \{ \delta_2, \ldots, \delta_k \} \).

Case 1. Suppose every element in \( D \) is locally monotone. Then \( \delta_1, \ldots, \delta_{k+1} \) form a monotone sequence. By Property E, every \((k-1)\)-tuple in the set \( \{\delta_1, \ldots, \delta_{k+1}\} \) is an edge in \( H' \) which is a contradiction since \( H' \) is \( K_{k-1}^{k-1} \)-free.

Case 2. Suppose there is at least one local extremum \( \delta_j \in D \) and at least two elements \( \delta_i, \delta_j \in D \) that are locally monotone. Then any \( k \)-tuple \( e \subset \{a_1, \ldots, a_{k+2}\} \) that includes the vertices

\[
i_{i-1}, a_i, a_{i+1}, a_{i+2}, a_{j-1}, a_j, a_{j+1}, a_{j+2}, a_{k-1}, a_k, a_{k+1}, a_{k+2}
\]

satisfies \( 1 \leq m(e) < k - 4 \). Therefore \( e \) is not an edge in \( H \) and we have a contradiction.

Case 3. Suppose there is exactly one element \( \delta_i \in D \) that is locally monotone (and therefore at least one local extremum). Since \( k \geq 15 \), either \(|\{a_1, \ldots, a_{i-1}\}| \geq 7\) or \(|\{a_{i+2}, \ldots, a_{k+2}\}| \geq 7\). Let us only consider the former case, the latter being symmetric. By Lemma 3.3, there is an element \( a_j \in \{a_2, \ldots, a_6\} \subset \{a_1, \ldots, a_{i-1}\} \) such that for \( e' = (a_1, \ldots, a_7) \), \( m(e' - a_j) = 2 \). Then any \( k \)-tuple \( e \subset \{a_1, \ldots, a_{k+2}\} \setminus \{a_j\} \) that includes vertices

\[
\{a_t : 1 \leq t \leq 7, t \neq j\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}
\]

satisfies \( 1 \leq m(e) < k - 4 \). Hence \( e \) is not an edge in \( H \) and we have a contradiction.

Case 4. Suppose every element in \( D \) is a local extremum. We then apply Lemma 3.3 to the set \( A = \{a_1, \ldots, a_7\} \) and \( B = \{a_8, \ldots, a_{14}\} \) to obtain vertices \( a_i \in A \) and \( a_j \in B \) such that \( m(\{a_1, \ldots, a_7\} \setminus \{a_i\}) = 2 \) and \( m(\{a_8, \ldots, a_{14}\} \setminus \{a_j\}) = 2 \). In particular, this implies that for \( e = \{a_1, \ldots, a_{k+2}\} \setminus \{a_i, a_j\} \), the corresponding sequence of \( \delta \)'s has at least two locally monotone elements. Since clearly \( e \) has at least one local extremum, we obtain \( 1 \leq m(e) < k - 4 \). Hence \( e \not\in E(H) \) and we have a contradiction.

Therefore we have shown that \( H \) is \( K_{k+2}^k \)-free.

Our final task is to show that \( \alpha_{k+1}(H) < k\alpha_k(H') \). Set \( n = kt \) where \( t = \alpha_k(H') \). Let us assume for contradiction that there are vertices \( a_1 < \cdots < a_n \) that induce a \((k+1)\)-independent set in \( H \). Let \( \delta_i = \delta(a_i, a_{i+1}) \) for \( 1 \leq i \leq n - 1 \). If the sequence \( \delta_1, \ldots, \delta_{n-1} \) contains fewer than \( k \)
local extrema, then there is a $j$ such that $\delta_j, \ldots, \delta_{j+t}$ is monotone. Since $t = \alpha_k(H')$, the $t + 1$ vertices $\{\delta_j, \ldots, \delta_{j+t}\}$ contain a copy of $K_{k-1}^k$ in $H'$. Say this copy is given by $\delta_j, \ldots, \delta_{j_k}$. Then by Property E, the vertices $a_{j_1} < \cdots < a_{j_k} < a_{j_k+1}$ induce a copy of $K_{k+1}^k$ which contradicts our assumption that $\{a_1, \ldots, a_n\}$ is a $(k + 1)$-independent set in $H$.

We may therefore assume that the sequence $\delta_1, \ldots, \delta_{n-1}$ contains at least $k$ local extrema. Now we make the following claim.

**Claim 3.4.** There is a set of $k + 1$ vertices $a_1^*, \ldots, a_{k+1}^* \in \{a_1, \ldots, a_n\}$ such that for $\delta_i^* = \delta(a_i^*, a_{i+1}^*)$, the sequence $\delta_1^*, \ldots, \delta_k^*$ has $k - 2$ local extrema.

**Proof.** Let $\delta_1, \ldots, \delta_k$ be the first $k$ extrema in the sequence $\delta_1, \ldots, \delta_{n-1}$.

**Case 1.** Suppose $\delta_{i_1}$ is a local minimum. If $k$ is odd, then consider the $k + 1$ distinct vertices

$$e = a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \ldots, a_{i_k}, a_{i_k+1}.$$  

Note that the pairs $(a_{i_1}, a_{i_1+1}), (a_{i_3}, a_{i_3+1}), (a_{i_5}, a_{i_5+1}), \ldots$ correspond to local minima. By Property B, $\delta(a_{i_1+1}, a_{i_1}) = \delta_{i_1}, \delta(a_{i_3+1}, a_{i_3}) = \delta_{i_3}, \ldots$. Since $\delta_{i_2}, \delta_{i_4}, \delta_{i_6}, \ldots$ were local maxima in the sequence $\delta_1, \ldots, \delta_{n-1}$, we have

$$\delta(a_{i_1}, a_{i_1+1}) < \delta(a_{i_1+1}, a_{i_1}) > \delta(a_{i_3}, a_{i_3+1}) < \delta(a_{i_3+1}, a_{i_3}) > \cdots.$$  

Hence the vertices in $e$ satisfy the claim. If $k$ is even, then by the same argument as above, the $k + 1$ vertices

$$a_1, a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \ldots, a_{i_{k-1}}, a_{i_k+1}$$

satisfy the claim.

**Case 2.** Suppose $\delta_{i_1}$ is a local maximum. If $k$ is odd, then the arguments above imply that the set of $k + 1$ vertices

$$a_1, a_2, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \ldots, a_{i_{k-1}}, a_{i_{k-1}} + 1$$

satisfies the claim. Likewise, if $k$ is even, the set of $k + 1$ vertices

$$a_1, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \ldots, a_{i_k}, a_{i_k+1}$$

satisfies the claim. □

By Claim 3.4, we obtain $k + 1$ vertices $h = (a_1^*, \ldots, a_{k+1}^*)$ along with $\delta_1^*, \ldots, \delta_k^*$ with the desired properties. Consider the $k$-tuple $e = h - a_1^*$. If $i = 1$ or $k + 1$, then it is easy to see that $m(e) = k - 3$, which implies $e \in E(H)$. For $i = 2$, $\delta_3^*$ is the only possible locally monotone element in the sequence $\delta(a_1^*, a_3^*), \delta_3^*, \ldots, \delta_k^*$. Therefore $m(e - a_i) \geq k - 4$ and $e \in E(H)$. A symmetric argument for the
case \( i = k \) implies that \( e \in E(H) \). Therefore we can assume \( 3 \leq i \leq k - 1 \). By Property B, we have \( \delta(a^*_1, a^*_i) = \max \{ \delta^*_i, \delta^*_i \} \). Let us consider the two cases.

**Case 1.** Suppose \( \delta(a^*_i, a^*_i) = \delta^*_i \). If \( \delta^*_i > \delta^*_i \), then \( \delta^*_i \) is the only element in the sequence \( \delta^*_i, \ldots, \delta^*_i, \delta^*_i, \ldots, \delta^*_k \) that is locally monotone. Hence \( m(e) = k - 4 \) and \( e \in E(H) \). If \( \delta^*_i < \delta^*_i \), then \( \delta^*_i \) is the only possible element in the sequence \( \delta^*_1, \ldots, \delta^*_i, \delta^*_i, \ldots, \delta^*_k \) that is locally monotone. More precisely, if \( i = k - 1 \) then \( m(e) = k - 3 \), and if \( 3 \leq i < k - 1 \) then \( m(e) = k - 4 \). Hence \( m(e) \geq k - 4 \) and therefore \( e \in E(H) \).

**Case 2.** Suppose \( \delta(a^*_i, a^*_i) = \delta^*_i \). If \( \delta^*_i > \delta^*_i \), then \( \delta^*_i \) is the only element in the sequence \( \delta^*_i, \ldots, \delta^*_i, \delta^*_i, \ldots, \delta^*_k \) that is locally monotone. Hence \( m(e) = k - 4 \) and \( e \in E(H) \). If \( \delta^*_i < \delta^*_i \), then \( \delta^*_i \) is the only possible element in the sequence \( \delta^*_1, \ldots, \delta^*_i, \delta^*_i, \ldots, \delta^*_k \) that is locally monotone. More precisely, if \( i = 3 \) then \( m(e) = k - 3 \), and if \( 3 < i < k - 1 \) then \( m(e) = k - 4 \). Hence \( m(e) \geq k - 4 \) and \( e \in E(H) \).

Therefore every \( k \)-tuple \( e = h - a_i \) is an edge in \( H \), and the \( k + 1 \) vertices \( h \) induces a \( K^k_{k+1} \) in \( H \). This is a contradiction and we have completed the proof.

## 4 Ramsey numbers for \( k \)-half-graphs versus cliques

Let \( K^3_4 \setminus e \) denote the 3-uniform hypergraph on four vertices, obtained by removing one edge from \( K^3_4 \). A simple argument of Erdős and Hajnal [12] implies \( r(K^3_4 \setminus e, K^3_n) < (n!)^2 \). On the other hand, they also gave a construction that shows \( r(K^3_4 \setminus e, K^3_n) > 2^{cn} \) for some constant \( c > 0 \). Improving either of these bounds is a very interesting open problem, as \( K^3_4 \setminus e \) is, in some sense, the smallest 3-uniform hypergraph whose Ramsey number with a clique is at least exponential.

A \( k \)-half-graph, denote by \( B^k \), is a \( k \)-uniform hypergraph on \( 2k - 2 \) vertices, whose vertex set is of the form \( S \cup T \), where \( |S| = |T| = k - 1 \), and whose edges are all \( k \)-subsets that contain \( S \), and one \( k \)-subset that contains \( T \). The hypergraph \( B^k \) can be viewed as a generalization of \( K^3_4 \setminus e \) as \( B^3 = K^3_4 \setminus e \).

The goal of this section is to obtain upper and lower bounds for \( r(B^k, K^k_n) \) that parallel the known state of affairs for \( K^3_4 \setminus e \). We begin by presenting a straightforward generalization of the argument of Erdős and Hajnal to establish an upper bound for Ramsey numbers for \( k \)-half-graphs versus cliques. Again for simplicity we write \( r(B^k, K^k_n) = r_k(B, n) \).

**Theorem 4.1.** For \( k \geq 4 \), we have \( r_k(B, n) \leq (n!)^{k-1} \).

First, let us recall an old lemma due to Spencer.

**Lemma 4.2** ([19]). Let \( H = (V, E) \) be a \( k \)-uniform hypergraph on \( N \) vertices. If \( |E(H)| > N/k \), then there exists a subset \( S \subseteq V(H) \) such that \( S \) is an independent set and

\[
|S| \geq \left( 1 - \frac{1}{k} \right) N \left( \frac{N}{k|E(H)|} \right)^{1/k}.
\]
Proof of Theorem 4.1. We proceed by induction on \( n \). The base case \( n = k \) is trivial. Let \( n > k \) and assume the statement holds for \( n' < n \). Let \( k \geq 4 \) and let \( \chi \) be a red/blue coloring on the edges of \( K_N^k \), where \( N = (n!)^{k-1} \). Let \( E_R \) denote the set of red edges in \( K_N^k \).

Case 1: Suppose \( |E_R| \leq N/k \). Then one can delete \( N/k \) vertices from \( H \) and obtain a blue clique of size \((1 - 1/k)N \geq n\).

Case 2: Suppose \( N/k < |E_R| < \frac{(1-\frac{1}{k})^{k-1}N}{kn^{k-1}} \). Then by Lemma 4.2, \( K_N^k \) contains a blue clique of size \( n \).

Case 3: Suppose \( |E_R| \geq \frac{(1-\frac{1}{k})^{k-1}N}{kn^{k-1}} \). Then by averaging, there is a \( (k-1) \)-element subset \( S \subseteq V \) such that \( N(S) = \{v \in V : S \cup \{v\} \in E_R\} \) satisfies

\[
|N(S)| \geq \frac{(1-\frac{1}{k})^{k-1}N}{n^{k-1}(\frac{N}{k-1})} \geq ((n-1)!)^{k-1}.
\]

The last inequality follows from the fact that \( k \geq 4 \). Fix a vertex \( u \in S \). If \( \{u\} \cup T \in E_R \) for some \( T \subseteq N(S) \) such that \( |T| = k-1 \), then \( S \cup T \) forms a red \( B^k \) and we are done. Therefore we can assume otherwise. By the induction hypothesis, \( N(S) \) contains a red copy of \( B^k \), or a blue copy of \( K^k_{n-1} \). We are done in the former case, and in the latter case, we can form a blue \( K_n^k \) by adding the vertex \( u \).

We now move to our main new contribution, which are constructions which show that \( r_k(B,n) \) is at least exponential in \( n \).

Theorem 4.3. For fixed \( k \geq 3 \), we have \( r_k(B,n) > 2^{\Omega(n)} \).

Proof. Surprisingly, we require different arguments for \( k \) even and \( k \) odd.

The case when \( k \) is odd. Assume \( k \) is odd, and set \( N = 2^{cn} \) where \( c = c(k) \) will be determined later. Then let \( T \) be a random tournament on the vertex set \([N]\), that is, for \( i,j \in [N] \), independently, either \((i,j) \in E \) or \((j,i) \in E \), where each of the two choices is equally likely. Then let \( \chi : \binom{[N]}{k} \rightarrow \{\text{red, blue}\} \) be a red/blue coloring on the \( k \)-subsets of \([N]\), where \( \chi(v_1,\ldots,v_k) = \text{red} \) if \( v_1,\ldots,v_k \) induces a regular tournament, that is, the indegree of every vertex is \((k-1)/2 \) (and hence the outdegree of every vertex is \((k-1)/2 \)). Otherwise we color it blue. We note that since \( k \) is odd, a regular tournament on \( k \) vertices is possible by the fact that \( K_k \) has an Eulerian circuit, and then by directing the edges according to the circuit we obtain a regular tournament.

Notice that the coloring \( \chi \) does not contain a red \( B^k \). Indeed, let \( S,T \subseteq [N] \) such that \( |S| = |T| = k-1 \), \( S \cap T = \emptyset \), and every \( k \)-tuple of the form \( S \cup \{v\} \) is red, for all \( v \in T \). Then for any \( u \in S \), all edges in the set \( u \times T \) must have the same direction, either all emanating out of \( u \) or all directed towards \( u \). Therefore it is impossible for \( u \cup T \) to have color red, for any choice \( u \in S \).

Next we estimate the expected number of monochromatic blue copies of \( K_n^k \) in \( \chi \). For a given \( k \)-tuple \( v_1,\ldots,v_k \in [N] \), the probability that \( \chi(v_1,\ldots,v_k) = \text{blue} \) is clearly at most \( 1 - 1/2^k \).
Let $T = \{v_1, \ldots, v_n\}$ be a set of $t$ vertices in $[n]$, where $v_1 < \cdots < v_n$. Let $S$ be a partial Steiner $(n, k, 2)$-system with vertex set $T$, that is, $S$ is a $k$-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in $S$. Moreover, $S$ satisfies $|S| = c'n^2$ where $c' = c'(k)$. It is known that such a system exists. Then the probability that every $k$-tuple in $T$ has color blue is at most the probability that every $k$-tuple in $S$ is blue. Since the edges in $S$ are independent, that is no two edges have more than one vertex in common, the probability that $T$ is a monochromatic blue clique is at most 

\[
\left(1 - \frac{1}{2^\binom{k}{2}}\right)^{|S|} \leq \left(1 - \frac{1}{2^\binom{k}{2}}\right)^{c'n^2}. 
\]

Therefore the expected number of monochromatic blue copies of $K^k_n$ in $\chi$ is at most

\[
\binom{N}{n} \left(1 - \frac{1}{2^\binom{k}{2}}\right)^{c'n^2} < 1,
\]

for an appropriate choice for $c = c(k)$. Hence, there is a coloring $\chi$ with no red $B^k$ and no blue $K^k_n$. Therefore

\[
r_k(B, n) > 2^{cn}. 
\]

The case when $k$ is even. Assume $k$ is even and set $N = 2^{cn}$ where $c = c(k)$ will be determined later. Consider the coloring $\phi : \binom{[N]}{2} \to \{1, \ldots, k-1\}$, where each edge has probability $1/(k-1)$ of being a particular color independent of all other edges (pairs). Using $\phi$, we define the coloring $\chi : \binom{[N]}{k} \to \{\text{red, blue}\}$, where the $k$-tuple $(v_1, \ldots, v_k)$ is red if $\phi$ is a proper edge-coloring on all pairs among $\{v_1, \ldots, v_k\}$, that is, each of the $k-1$ colors appears as a perfect matching. Otherwise we color it blue.

Notice that the coloring $\chi$ does not contain a red $B^k$. Indeed let $S, T \subset [N]$ such that $|S| = |T| = k-1$ and $S \cap T = \emptyset$. If, for all $v \in T$, the $k$-tuples of the form $S \cup \{v\}$ are red, then the set of edges $\{u\} \times T$ is monochromatic with respect to $\phi$ for any $u \in S$. Hence, $\chi$ could not have colored $\{u\} \cup T$ red for any $u \in S$.

For a given $k$-tuple $v_1, \ldots, v_k \in [N]$, the probability that $\chi(v_1, \ldots, v_k) = \text{blue}$ is at most $1 - (1/(k-1))^{\binom{k}{2}}$. By the same argument as above, the expected number of monochromatic blue copies of $K^k_n$ with respect to $\chi$ is less than 1 for an appropriate choice of $c = c(k)$. Hence, there is a coloring $\chi$ with no red $B^k$ and no blue $K^k_n$. Therefore

\[
r_k(B, n) > 2^{cn}
\]

and the proof is complete. \qed

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References


