Constructions in Ramsey theory

Dhruv Mubayi* Andrew Suk†

Abstract

We provide several constructions for problems in Ramsey theory. First, we prove a superexponential lower bound for the classical 4-uniform Ramsey number $r_4(5,n)$, and the same for the iterated $(k-4)$-fold logarithm of the k-uniform version $r_k(k+1,n)$. This is the first improvement of the original exponential lower bound for $r_4(5,n)$ implicit in work of Erdős and Hajnal from 1972 and also improves the current best known bounds for larger k due to the authors. Second, we prove an upper bound for the hypergraph Erdős-Rogers function $f^k_{k+1,k+2}(N)$ that is an iterated $(k-13)$-fold logarithm in N. This improves the previous upper bounds that were only logarithmic and addresses a question of Dudek and the first author that was reiterated by Conlon, Fox and Sudakov. Third, we generalize the results of Erdős and Hajnal about the 3-uniform Ramsey number of K_4 minus an edge versus a clique to k-uniform hypergraphs.

1 Introduction

A k-uniform hypergraph H (k-graph for short) with vertex set V is a collection of k-element subsets of V. We write K^k_n for the complete k-uniform hypergraph on an n-element vertex set. Given k-graphs F, G, the *Ramsey number* $r(F,G)$ is the minimum N such that every red/blue coloring of the edges of K^k_N results in a monochromatic red copy of F or a monochromatic blue copy of G.

In this paper, we study several problems in hypergraph Ramsey theory. We describe each problem in detail in its relevant section. Here we provide a brief summary. In Section 2, we give new lower bounds on the classical Ramsey number $r(K^k_{k+1},K^k_n)$, improving the previous best known bounds obtained by the authors [18]. In particular, we give the first superexponential lower bound for $r(K^4_5,K^4_n)$ since the problem was first explicitly stated by Erdős and Hajnal [12] in 1972. In Section 3, we establish a new upper bound for the hypergraph Erdős-Rogers function $f^k_{k+1,k+2}(N)$ that is an iterated logarithm function in N. More precisely, we construct k-graphs on N vertices, with no copy of K^k_{k+2}, yet every set of n vertices contains a copy of K^k_{k+1} where n is the $(k-13)$-fold iterated logarithm of N. This addresses questions posed by Dudek and the first author [8] as well as by Conlon, Fox, and Sudakov [7] and significantly improves the previous best known bound in [8] of $n = O((\log N)^{1/(k-1)})$. In Section 4 we study the Ramsey numbers for k-half-graphs versus

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA.
Research partially supported by NSF grant DMS-1300138. Email: mubayi@uic.edu

†Department of Mathematics, University of California at San Diego, La Jolla, CA, 92093 USA. Supported by NSF grant DMS-1500153, an NSF CAREER award, and an Alfred Sloan Fellowship. Email: asuk@ucsd.edu
MSC (2010): 05C15, 05C55, 05C65, 05D10, 05D40

cliques, generalizing the results of Erdős and Hajnal [12] about the 3-uniform Ramsey number of K_4 minus an edge versus a clique. The upper bound is a straightforward extension of the method in [12], while the constructions are new.

All logarithms are base 2 unless otherwise stated. For the sake of clarity of presentation, we systematically omit floor and ceiling signs whenever they are not crucial.

2 A new lower bound for $r_k(k+1,n)$

In order to avoid the excessive use of superscripts, we use the simpler notation $r(K_s^k, K_n^k) = r_k(s,n)$. Estimating the Ramsey number $r_k(s,n)$ is a classical problem in extremal combinatorics and has been extensively studied [13, 14, 16]. Here we study the *off-diagonal* Ramsey number, that is, $r_k(s,n)$ with k,s fixed and n tending to infinity. It is known that for fixed $s \geq k+1$, $r_2(s,n)$ grows polynomially in n [1, 2, 3] and $r_3(s,n)$ grows exponentially in a power of n [6]. In 1972, Erdős and Hajnal [12] raised the question of determining the correct tower growth rate for $r_k(s,n)$. We define the *tower function* $\mathrm{twr}_k(x)$ by

$$\mathrm{twr}_1(x) = x \qquad \text{and} \qquad \mathrm{twr}_{i+1} = 2^{\mathrm{twr}_i(x)}.$$

By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting (see [17]), it follows that $r_k(s,n) \geq \mathrm{twr}_{k-1}(\Omega(n))$, for $k \geq 4$ and for all $s \geq 2^{k-1} - k + 3$. However they conjectured the following.

Conjecture 2.1. (Erdős-Hajnal [12]) *For $s \geq k+1 \geq 5$ fixed, $r_k(s,n) \geq \mathrm{twr}_{k-1}(\Omega(n))$.*

In [5], Conlon, Fox, and Sudakov modified the Erdős-Hajnal stepping-up lemma to show that Conjecture 2.1 holds for all $s \geq \lceil 5k/2 \rceil - 3$. Recently the authors nearly proved the conjecture by establishing the following.

Theorem 2.2 ([18])**.** *There is a positive constant $c > 0$ such that the following holds. For $k \geq 4$ and $n > 3k$, we have*

1. $r_k(k+3,n) \geq \mathrm{twr}_{k-1}(cn)$,
2. $r_k(k+2,n) \geq \mathrm{twr}_{k-1}(c\log^2 n)$,
3. $r_k(k+1,n) \geq \mathrm{twr}_{k-2}(cn^2)$.

Implicit in work of Erdős and Hajnal [12] is the bound $r_4(5,n) > 2^{cn}$ for some absolute positive constant c. While the authors [18] recently improved this to 2^{cn^2} above, there has been no superexponential lower bound given for this basic problem. Here we provide such a lower bound.

Theorem 2.3. *There is an absolute constant $c > 0$ such that*

$$r_4(5,n) > 2^{n^{c\log\log n}},$$

and more generally for $k > 4$,

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c\log\log n}).$$

One of the building blocks we will use in our construction is the following lower bound of Conlon, Fox, and Sudakov [6]: there is an absolute positive constant $c > 0$ such that

$$r_3(4,t) > 2^{ct\log t}. \tag{1}$$

Our lower bound for $r_4(5,n)$ is proved via the following theorem.

Theorem 2.4. *For n sufficiently large, we have*

$$r_4(5,n) > 2^{r_3(4,\lfloor(\log n)/2\rfloor)-1}.$$

Proof. The idea is to apply a variant of the Erdős-Hajnal stepping up lemma (see [17]). Set $t = \lfloor \frac{\log n}{2} \rfloor$. Let ϕ be a red/blue coloring of the edges of the complete 3-uniform hypergraph on the vertex set $\{0,1,\ldots,r_3(4,t)-2\}$ without a red K_4^3 and without a blue K_t^3. We use ϕ to define a red/blue coloring χ of the edges of the complete 4-uniform hypergraph K_N^4 on the vertex set $V = \{0,1,\ldots,N-1\}$ with $N = 2^{r_3(4,t)-1}$, as follows.

For any $a \in V$, write $a = \sum_{i=0}^{r_3(4,t)-2} a(i)2^i$ with $a(i) \in \{0,1\}$ for each i. For $a \neq b$, let $\delta(a,b)$ denote the largest i for which $a(i) \neq b(i)$. Notice that we have the following stepping-up properties (again see [17])

Property A: For every triple $a < b < c$, $\delta(a,b) \neq \delta(b,c)$.

Property B: For $a_1 < \cdots < a_r$, $\delta(a_1,a_r) = \max_{1\le j\le r-1} \delta(a_j,a_{j+1})$.

Given any 4-tuple $a_1 < \cdots < a_4$ of V, consider the integers $\delta_i = \delta(a_i,a_{i+1}), 1 \le i \le 3$. Say that $\delta_1,\delta_2,\delta_3$ forms a monotone sequence if $\delta_1 < \delta_2 < \delta_3$ or $\delta_1 > \delta_2 > \delta_3$. Now, define χ as follows:

$$\chi(a_1,a_2,a_3,a_4) = \begin{cases} \phi(\delta_1,\delta_2,\delta_3) & \text{if } \delta_1,\delta_2,\delta_3 \text{ is monotone} \\ blue & \text{if } \delta_1,\delta_2,\delta_3 \text{ is not monotone} \end{cases}$$

Hence we have the following property which can be easily verified using Properties A and B (see [17]).

Property C: For $a_1 < \cdots < a_r$, set $\delta_j = \delta(a_j,a_{j+1})$ and suppose that $\delta_1,\ldots,\delta_{r-1}$ form a monotone sequence. If χ colors every 4-tuple in $\{a_1,\ldots,a_r\}$ red (blue), then ϕ colors every triple in $\{\delta_1,\ldots,\delta_{r-1}\}$ red (blue).

For sake of contradiction, suppose that the coloring χ produces a red K_5^4 on vertices $a_1 < \cdots < a_5$, and let $\delta_i = \delta(a_i,a_{i+1})$, $1 \le i \le 4$. Then $\delta_1,\ldots,\delta_4$ form a monotone sequence and, by Property C, ϕ colors every triple in $\{\delta_1,\ldots,\delta_4\}$ red which is a contradiction. Therefore, there is no red K_5^4 in coloring χ.

Next we show that there is no blue K_n^4 in coloring χ. Our argument is reminiscent of the standard argument for the bound $r_2(n,n) < 4^n$, though it must be adapted to this setting. For sake of contradiction, suppose we have vertices $a_1, \ldots, a_n \in V$ such that $a_1 < \cdots < a_n$ and χ colors every 4-tuple in the set $\{a_1, \ldots, a_n\}$ blue. Let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \leq i \leq n-1$. We greedily construct a set $D_h = \{\delta_{i_1}, \ldots, \delta_{i_h}\} \subset \{\delta_1, \ldots, \delta_{n-1}\}$ and a set $S_h \subset \{a_1, \ldots, a_n\}$ such that the following holds.

1. We have $\delta_{i_1} > \cdots > \delta_{i_h}$.
2. For each $\delta_{i_j} = \delta(a_{i_j}, a_{i_j+1}) \in D_h = \{\delta_{i_1}, \ldots, \delta_{i_h}\}$, consider the set of vertices

$$A = \{a_{i_{j+1}}, a_{i_{j+1}+1}, \ldots, a_{i_h}, a_{i_h+1}\} \cup S_h.$$

   Then either every element in A is greater than a_{i_j} or every element in A is less than a_{i_j+1}. In the former case we will label δ_{i_j} *white*, in the latter case we label it *black*.
3. The indices of the vertices in S_h are consecutive, that is, $S_h = \{a_r, a_{r+1}, \ldots, a_{s-1}, a_s\}$ for $1 \leq r < s \leq n$.

We start with the $D_0 = \emptyset$ and $S_0 = \{a_1, \ldots, a_n\}$. Having obtained $D_h = \{\delta_{i_1}, \ldots, \delta_{i_h}\}$ and $S_h = \{a_r, \ldots, a_s\}$, $1 \leq r < s \leq n$, we construct D_{h+1} and S_{h+1} as follows. Let $\delta_{i_{h+1}} = \delta(a_\ell, a_{\ell+1})$ be the unique largest element in $\{\delta_r, \delta_{r+1}, \ldots, \delta_{s-1}\}$, and set $D_{h+1} = D_h \cup \delta_{i_{h+1}}$. The uniqueness of $\delta_{i_{h+1}}$ follows from Properties A and B. If $|\{a_r, a_{r+1}, \ldots, a_\ell\}| \geq |S_h|/2$, then we set $S_{h+1} = \{a_r, a_{r+1}, \ldots, a_\ell\}$. Otherwise by the pigeonhole principle, we have $|\{a_{\ell+1}, a_{\ell+2}, \ldots, a_s\}| \geq |S_h|/2$ and we set $S_{h+1} = \{a_{\ell+1}, a_{\ell+2}, \ldots, a_s\}$.

Since $|S_0| = n$, $t = \lfloor \frac{\log n}{2} \rfloor$ and $|S_{h+1}| \geq |S_h|/2$ for $h \geq 0$, we can construct $D_{2t} = \{\delta_{i_1}, \ldots, \delta_{i_{2t}}\}$ with the desired properties. By the pigeonhole principle, at least t elements in D_{2t} have the same label, say *white*. The other case will follow by a symmetric argument. We remove all black labeled elements in D_{2t}, and let $\{\delta_{j_1}, \ldots, \delta_{j_t}\}$ be the resulting set. Now consider the vertices $a_{j_1}, a_{j_2}, \ldots, a_{j_t} \in V$. By construction and by Property B, we have $a_{j_1} < a_{j_2} < \cdots < a_{j_t}$ and $\delta(a_{j_1}, a_{j_2}) = \delta_{i_{j_1}}, \delta(a_{j_2}, a_{j_3}) = \delta_{i_{j_2}}, \ldots, \delta(a_{j_t}, a_{j_{t+1}}) = \delta_{i_{j_t}}$. Therefore we have a monotone sequence

$$\delta(a_{j_1}, a_{j_2}) > \delta(a_{j_2}, a_{j_3}) > \cdots > \delta(a_{j_t}, a_{j_{t+1}}).$$

By Property C, ϕ colors every triple from this set blue which is a contradiction. Therefore there is no red K_5^4 and no blue K_n^4 in coloring χ. $\square$

Applying the lower bound in (1), we obtain that

$$r_4(5, n) \geq 2^{r_3(4, \lfloor \log n/2 \rfloor) - 1} > 2^{2^{c \log n \log \log n}} = 2^{n^{c \log \log n}}$$

for some absolute positive constant c and this establishes the first part of Theorem 2.3.

We next prove Theorem 2.3 for $k \geq 5$. Independently, Conlon, Fox and Sudakov [4] gave a different proof of Theorem 2.2 part 1. Their approach was to begin with a known 4-uniform construction that yields $r_4(7, n) > 2^{2^{cn}}$ and then use a variant of the stepping up lemma to give tower-type lower bounds for larger k. Unfortunately, this variant of the stepping up lemma does not work if one begins instead with a lower bound for $r_4(5, n)$ which is our case. However, a further variant of the approach does work, and this is what we do below.

Lemma 2.5. *For $k \geq 5$ and n sufficiently large, we have*

$$r_k(k+1,n) \geq 2^{r_{k-1}(k,\lfloor n/6 \rfloor)-1}.$$

Proof. Again we apply a variant of the stepping-up lemma. Let ϕ be a red/blue coloring of the edges of the complete $(k-1)$-uniform hypergraph on the vertex set $\{0,1,\ldots,r_{k-1}(k,\lfloor n/6\rfloor)-2\}$ without a red K_k^{k-1} and without a blue $K_{\lfloor n/6 \rfloor}^{k-1}$. We use ϕ to define a red/blue coloring χ of the edges of the complete k-uniform hypergraph K_N^k on the vertex set $V=\{0,1,\ldots,N-1\}$ with $N=2^{r_{k-1}(k,\lfloor n/6\rfloor)-1}$, as follows.

Just as above, for any $a \in V$, write $a=\sum_{i=0}^{r_{k-1}(k,\lfloor n/6\rfloor)-2} a(i)2^i$ with $a(i)\in\{0,1\}$ for each i. For $a \neq b$, let $\delta(a,b)$ denote the largest i for which $a(i) \neq b(i)$. Hence Properties A and B hold.

Given any k-tuple $a_1 < a_2 < \ldots < a_k$ of V, consider the integers $\delta_i = \delta(a_i,a_{i+1}), 1 \leq i \leq k-1$. We say that δ_i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. We say that δ_i is *locally monotone* if $\delta_{i-1} < \delta_i < \delta_{i+1}$ or $\delta_{i-1} > \delta_i > \delta_{i+1}$. Since $\delta_{i-1} \neq \delta_i$ for every i, every nonmonotone sequence $\delta_1,\ldots,\delta_{k-1}$ has a local extremum. If $\delta_1,\ldots,\delta_{k-1}$ form a monotone sequence, then let $\chi(a_1,a_2,\ldots,a_k)=\phi(\delta_1,\delta_2,\ldots,\delta_{k-1})$. Otherwise if $\delta_1,\ldots,\delta_{k-1}$ is not monotone, then let $\chi(a_1,a_2,\ldots,a_k)$ be red if and only if δ_2 is a local maximum and δ_3 is a local minimum. Hence the following generalization of Property C holds.

Property D: For $a_1 < \cdots < a_r$, set $\delta_j = \delta(a_j,a_{j+1})$ and suppose that $\delta_1,\ldots,\delta_{r-1}$ form a monotone sequence. If χ colors every k-tuple in $\{a_1,\ldots,a_r\}$ red (blue), then ϕ colors every $(k-1)$-tuple in $\{\delta_1,\ldots,\delta_{r-1}\}$ red (blue).

For sake of contradiction, suppose that the coloring χ produces a red K_{k+1}^k on vertices $a_1 < \cdots < a_{k+1}$, and let $\delta_i = \delta(a_i,a_{i+1})$, $1 \leq i \leq k$. We have two cases.

Case 1. Suppose $\delta_1,\ldots,\delta_{k-1}$ is monotone. Then if $\delta_2,\ldots,\delta_k$ is also a monotone sequence, ϕ colors every $(k-1)$-tuple in $\{\delta_1,\ldots,\delta_k\}$ red by Property D, which is a contradiction. Otherwise, δ_{k-1} is the only local extremum and $\chi(a_2,\ldots,a_{k+1})$ is blue, which is again a contradiction.

Case 2. Suppose $\delta_1,\ldots,\delta_{k-1}$ is not monotone. Then we know that δ_2 is a local maximum and δ_3 is a local minimum. However this implies that $\chi(a_2,\ldots,a_{k+1})$ is blue, which is a contradiction. Hence there is no red K_{k+1}^k in coloring χ.

Next we show that there is no blue K_n^k in coloring χ. For sake of contradiction, suppose we have vertices $a_1,\ldots,a_n \in V$ such that $a_1 < \cdots < a_n$ and χ colors every k-tuple blue, and let $\delta_i = \delta(a_i,a_{i+1})$ for $1 \leq i \leq n-1$. By Property D, there is no integer r such that $\delta_r,\delta_{r+1},\ldots,\delta_{r+\lfloor n/6\rfloor}$ is monotone, since this implies that ϕ colors every $(k-1)$-tuple in the set $\{\delta_r,\delta_{r+1},\ldots,\delta_{r+\lfloor n/6\rfloor}\}$ blue which is a contradiction. Therefore the sequence $\delta_1,\ldots,\delta_{n-1}$ contains at least four local extrema. Let δ_{j_1} be the first local maximum, and let δ_{j_2} be the next local extremum, which must be a local minimum. Recall that $\delta_{j_1} = \delta(a_{j_1},a_{j_1+1})$ and $\delta_{j_2} = \delta(a_{j_2},a_{j_2+1})$. Consider the k vertices

$$a_{j_1-1},a_{j_1},a_{j_2},a_{j_2+1},a_{j_2+2},\ldots,a_{j_2+k-3}$$

and the sequence

$$\delta(a_{j_1-1},a_{j_1}),\delta(a_{j_1},a_{j_2}),\delta(a_{j_2},a_{j_2+1}),\ldots,\delta(a_{j_2+k-4},a_{j_2+k-3}).$$

By Property B we have $\delta(a_{j_1},a_{j_2})=\delta_{j_1}$, and therefore $\delta(a_{j_1},a_{j_2})$ is a local maximum and $\delta(a_{j_2},a_{j_2+1})$ is a local minimum. Therefore $\chi(a_{j_1-1},a_{j_1},a_{j_2},a_{j_2+1},\ldots,a_{j_2+k-3})$ is red and we have our contradiction. Hence there is no blue K_n^k in coloring χ. □

By combining Theorem 2.4 with Lemma 2.5, we establish Theorem 2.3.

3 The Erdős-Rogers function for hypergraphs

An s-independent set in a k-graph H is a vertex subset that contains no copy of K_s^k. So if $s=k$, then it is just an independent set. Let $\alpha_s(H)$ denote the size of the largest s-independent set in H.

Definition 3.1. For $k\le s<t<N$, the Erdős-Rogers function $f_{s,t}^k(N)$ is the minimum of $\alpha_s(H)$ taken over all K_t^k-free k-graphs H of order N.

To prove the lower bound $f_{s,t}^k(N)\ge n$ one must show that every K_t^k-free k-graph of order N contains an s-independent set with n vertices. On the other hand, to prove the upper bound $f_{s,t}^k(N)<n$, one must construct a K_t^k-free k-graph H of order N with $\alpha_s(H)<n$.

The problem of determining $f_{s,t}^k(n)$ extends that of finding Ramsey numbers. Formally,

$$r_k(s,n)=\min\{N: f_{k,s}^k(N)\ge n\}.$$

For $k=2$ the above function was first considered by Erdős and Rogers [15] only for $t=s+1$, which might be viewed as the most restrictive case. Since then the function has been studied by several researchers culminating in the work of Wolfowitz [20] and Dudek, Retter and Rödl [9] who proved the upper bound that follows (the lower bound is due to Dudek and the first author [8]): for every $s\ge 3$ there are positive constants c_1 and $c_2(s)$ such that

$$c_1\left(\frac{N\log N}{\log\log N}\right)^{1/2}<f_{s,s+1}^2(N)<c_2(\log N)^{4s^2}N^{1/2}.$$

The problem of estimating the Erdős-Rogers function for $k>2$ appears to be much harder. Let us denote

$$g(k,N)=f_{k+1,k+2}^k(N)$$

so that the above result (for $s=3$) becomes $g(2,N)=N^{1/2+o(1)}$. Dudek and the first author [8] proved that $(\log N)^{1/4+o(1)}<g(3,N)<O(\log N)$ and more generally that there are positive constants c_1 and c_2 with

$$c_1(\log_{(k-2)}N)^{1/4}<g(k,N)<c_2(\log N)^{1/(k-2)} \tag{2}$$

where $\log_{(i)}$ is the log function iterated i times. The exponent $1/4$ was improved to $1/3$ by Conlon, Fox, Sudakov [7]. Both sets of authors asked whether the upper bound could be improved (presumably to an iterated log function). Here we prove this where the number of iterations is $k - O(1)$. It remains an open problem to determine the correct number of iterations (which may well be $k-2$).

Theorem 3.2. *Fix $k \geq 14$. Then $g(k, N) < O(\log_{(k-13)} N)$.*

Proof. We will proceed by induction on k. The base case of $k = 14$ follows from the upper bound in (2). For the inductive step, let $k > 14$ and assume that the result holds for $k-1$. We will show that

$$g(k, 2^N) < k \cdot g(k-1, N),$$

and this recurrence clearly implies the theorem. Indeed, it easily implies the upper bound

$$g(k, N) < 2^k k! \log_{(k-13)} N$$

by induction on k, as $g(k+1, N)$ is at most

$$\begin{aligned} g(k+1, 2^{\lceil \log N \rceil}) &< (k+1) g(k, \lceil \log N \rceil) \\ &< 2^k (k+1)! \log_{(k-13)} \lceil \log N \rceil \\ &\leq 2^{k+1} (k+1)! \log_{(k-12)} N. \end{aligned}$$

Our strategy is to apply a variant of the stepping-up lemma. Let us begin with a K_{k+1}^{k-1}-free $(k-1)$-graph H' on N vertices for which $\alpha_k(H') = g(k-1, N)$. Note that this exists by definition of $g(k-1, N)$. We will use H' to produce a K_{k+2}^{k}-free k-graph H on 2^N vertices with $\alpha_{k+1}(H) < k\alpha_k(H') = kg(k-1, N)$.

Let $V(H') = \{0, 1, \ldots, N-1\}$ and $V(H) = \{0, 1, \ldots, 2^N - 1\}$. For any $a \in V(H)$, write $a = \sum_{i=0}^{N-1} a(i) 2^i$ with $a(i) \in \{0, 1\}$ for each i. For $a \neq b$, let $\delta(a, b)$ denote the largest i for which $a(i) \neq b(i)$. Therefore Properties A and B in the previous section hold.

Given any set of s vertices $a_1 < a_2 < \ldots < a_s$ of $V(H)$, consider the integers $\delta_i = \delta(a_i, a_{i+1}), 1 \leq i \leq s-1$. For $e = (a_1, \ldots, a_s)$, let $m(e)$ denote the number of local extrema in the sequence $\delta_1, \ldots, \delta_{s-1}$. In the case $s = k$, we define the edges of H as follows. If $\delta_1, \ldots, \delta_{k-1}$ form a monotone sequence, then let $(a_1, a_2, \ldots, a_k) \in E(H)$ if and only if $(\delta_1, \delta_2, \ldots, \delta_{k-1}) \in E(H')$. Otherwise if $\delta_1, \ldots, \delta_{k-1}$ is not monotone, then $(a_1, a_2, \ldots, a_k) \in E(H)$ if and only if $m(e) \in \{k-4, k-3\}$. In other words, given that $\delta_1, \ldots, \delta_{k-1}$ is not monotone, $(a_1, a_2, \ldots, a_k) \in E(H)$ if and only if $\delta_1, \ldots, \delta_{k-1}$ has at most one locally monotone element. Note that we have the following variant of Property D.

Property E: For $a_1 < \cdots < a_r$, set $\delta_j = \delta(a_j, a_{j+1})$ and suppose that $\delta_1, \ldots, \delta_{r-1}$ form a monotone sequence. If every k-tuple in $\{a_1, \ldots, a_r\}$ is in $E(H)$ (in $\overline{E}(H)$), then every $(k-1)$-tuple in $\{\delta_1, \ldots, \delta_{r-1}\}$ is in $E(H')$ (in $\overline{E}(H')$).

We are to show that H contains no $(k+2)$-clique and $\alpha_{k+1}(H) < k\alpha_k(H')$. First let us establish the following lemma.

Lemma 3.3. *Given $e = (a_1, \dots, a_7)$ with $a_1 < \dots < a_7$, let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \le i \le 6$. If $m(e) = 4$, then there is an a_i such that $2 \le i \le 6$ and $m(e - a_i) = 2$.*

Proof. Suppose first that δ_2 is a local minimum, so $\delta_1 > \delta_2 < \delta_3 > \cdots$. Then we have $m(e-a_4) = 2$. Indeed, since δ_4 is a local minimum, Property B implies $\delta(a_3, a_5) = \delta_3$. If $\delta_5 > \delta_3$, then we have $\delta_2 < \delta(a_3, a_5) < \delta_5$ and therefore $m(e - a_4) = 2$. If $\delta_5 < \delta_3$, then we have $\delta(a_3, a_5) > \delta_5 > \delta_6$ which again implies that $m(e - a_4) = 2$.

Now suppose that δ_2 is a local maximum, so $\delta_1 < \delta_2 > \delta_3 < \cdots$. Then we have $m(e - a_3) = 2$. Indeed, by Property B we have $\delta(a_2, a_4) = \delta_2$. If $\delta_4 < \delta_2$, then we have $\delta(a_2, a_4) > \delta_4 > \delta_5$ which implies $m(e - a_3) = 2$. If $\delta_4 > \delta_2$, then we have $\delta_1 < \delta(a_2, a_4) < \delta_4$ which again implies $m(e - a_3) = 2$. □

For sake of contradiction, suppose there are $k + 2$ vertices $a_1 < \dots < a_{k+2}$ that induce a K^k_{k+2} in H. Define $\delta_i = \delta(a_i, a_{i+1})$ for all $1 \le i \le k + 1$. Given the sequence $\delta_1, \delta_2, \dots, \delta_{k+1}$, let us consider the number of locally monotone elements in $D = \{\delta_2, \dots, \delta_k\}$.

Case 1. Suppose every element in D is locally monotone. Then $\delta_1, \dots, \delta_{k+1}$ form a monotone sequence. By Property E, every $(k - 1)$-tuple in the set $\{\delta_1, \dots, \delta_{k+1}\}$ is an edge in H' which is a contradiction since H' is K^{k-1}_{k+1}-free.

Case 2. Suppose there is at least one local extremum $\delta_\ell \in D$ and at least two elements $\delta_i, \delta_j \in D$ that are locally monotone. Then any k-tuple $e \subset \{a_1, \dots, a_{k+2}\}$ that includes the vertices

$$a_{i-1}, a_i, a_{i+1}, a_{i+2}, a_{j-1}, a_j, a_{j+1}, a_{j+2}, a_{\ell-1}, a_\ell, a_{\ell+1}, a_{\ell+2}$$

satisfies $1 \le m(e) < k - 4$. Therefore e is not an edge in H and we have a contradiction.

Case 3. Suppose there is exactly one element $\delta_i \in D$ that is locally monotone (and therefore at least one local extremum). Since $k \ge 15$, either $|\{a_1, \dots, a_{i-1}\}| \ge 7$ or $|\{a_{i+2}, \dots, a_{k+2}\}| \ge 7$. Let us only consider the former case, the latter being symmetric. By Lemma 3.3, there is an element $a_j \in \{a_2, \dots, a_6\} \subset \{a_1, \dots, a_{i-1}\}$ such that for $e' = (a_1, \dots, a_7)$, $m(e' - a_j) = 2$. Then any k-tuple $e \subset \{a_1, \dots, a_{k+2}\} \setminus \{a_j\}$ that includes vertices

$$\{a_t : 1 \le t \le 7, t \ne j\} \cup \{a_{i-1}, a_i, a_{i+1}, a_{i+2}\}$$

satisfies $1 \le m(e) < k - 4$. Hence e is not an edge in H and we have a contradiction.

Case 4. Suppose every element in D is a local extremum. We then apply Lemma 3.3 to the set $A = \{a_1, \dots, a_7\}$ and $B = \{a_8, \dots, a_{14}\}$ to obtain vertices $a_i \in A$ and $a_j \in B$ such that $m(\{a_1, \dots, a_7\} \setminus \{a_i\}) = 2$ and $m(\{a_8, \dots, a_{14}\} \setminus \{a_j\}) = 2$. In particular, this implies that for $e = \{a_1, \dots, a_{k+2}\} \setminus \{a_i, a_j\}$, the corresponding sequence of δ's has at least two locally monotone elements. Since clearly e has at least one local extremum, we obtain $1 \le m(e) < k - 4$. Hence $e \notin E(H)$ and we have a contradiction.

Therefore we have shown that H is K^k_{k+2}-free.

Our final task is to show that $\alpha_{k+1}(H) < k\alpha_k(H')$. Set $n = kt$ where $t = \alpha_k(H')$. Let us assume for contradiction that there are vertices $a_1 < \dots < a_n$ that induce a $(k + 1)$-independent set in H. Let $\delta_i = \delta(a_i, a_{i+1})$ for $1 \le i \le n - 1$. If the sequence $\delta_1, \dots, \delta_{n-1}$ contains fewer than k

local extrema, then there is a j such that $\delta_j, \ldots, \delta_{j+t}$ is monotone. Since $t = \alpha_k(H')$, the $t+1$ vertices $\{\delta_j, \ldots, \delta_{j+t}\}$ contain a copy of K_k^{k-1} in H'. Say this copy is given by $\delta_{j_1}, \ldots, \delta_{j_k}$. Then by Property E, the vertices $a_{j_1} < \cdots < a_{j_k} < a_{j_k+1}$ induce a copy of K_{k+1}^k which contradicts our assumption that $\{a_1, \ldots, a_n\}$ is a $(k+1)$-independent set in H.

We may therefore assume that the sequence $\delta_1, \ldots, \delta_{n-1}$ contains at least k local extrema. Now we make the following claim.

Claim 3.4. *There is a set of $k+1$ vertices $a_1^*, \ldots, a_{k+1}^* \in \{a_1, \ldots, a_n\}$ such that for $\delta_i^* = \delta(a_i^*, a_{i+1}^*)$, the sequence $\delta_1^*, \ldots, \delta_k^*$ has $k-2$ local extrema.*

Proof. Let $\delta_{i_1}, \ldots, \delta_{i_k}$ be the first k extrema in the sequence $\delta_1, \ldots, \delta_{n-1}$.

Case 1. Suppose δ_{i_1} is a local minimum. If k is odd, then consider the $k+1$ distinct vertices

$$e = a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \ldots, a_{i_k}, a_{i_k+1}.$$

Note that the pairs $(a_{i_1}, a_{i_1+1}), (a_{i_3}, a_{i_3+1}), (a_{i_5}, a_{i_5+1}), \ldots$ correspond to local minima. By Property B, $\delta(a_{i_1+1}, a_{i_3}) = \delta_{i_2}$, $\delta(a_{i_3+1}, a_{i_5}) = \delta_{i_4}, \ldots$. Since $\delta_{i_2}, \delta_{i_4}, \delta_{i_6}, \ldots$ were local maxima in the sequence $\delta_1, \ldots, \delta_{n-1}$, we have

$$\delta(a_{i_1}, a_{i_1+1}) < \delta(a_{i_1+1}, a_{i_3}) > \delta(a_{i_3}, a_{i_3+1}) < \delta(a_{i_3+1}, a_{i_5}) > \cdots .$$

Hence the vertices in e satisfy the claim. If k is even, then by the same argument as above, the $k+1$ vertices

$$a_1, a_{i_1}, a_{i_1+1}, a_{i_3}, a_{i_3+1}, a_{i_5}, a_{i_5+1}, \ldots, a_{i_{k-1}}, a_{i_{k-1}+1}$$

satisfy the claim.

Case 2. Suppose δ_{i_1} is a local maximum. If k is odd, then the arguments above imply that the set of $k+1$ vertices

$$a_1, a_2, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \ldots, a_{i_{k-1}}, a_{i_{k-1}+1}$$

satisfies the claim. Likewise, if k is even, the set of $k+1$ vertices

$$a_1, a_{i_2}, a_{i_2+1}, a_{i_4}, a_{i_4+1}, \ldots, a_{i_k}, a_{i_k+1}$$

satisfies the claim. □

By Claim 3.4, we obtain $k+1$ vertices $h = (a_1^*, \ldots, a_{k+1}^*)$ along with $\delta_1^*, \ldots, \delta_k^*$ with the desired properties. Consider the k-tuple $e = h - a_i^*$. If $i = 1$ or $k+1$, then it is easy to see that $m(e) = k-3$, which implies $e \in E(H)$. For $i = 2$, δ_3^* is the only possible locally monotone element in the sequence $\delta(a_1^*, a_3^*), \delta_3^*, \ldots, \delta_k^*$. Therefore $m(e - a_i) \geq k-4$ and $e \in E(H)$. A symmetric argument for the case $i = k$ implies that $e \in E(H)$. Therefore we can assume $3 \le i \le k-1$. By Property B, we have $\delta(a^*_{i-1}, a^*_{i+1}) = \max\{\delta^*_{i-1}, \delta^*_i\}$. Let us consider the two cases.

Case 1. Suppose $\delta(a^*_{i-1}, a^*_{i+1}) = \delta^*_{i-1}$. If $\delta^*_{i+1} > \delta^*_{i-1}$, then δ^*_{i-1} is the only element in the sequence $\delta^*_1, \ldots, \delta^*_{i-1}, \delta^*_{i+1}, \ldots, \delta^*_k$ that is locally monotone. Hence $m(e) = k-4$ and $e \in E(H)$. If $\delta^*_{i+1} < \delta^*_{i-1}$, then δ^*_{i+1} is the only possible element in the sequence $\delta^*_1, \ldots, \delta^*_{i-1}, \delta^*_{i+1}, \ldots, \delta^*_k$ that is locally monotone. More precisely, if $i = k-1$ then $m(e) = k-3$, and if $3 \le i < k-1$ then $m(e) = k-4$. Hence $m(e) \ge k-4$ and therefore $e \in E(H)$.

Case 2. Suppose $\delta(a^*_{i-1}, a^*_{i+1}) = \delta^*_i$. If $\delta^*_{i-2} > \delta^*_i$, then δ^*_i is the only element in the sequence $\delta^*_1, \ldots, \delta^*_{i-2}, \delta^*_i, \ldots, \delta^*_k$ that is locally monotone. Hence $m(e) = k-4$ and $e \in E(H)$. If $\delta^*_{i-2} < \delta^*_i$, then δ^*_{i-2} is the only possible element in the sequence $\delta^*_1, \ldots, \delta^*_{i-2}, \delta^*_i, \ldots, \delta^*_k$ that is locally monotone. More precisely, if $i = 3$ then $m(e) = k-3$, and if $3 < i \le k-1$ then $m(e) = k-4$. Hence $m(e) \ge k-4$ and $e \in E(H)$.

Therefore every k-tuple $e = h - a_i$ is an edge in H, and the $k+1$ vertices h induces a K^k_{k+1} in H. This is a contradiction and we have completed the proof. □

4 Ramsey numbers for k-half-graphs versus cliques

Let $K^3_4 \setminus e$ denote the 3-uniform hypergraph on four vertices, obtained by removing one edge from K^3_4. A simple argument of Erdős and Hajnal [12] implies $r(K^3_4 \setminus e, K^3_n) < (n!)^2$. On the other hand, they also gave a construction that shows $r(K^3_4 \setminus e, K^3_n) > 2^{cn}$ for some constant $c > 0$. Improving either of these bounds is a very interesting open problem, as $K^3_4 \setminus e$ is, in some sense, the smallest 3-uniform hypergraph whose Ramsey number with a clique is at least exponential.

A *k-half-graph*, denote by B^k, is a k-uniform hypergraph on $2k-2$ vertices, whose vertex set is of the form $S \cup T$, where $|S| = |T| = k-1$, and whose edges are all k-subsets that contain S, and one k-subset that contains T. The hypergraph B^k can be viewed as a generalization of $K^3_4 \setminus e$ as $B^3 = K^3_4 \setminus e$.

The goal of this section is to obtain upper and lower bounds for $r(B^k, K^k_n)$ that parallel the known state of affairs for $K^3_4 \setminus e$. We begin by presenting a straightforward generalization of the argument of Erdős and Hajnal to establish an upper bound for Ramsey numbers for k-half-graphs versus cliques. Again for simplicity we write $r(B^k, K^k_n) = r_k(B, n)$.

Theorem 4.1. *For $k \ge 4$, we have $r_k(B, n) \le (n!)^{k-1}$.*

First, let us recall an old lemma due to Spencer.

Lemma 4.2 ([19])**.** *Let $H = (V, E)$ be a k-uniform hypergraph on N vertices. If $|E(H)| > N/k$, then there exists a subset $S \subset V(H)$ such that S is an independent set and*

$$|S| \ge \left(1 - \frac{1}{k}\right) N \left(\frac{N}{k|E(H)|}\right)^{\frac{1}{k-1}}.$$

Proof of Theorem 4.1. We proceed by induction on n. The base case $n = k$ is trivial. Let $n > k$ and assume the statement holds for $n' < n$. Let $k \geq 4$ and let χ be a red/blue coloring on the edges of K_N^k, where $N = (n!)^{k-1}$. Let E_R denote the set of red edges in K_N^k.

Case 1: Suppose $|E_R| \leq N/k$. Then one can delete N/k vertices from H and obtain a blue clique of size $(1 - 1/k)N \geq n$.

Case 2: Suppose $N/k < |E_R| < \frac{\left(1-\frac{1}{k}\right)^{k-1} N^k}{kn^{k-1}}$. Then by Lemma 4.2, K_N^k contains a blue clique of size n.

Case 3: Suppose $|E_R| \geq \frac{\left(1-\frac{1}{k}\right)^{k-1} N^k}{kn^{k-1}}$. Then by averaging, there is a $(k-1)$-element subset $S \subset V$ such that $N(S) = \{v \in V : S \cup \{v\} \in E_R\}$ satisfies

$$|N(S)| \geq \frac{\left(1 - \frac{1}{k}\right)^{k-1} N^k}{n^{k-1}\binom{N}{k-1}} \geq ((n-1)!)^{k-1}.$$

The last inequality follows from the fact that $k \geq 4$. Fix a vertex $u \in S$. If $\{u\} \cup T \in E_R$ for some $T \subset N(S)$ such that $|T| = k - 1$, then $S \cup T$ forms a red B^k and we are done. Therefore we can assume otherwise. By the induction hypothesis, $N(S)$ contains a red copy of B^k, or a blue copy of K_{n-1}^k. We are done in the former case, and in the latter case, we can form a blue K_n^k by adding the vertex u. $\square$

We now move to our main new contribution, which are constructions which show that $r_k(B, n)$ is at least exponential in n.

Theorem 4.3. *For fixed $k \geq 3$, we have $r_k(B, n) > 2^{\Omega(n)}$.*

Proof. Surprisingly, we require different arguments for k even and k odd.

The case when k is odd. Assume k is odd, and set $N = 2^{cn}$ where $c = c(k)$ will be determined later. Then let T be a random tournament on the vertex set $[N]$, that is, for $i, j \in [N]$, independently, either $(i, j) \in E$ or $(j, i) \in E$, where each of the two choices is equally likely. Then let $\chi : \binom{[N]}{k} \rightarrow \{\text{red}, \text{blue}\}$ be a red/blue coloring on the k-subsets of $[N]$, where $\chi(v_1, \ldots, v_k) = \text{red}$ if $v_1, \ldots, v_k$ induces a *regular* tournament, that is, the indegree of every vertex is $(k-1)/2$ (and hence the outdegree of every vertex is $(k-1)/2$). Otherwise we color it blue. We note that since k is odd, a regular tournament on k vertices is possible by the fact that K_k has an Eulerian circuit, and then by directing the edges according to the circuit we obtain a regular tournament.

Notice that the coloring χ does not contain a red B^k. Indeed, let $S, T \subset [N]$ such that $|S| = |T| = k-1$, $S \cap T = \emptyset$, and every k-tuple of the form $S \cup \{v\}$ is red, for all $v \in T$. Then for any $u \in S$, all edges in the set $u \times T$ must have the same direction, either all emanating out of u or all directed towards u. Therefore it is impossible for $u \cup T$ to have color red, for any choice $u \in S$.

Next we estimate the expected number of monochromatic blue copies of K_n^k in χ. For a given k-tuple $v_1, \ldots, v_k \in [N]$, the probability that $\chi(v_1, \ldots, v_k) = \text{blue}$ is clearly at most $1 - 1/2^{\binom{k}{2}}$.

Let $T = \{v_1, \dots, v_n\}$ be a set of t vertices in $[n]$, where $v_1 < \cdots < v_n$. Let S be a partial Steiner $(n, k, 2)$-system with vertex set T, that is, S is a k-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S. Moreover, S satisfies $|S| = c'n^2$ where $c' = c'(k)$. It is known that such a system exists. Then the probability that every k-tuple in T has color blue is at most the probability that every k-tuple in S is blue. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that T is a monochromatic blue clique is at most $\left(1 - 1/2^{\binom{k}{2}}\right)^{|S|} \leq \left(1 - 1/2^{\binom{k}{2}}\right)^{c'n^2}$. Therefore the expected number of monochromatic blue copies of K_n^k in χ is at most

$$\binom{N}{n}\left(1 - 1/2^{\binom{k}{2}}\right)^{c'n^2} < 1,$$

for an appropriate choice for $c = c(k)$. Hence, there is a coloring χ with no red B^k and no blue K_n^k. Therefore

$$r_k(B, n) > 2^{cn}.$$

The case when k is even. Assume k is even and set $N = 2^{cn}$ where $c = c(k)$ will be determined later. Consider the coloring $\phi : \binom{[N]}{2} \to \{1, \dots, k-1\}$, where each edge has probability $1/(k-1)$ of being a particular color independent of all other edges (pairs). Using ϕ, we define the coloring $\chi : \binom{[N]}{k} \to \{\text{red}, \text{blue}\}$, where the k-tuple $(v_1, \dots, v_k)$ is red if ϕ is a proper edge-coloring on all pairs among $\{v_1, \dots, v_k\}$, that is, each of the $k-1$ colors appears as a perfect matching. Otherwise we color it blue.

Notice that the coloring χ does not contain a red B^k. Indeed let $S, T \subset [N]$ such that $|S| = |T| = k-1$ and $S \cap T = \emptyset$. If, for all $v \in T$, the k-tuples of the form $S \cup \{v\}$ are red, then the set of edges $\{u\} \times T$ is monochromatic with respect to ϕ for any $u \in S$. Hence, χ could not have colored $\{u\} \cup T$ red for any $u \in S$.

For a given k-tuple $v_1, \dots, v_k \in [N]$, the probability that $\chi(v_1, \dots, v_k) = $ blue is at most $1 - (1/(k-1))^{\binom{k}{2}}$. By the same argument as above, the expected number of monochromatic blue copies of K_n^k with respect to χ is less than 1 for an appropriate choice of $c = c(k)$. Hence, there is a coloring χ with no red B^k and no blue K_n^k. Therefore

$$r_k(B, n) > 2^{cn}$$

and the proof is complete. □

Acknowledgment. We thank the referee for helpful comments.

References

[1] M. Ajtai, J. Komlós, E. Szemerédi, A note on Ramsey numbers, *J. Combin. Theory Ser. A* **29** (1980), 354–360.

[2] T. Bohman, The triangle-free process, *Adv. Math.* **221** (2009), 1653–1677.

[3] T. Bohman, P. Keevash, The early evolution of the H-free process, *Invent. Math.* **181** (2010), 291–336.

[4] D. Conlon, J. Fox, B. Sudakov, personal communication.

[5] D. Conlon, J. Fox, and B. Sudakov, An improved bound for the stepping-up lemma, *Discrete Applied Mathematics* **161** (2013), 1191–1196.

[6] D. Conlon, J. Fox, and B. Sudakov, Hypergraph Ramsey numbers, *J. Amer. Math. Soc.* **23** (2010), 247–266.

[7] D. Conlon, J. Fox, and B. Sudakov, Short proofs of some extremal results, *Combin. Probab. Comput.* **23** (2014), 8–28.

[8] A. Dudek, D. Mubayi, On generalized Ramsey numbers for 3-uniform hypergraphs, *J. Graph Theory* **76** (2014), 217–223.

[9] A. Dudek, T. Retter, V. Rödl, On generalized Ramsey numbers of Erdős and Rogers, *J. Combin. Theory Ser. B* 109 (2014), 213–227.

[10] J. Fox, J. Pach, B. Sudakov, A. Suk, Erdős-Szekeres-type theorems for monotone paths and convex bodies, *Proc. Lond. Math. Soc.* **105** (2012), 953–982.

[11] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[12] P. Erdős, A. Hajnal, On Ramsey like theorems, problems and results, in Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), pp. 123–140, Inst. Math. Appl., Southhend-on-Sea, 1972.

[13] P. Erdős, A. Hajnal, R. Rado, Partition relations for cardinal numbers, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 93–196.

[14] P. Erdős, R. Rado, Combinatorial theorems on classifications of subsets of a given set, *Proc. Lond. Math. Soc.* **3** (1952), 417–439.

[15] P. Erdős, C.A. Rogers, *The construction of certain graphs*, Canad. J. Math. **14** (1962), 702–707.

[16] P. Erdős, G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* **2** (1935), 463–470.

[17] R. L. Graham, B. L. Rothschild, J. H. Spencer: *Ramsey Theory, 2nd ed.*, Wiley, New York, 1990.

[18] D. Mubayi, A. Suk, Off-diagonal hypergraph Ramsey numbers, *J. Combin. Theory Ser. B* **125** (2017), 168–177.

[19] J. Spencer, Turán theorem for k-graphs, *Disc. Math.* **2** (1972), 183–186.

[20] G. Wolfowitz, K_4-free graphs without large induced triangle-free subgraphs, *Combinatorica* **33** (2013), no. 5, 623–631.