On a generalized Erdős–Rademacher problem

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Abstract

The triangle covering number of a graph is the minimum number of vertices that hit all triangles. Given positive integers s, t and an n-vertex graph G with $\lfloor n^2/4 \rfloor + t$ edges and triangle covering number s, we determine (for large n) sharp bounds on the minimum number of triangles in G and also describe the extremal constructions. Similar results are proved for cliques of larger size and color critical graphs.

This extends classical work of Rademacher, Erdős, and Lovász-Simonovits whose results apply only to $s \le t$. Our results also address two conjectures of Xiao and Katona. We prove one of them and give a counterexample and prove a modified version of the other conjecture.

1 Introduction

A classical result of Mantel [6] states that every graph on n vertices with $\lfloor n^2/4 \rfloor + 1$ edges contains at least one copy of K_3. Rademacher showed that there are actually at least $\lfloor n/2 \rfloor$ copies of K_3 in such graphs. Later, Erdős [2, 3] proved that if $t \le cn$ for some small constant $c > 0$, then every graph on n vertices with $\lfloor n^2/4 \rfloor + t$ edges contains at least $t \lfloor n/2 \rfloor$ copies of K_3. Erdős also conjectured that the same conclusion holds for all $t < n/2$. Later, Lovász and Simonovits [5] proved Erdős' conjecture and they also proved a similar result for K_k with $k \ge 4$. In [7], the second author extended their results by proving tight bounds on the number of copies of color critical graphs in a graph with a prescribed number of vertices and edges.

Given a graph G we use $V(G)$ to denote its vertex set and use $E(G)$ to denote its edge set. Let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. Sometimes we abuse notation and let $G = E(G)$ and $|G| = e(G)$. For a fixed graph F let $N_F(G)$ denote the number of copies of F in G. The F-covering number $\tau_F(G)$ of G is the minimum size of $S \subset V(G)$ such that every copy of F in G has at least one vertex in S. If $F = K_k$, then we simply use $N_k(G)$ and $\tau_k(G)$ to denote $N_{K_k}(G)$ and $\tau_{K_k}(G)$, respectively.

The classical Erdős–Rademacher problem is to determine the minimum value of $N_F(G)$ for graphs G with fixed number of vertices and edges. Very recently, Xiao and Katona [10] posed a generalized Erdős–Rademacher problem by putting constraints on $\tau_F(G)$. More precisely, they asked for the minimum value of $N_F(G)$ for graphs G with a fixed number

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of vertices and edges and a fixed F-covering number. In particular, they proved that every graph G on n vertices with $\lfloor n^2/4 \rfloor + 1$ edges and $\tau_3(G) = 2$ must contain at least $n-2$ copies of K_3, which is substantially greater than the bound guaranteed by Rademacher's result. This phenomenon motivatedp them to pose the following conjectures for the general case.

Conjecture 1.1 (Xiao–Katona [10]). *Let $s > t \geq 1$ be fixed integers and let $n \geq n_0 = n_0(s,t)$ be sufficiently large. Then every graph G on n vertices with $\lfloor n^2/4 \rfloor + t$ edges and $\tau_3(G) \geq s$ contains at least $(s-1)\lfloor n/2 \rfloor + \lceil n/2 \rceil - 2(s-t)$ copies of K_3.*

Let V be a set of size n. Then a partition $V = V_1 \cup \cdots \cup V_{k-1}$ is called balanced if $\lceil n/(k-1) \rceil \geq |V_i| \geq \lfloor n/(k-1) \rfloor$ for all $i \in [k-1]$. For $k \geq 2$ define $t_k(n) = \prod_{1 \leq i < j \leq k} |V_i||V_j|$, where $V_1 \cup \cdots \cup V_k = [n]$ is a balanced partition.

Conjecture 1.2 (Xiao–Katona [10]). *Let $s > t \geq 1, k \geq 4$ be fixed integers. Then every graph G on n vertices with $t_{k-1}(n) + 1$ edges and $\tau_k(G) \geq 2$ contains at least $(|V_1| + |V_2| - 2)\prod_{i=3}^{k-1} |V_i|$ copies of K_k, where $V_1 \cup \cdots \cup V_{k-1}$ is a balanced partition of $[n]$ with $|V_1| \geq \cdots \geq |V_{k-1}|$.*

Xiao and Katona claimed that there is a common generalization of Conjectures 1.1 and 1.2 without writing it explicitly. They also observed that the case $s \leq t$ of these questions is a consequence of the previously mentioned results of Rademacher, Erdős [2, 3] and Lovász–Simonovits [5]. Indeed, it follows from a result of Lovász and Simonovits [5] that the graph obtained from the balanced complete $(k-1)$-partite n-vertex graph by adding t pairwise vertex-disjoint edges into a largest part minimizes the number of copies of K_k among all n-vertex graph with $t_{k-1}(n) + t$ edges. Moreover, this graph clearly has K_k-covering number $t \geq s$. It therefore suffices to consider only the case $s > t$ for these questions.

We show that Conjecture 1.1 is not true in general and give the correct bound on the number of copies of K_3 for all s, t and sufficiently large n. On the other hand, we prove Conjecture 1.2 for sufficiently large n and we also prove several generalizations of Conjecture 1.2 for graphs G with $t_{k-1}(n) + t$ edges and $\tau_k(G) \geq s$. Our method also gives a bound, which is tight up to a smaller order error term, for the number of color critical graphs F in a graph with a fixed number of vertices and edges and a fixed F-covering number.

1.1 Triangles

To motivate the following definitions let us look at a simple construction first. Suppose that n is an even integer and $s-t$ is a square. Then the graph G obtained from the complete bipartite graph with part sizes $n/2 + (s-t)^{1/2}$ and $n/2 - (s-t)^{1/2}$ by adding s pairwise vertex-disjoint edges to the larger part satisfies $\tau_3(G) = s$ and $e(G) = n^2/4 + t$. Moreover, $N_3(G) = s\left(n/2 - (s-t)^{1/2}\right)$, which is smaller than the bound in Conjecture 1.1 for all $t \geq 2$.

Now let us present the definitions we need in this section. Let $\mathbb{N} = \{0, 1, \ldots\}$ be the set of nonnegative integers. For $s > t \geq 1$ and $n \in \mathbb{N}$ let $e(n) = n^2 - 4t_2(n) = n^2 - 4\lfloor n^2/4 \rfloor \in \{0,1\}$ and

$$M_{s,t} = M_{s,t}(n) = \left\{ m \in \mathbb{N} : (4s - 4t - 4m + e(n))^{1/2} \in \mathbb{N} \right\}.$$

Note that $M_{s,t} \neq \emptyset$ since $s-t \in M_{s,t}$. Let

$$m_{s,t} = m_{s,t}(n) = \min M_{s,t},$$

and let

$$R_3(n,s,t) = (4s - 4t - 4m_{s,t} + e(n))^{1/2} \in \mathbb{N}.$$

Define

$$n^+_{s,t} = \frac{1}{2}\left(n + R_3(n,s,t)\right) \quad \text{and} \quad n^-_{s,t} = \frac{1}{2}\left(n - R_3(n,s,t)\right).$$

Let $B_{s,t}(n)$ be the complete bipartite graph on n vertices with two parts V_1 and V_2 such that $|V_1| = n^+_{s,t}$ and $|V_2| = n^-_{s,t}$.

Let $\mathcal{BM}_{s,t}(n)$ consist of all graphs obtained from $B_{s,t}(n)$ as follows: take distinct vertices $u_1, \dots, u_s, v_1, \dots, v_s$ in V_1, add the edges $u_1v_1, \dots, u_sv_s$ and remove $m_{s,t}$ distinct edges $e_1, \dots, e_{m_{s,t}}$ such that every e_i has one endpoint in $\{u_1, \dots, u_s, v_1, \dots, v_s\}$ and the other endpoint in V_2, and there is no triangle with three edges in $\{e_1, \dots, e_{m_{s,t}}, u_1v_1, \dots, u_sv_s\}$ (see Figure 1 (a) and (b)).

Let $\mathcal{BS}_{s,t}(n)$ consists of all graphs obtained from $B_{s,t}(n)$ as follows: take distinct vertices $u'_1, \dots, u'_{s-1}, v'_1, \dots, v'_{s-1}$ in V_1 and distinct vertices u'_s, v'_s in V_2, add the edges $u'_1v'_1, \dots, u'_sv'_s$ and remove $m_{s,t}$ distinct edges $e'_1, \dots, e'_{m_{s,t}}$ such that every e'_i has one endpoint in $\{u'_1, \dots, u'_{s-1}, v'_1, \dots, v'_{s-1}\}$ and the other endpoint in $\{u'_s, v'_s\}$ and there is no triangle with three edges in $\{e'_1, \dots, e'_{m_{s,t}}, u'_1v'_1, \dots, u'_sv'_s\}$ (see Figure 1 (c) and (d)).

We abuse notation by letting $BM_{s,t}(n)$ and $BS_{s,t}(n)$ denote a generic member of $\mathcal{BM}_{s,t}(n)$ and $\mathcal{BS}_{s,t}(n)$ respectively.

Remark. To compare with the original Erdős-Rademacher problem, i.e. without the $\tau_3(G) \geq s$ constraint, recall that the extremal graphs for that problem are graphs obtained from the balanced complete bipartite graph by adding a triangle-free graph with t edges into the larger part.

Fact 1.3. *The following holds.*

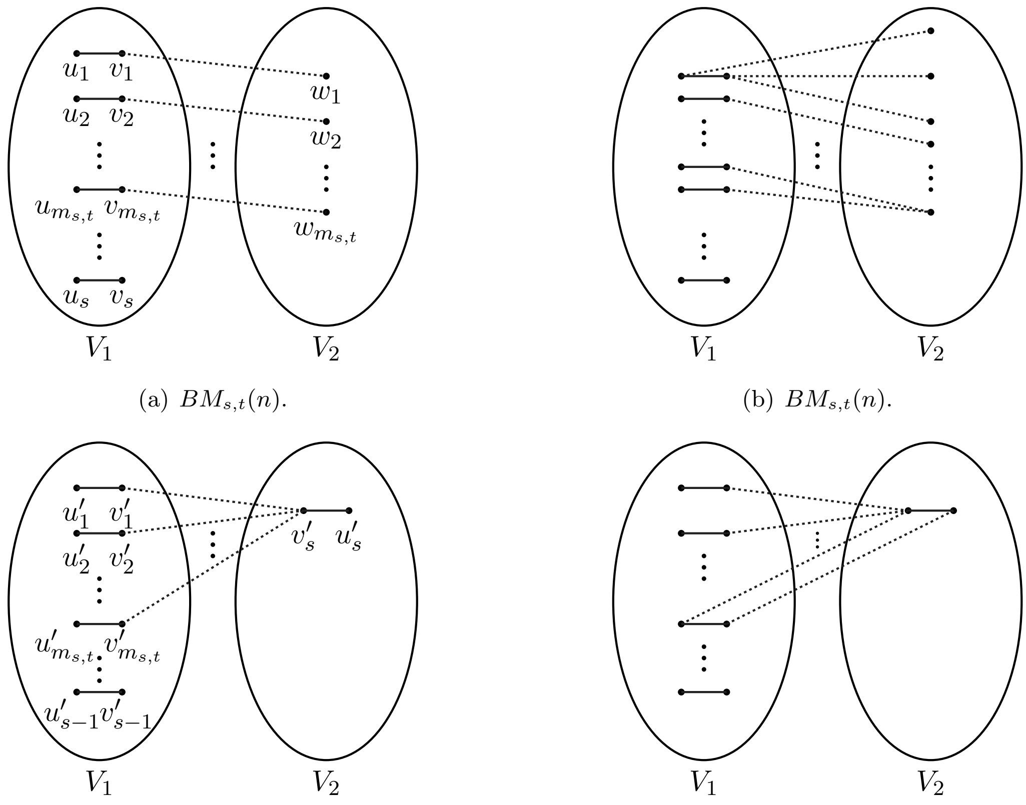
- $e\left(BM_{s,t}(n)\right) = e\left(BS_{s,t}(n)\right) = t_2(n) + t.$
- $\tau_3\left(BM_{s,t}(n)\right) = \tau_3\left(BS_{s,t}(n)\right) = s.$
- $N_3\left(BM_{s,t}(n)\right) = s \cdot n^-_{s,t} - m_{s,t}.$
- $N_3\left(BS_{s,t}(n)\right) = (s-1)n^-_{s,t} + n^+_{s,t} - 2m_{s,t} = s \cdot n^-_{s,t} - m_{s,t} + (n^+_{s,t} - n^-_{s,t} - m_{s,t}).$

By Lemma 2.11, if for some $p \in \mathbb{N}$

$$s - t = \begin{cases} p^2 - 1, & \text{if } n \text{ is even,} \\ p(p+1) - 1, & \text{if } n \text{ is odd,} \end{cases}$$

then $N_3\left(BM_{s,t}(n)\right) = N_3\left(BS_{s,t}(n)\right) = s \cdot n^-_{s,t} - m_{s,t}$.

Our first result shows that $BM_{s,t}(n)$ (and also $BS_{s,t}(n)$ for some special values of s, t) contains the least number of copies of K_3 among all n-vertex graphs with $t_2(n) + t$ edges and K_3-covering number at least s.



(a) $BM_{s,t}(n)$.

(b) $BM_{s,t}(n)$.

(c) $BS_{s,t}(n)$.

(d) $BS_{s,t}(n)$.

Figure 1: Several examples of graphs in $\mathcal{BM}_{s,t}(n)$ and $\mathcal{BS}_{s,t}(n)$.

Theorem 1.4. *Let $s > t \geq 1$. Then there exists $n_0 = n_0(s,t)$ such that the following holds for all $n \geq n_0$. Let G be a graph on n vertices with $t_2(n) + t$ edges. If $\tau_3(G) = s$, then*

$$N_3(G) \geq s \cdot n_{s,t}^{-} - m_{s,t}$$

Moreover, equality holds only if $G \cong BM_{s,t}(n)$ or $G \cong BS_{s,t}(n)$ except when $(s,t) \in \{(2,1),(3,1),(4,1)\}$ and n is even, or $(s,t) \in \{(3,2),(4,1),(5,1),(6,1)\}$ and n is odd. For these exceptional cases there are other examples showing that the bound is best possible.

Note that Theorem 1.4 shows that Conjecture 1.1 is not true in general. For example, let n be even, $(s-t)^{1/2} \in \mathbb{N}$ and $s - t > 4$. Then

$$N_3\left(BM_{s,t}(n)\right) = s \cdot n_{s,t}^{-} - m_{s,t} = s \cdot n_{s,t}^{-} = \frac{sn}{2} - (s-t)^{1/2}s,$$

which is strictly less that $sn/2 - 2(s-t)$.

1.2 k-cliques for $s = t + 1$

Let $V_1 \cup \cdots \cup V_{k-1}$ be a partition of $[n]$ with $|V_1| \geq \cdots \geq |V_{k-1}|$. Let $K[V_1, \ldots, V_{k-1}]$ be the complete $(k-1)$-partite graph on $[n]$ with parts $V_1, \ldots, V_{k-1}$. If $V_1 \cup \cdots \cup V_{k-1}$ is a balanced partition, then $K[V_1, \ldots, V_{k-1}]$ is called the Turán graph $T_{k-1}(n)$. Notice that $t_{k-1}(n) = |T_{k-1}(n)|$. The celebrated Turán theorem [9] states that the maximum number of edges of an n-vertex K_k-free graph is uniquely achieved by $T_{k-1}(n)$.

For $s > m \geq 0$ and $\vec{x} = (x_1, \ldots, x_{k-1}) \in \mathbb{N}^{k-1}$ with $\sum_{i=1}^{k-1} x_i = n$ let $V_1 \cup \cdots \cup V_{k-1}$ be a partition of $[n]$ with $|V_i| = x_i$ for $i \in [k-1]$. Let $\mathcal{KM}_{m,s}(\vec{x})$ consist of all graphs that are obtained from $K[V_1, \ldots, V_{k-1}]$ as follows: take distinct vertices $u_1, \ldots, u_s$, $v_1, \ldots, v_s$ in V_1, add the edges $u_1v_1, \ldots, u_sv_s$ and remove m distinct edges $e_1, \ldots, e_m$ such that every e_i contains one vertex from $\{u_1, \ldots, u_s, v_1, \ldots, v_s\}$ and one vertex from V_{k-1} and there is no triangle with edges in $\{e_1, \ldots, e_m, u_1v_1, \ldots, u_sv_s\}$. We abuse notation by letting $KM_{s,t}(\vec{x})$ denote a generic member in $\mathcal{KM}_{m,s}(\vec{x})$. It is easy to see that

$$e\left(KM_{m,s}(\vec{x})\right) = \sum_{1 \leq i < j < k} x_i x_j + s - m \quad \text{and} \quad N_k\left(KM_{m,s}(\vec{x})\right) = s\prod_{i=2}^{k-1} x_i - m\prod_{i=2}^{k-2} x_i.$$

Let us now consider some special cases of $KM_{m,s}(\vec{x})$ in more detail.

For $n \in \mathbb{N}$, write

$$n = q_{n,k}(k-1) + r_{n,k} \qquad \text{where} \qquad 0 \leq r_{n,k} < k-1.$$

Writing $r = r_{n,k}$ and $q = q_{n,k}$, let $\vec{y}_r \in \mathbb{N}^{k-1}$ be defined as follows:

$$\vec{y}_r = \begin{cases} (q+1, q, \ldots, q, q-1) & \text{if } r = 0 \\ (q+1, q, \ldots, q) & \text{if } r = 1 \\ (q+2, \underbrace{q+1, \ldots, q+1}_{r-2 \text{ times}}, \underbrace{q, \ldots, q}_{k-r \text{ times}}) & \text{if } r \geq 2. \end{cases}$$

Define

$$N_k(n,s) = \begin{cases} s \cdot q^{k-3}\,(q-1) & \text{if } r = 0, \\ s \cdot q^{k-2} - q^{k-3} & \text{if } r = 1, \\ s \cdot (q+1)^{r-2}\, q^{k-r} & \text{if } r \geq 2. \end{cases}$$

Observe that

$$e\left(KM_{0,s}(\vec{y}_r)\right) = e\left(KM_{1,s}(\vec{y}_1)\right) = t_{k-1}(n) + s - 1 \quad \text{ for } \; r \neq 1$$

and

$$N_k\left(KM_{m,s}(\vec{y}_r)\right) = N_k(n,s)$$

for $m = 0, r \neq 1$ and $m = 1, r = 1$.

Our next result shows that the constructions defined above contain the least number of copies of K_k in an n-vertex graph G with $t_{k-1}(n) + s - 1$ edges and $\tau_k(G) = s$.

Theorem 1.5. *Let $k \geq 4$ and $s \geq 2$ be fixed integers. Then there exists $n_1 = n_1(k,s)$ such that the following holds for all $n \geq n_1$. Let G be a graph on n vertices with $t_{k-1}(n) + s - 1$ edges. If $\tau_k(G) = s$, then $N_k(G) \geq N_k(n,s)$. Moreover, for $s \geq 3$ equality holds iff $G \cong KM_{0,s}(\vec{y}_{r_{n,k}})$ if $r_{n,k} \neq 1$ and $G \cong KM_{1,s}(\vec{y}_1)$ if $r_{n,k} = 1$.*

For $s \geq 2$, the following construction which was defined in [10] also achieves the bound $N_k(n,2)$. Let $V_1 \cup \cdots \cup V_{k-1}$ be a balanced partition of $[n]$ with $|V_1| \geq \cdots \geq |V_{k-1}|$. Let $T_{k-1}^{\sqsubset}$ be obtained from $K[V_1, \ldots, V_{k-1}]$ as follows: take two distinct vertices $u_1, v_1 \in V_1$ and two distinct vertices $u_2, v_2 \in V_2$, add edges u_1v_1, u_2v_2 and remove the edge v_1v_2. One can easily check that $N_k(T_{k-1}^{\sqsubset}) = (|V_1| + |V_2| - 2)\prod_{i=3}^{k-1}|V_i| = N_k(n,s)$. Therefore, Theorem 1.5 shows that Conjecture 1.2 is true for large n.

1.3 k-cliques for large s

Recall that for given n and k, $q_{n,k} = \lfloor n/(k-1) \rfloor$ and $r_{n,k} = n - (k-1)q_{n,k}$. Given $s > t \geq 1$ and $k \geq 3$, let

$$R_k(n,s,t) = \left(\frac{2(k-1)(s-t) + (k-1-r_{n,k})r_{n,k}}{k-2} \right)^{1/2}.$$

We note that while $R_k(n,s,t)$ depends on n it is bounded from above by a function of only k, s, t. Let

$$n^+_{k,s,t} = \frac{n + (k-2)R_k(n,s,t)}{k-1} \quad \text{and} \quad n^-_{k,s,t} = \frac{n - R_k(n,s,t)}{k-1}.$$

Suppose that $n^-_{k,s,t} \in \mathbb{N}$. Then let $V_1 \cup \cdots \cup V_{k-1}$ be a partition of $[n]$ with $|V_1| = n^+_{k,s,t}$ and $|V_i| = n^-_{k,s,t}$ for $2 \leq i \leq k-1$. Let $KM(n,k,s,t)$ be obtained from $K[V_1, \ldots, V_{k-1}]$ by taking distinct vertices $u_1, \ldots, u_s, v_1, \ldots, v_s$ in V_1 and then adding $u_1v_1, \ldots, u_sv_s$. Using Lemma 2.2 one can easily check that

$$e\left(KM(n,k,s,t)\right) = t_{k-1}(n) + t \quad \text{and} \quad N_k\left(KM(n,k,s,t)\right) = s \cdot (n^-_{k,s,t})^{k-2}.$$

The following result shows that if s is large, then $KM(n,k,s,t)$ minimizes the number of copies of K_k among all n-vertex graphs G with $t_{k-1}(n) + t$ edges and $\tau_k(G) = s$.

Theorem 1.6. *Let $s > t \geq 1$ and $k \geq 4$ be fixed integers. There exists $n_2 = n_2(k,s,t)$ such that the following holds for all $n \geq n_2$ and $s > 2R_k(n,s,t)$. If G is a graph on n vertices with $t_{k-1}(n) + t$ edges and $\tau_k(G) = s$, then*

$$N_k(G) \geq s \cdot (n^-_{k,s,t})^{k-2}.$$

Moreover, if $n^-_{k,s,t} \in \mathbb{N}$, then equality holds iff $G \cong KM(n,k,s,t)$.

Note that we are not able to determine the exact minimum value of $N_k(G)$ for small s because, similar to the situation in Theorem 1.4, when s is small there could be many constructions that achieve the minimum value of $N_k(G)$. On the other hand, for the case $n^-_{k,s,t} \notin \mathbb{N}$ our bound might be not tight and actually, we think there might be a better bound for $N_k(G)$ in this case.

Let $R_k(s,t) = (2(k-1)(s-t)/(k-2))^{1/2}$. If $r_{n,k} = 0$, then $R_k(n,s,t) = R_k(s,t)$. Since $k \geq 4$ and $t \geq 1$, $s > 2R_k(s,t)$ holds for all $s \geq 11$. Therefore, Theorem 1.6 gives the following corollary.

Corollary 1.7. *Let $s > t \geq 1$ and $k \geq 4$ be fixed integers. Suppose that $s \geq 11$. Then there exists $n_3 = n_3(k,s,t)$ such that the following holds for all $n \geq n_3$ and $n \equiv 0 \mod k-1$. If G is a graph on n vertices with $t_{k-1}(n) + t$ edges and $\tau_k(G) = s$, then $N_k(G) \geq s \cdot (n^-_{k,s,t})^{k-2}$. Moreover, if $n^-_{k,s,t} \in \mathbb{N}$, then equality holds iff $G \cong KM(n,k,s,t)$.*

After this work was done we found that similar results as in Theorems 1.4, 1.5, and 1.6 were recently proved by Balogh and Clemen [1].

1.4 Color critical graphs

Given a graph G let $\chi(G)$ denote the chromatic number of G. Let H be a subgraph of G. Then the graph $G - H$ is obtained from G by removing all edges that are contained in H. In particular, if $e \in E(G)$, then $G - e$ is obtained from G by removing e.

Definition 1.8. *Let $k \geq 3$. A graph F is k-critical if $\chi(F) = k$ and there exists $e \in E(F)$ such that $\chi(F - e) < k$.*

Let $k \geq 3$ and let F be a k-critical graph. Let $c(n, F)$ denote the minimum number of copies of F in the graph obtained from $T_{k-1}(n)$ by adding one edge. The number $c(n, F)$ can be calculated using a formula in [7] and in particular there exists a constant $\alpha_F > 0$ depending only on F such that $c(n, F) = \alpha_F n^{f-2} + \Theta(n^{f-3})$.

The second author proved [7] that for any k-critical graph F there exists a constant $\delta = \delta_F > 0$ such that for every $1 \leq t \leq \delta n$ every n-vertex graph G with $t_{k-1}(n) + t$ edges contains at least $t \cdot c(n, F)$ copies of F. We prove the analogous theorem for $\tau_F(G) = s$.

Theorem 1.9. *Let $s > t \geq 1$ and $k \geq 3$ be fixed integers. Let F be a fixed k-critical graph on f vertices. Then there exists constants $C = C(F, s, t)$ and $n_4 = n_4(F, s, t)$ such that the following holds for all $n \geq n_4$. If G is a graph on n vertices with $t_{k-1}(n) + t$ edges and $\tau_k(G) = s$, then $N_F(G) \geq s \cdot c(n, F) - Cn^{f-3}$.*

Remark. For graphs that are not color critical it remains open in general to determine even their Turán numbers exactly. Therefore, one could expect that a Erdős-Rademacher-type result (or result as Theorem 1.9) for these graphs can be very hard in general.

This bound is tight up to an error term since the graph obtained from $T_{k-1}(n)$ by adding s pairwise disjoint edges into one part of $T_{k-1}(n)$ contains at most $s \cdot c(n, F) + C'n^{f-3}$ copies of F for some constant $C' > 0$.

2 Proofs

2.1 Lemmas

In this section we prove several lemmas that will be used in our proofs.

Definition 2.1. *Let $k \geq 3$ and let F be a k-critical graph. Let $c(x_1, \ldots, x_{k-1}, F)$ be the number of copies of F in the graph obtained from the complete $(k-1)$-partite graph with parts of sizes $x_1, \ldots, x_{k-1}$ by adding one edge to the part of size x_1.*

The following explicit expression for $t_{k-1}(n)$ is very useful in our calculations.

Lemma 2.2 (e.g. see [5])**.** *Let $k \geq 3$ and suppose that $n \equiv r \mod (k-1)$ for some $0 \leq r \leq k-2$. Then*

$$t_{k-1}(n) = \frac{(k-2)}{2(k-1)}n^2 - \frac{(k-1-r)r}{2(k-1)}.$$

The following lemma gives a relation between $c(x_1, \ldots, x_{k-1}, F)$ and $c(n, F)$.

Lemma 2.3 ([7])**.** *Let $k \geq 3$ and F be a k-critical graph. Then there exists a constant $\gamma_F > 0$ depending only on F such that the following holds for all sufficiently large n. If $\sum_{i=1}^{k-1} x_i = n$ and $\lfloor n/(k-1) \rfloor - d \leq x_i \leq \lceil n/(k-1) \rceil + d$ for all $i \in [k-1]$and $d \leq \frac{n}{3(k-1)}$, then*

$$c(x_1, \ldots, x_{k-1}, F) \geq c(n, F) - \gamma_F d n^{f-3}.$$

The following lemma, which can be found in several places (e.g. see [7]), gives a bound on the size of each part for a $(k-1)$-partite graph whose number of edges is close to $t_{k-1}(n)$.

Lemma 2.4 (e.g. see [7])**.** *Suppose that $k \geq 3$ is fixed, n is sufficiently large, $d < n$ and $\sum_{i=1}^{k-1} x_i = n$. If*

$$\sum_{1 \leq i < j \leq k-1} x_i x_j \geq t_{k-1}(n) - d,$$

then $\lfloor n/(k-1) \rfloor - d \leq x_i \leq \lceil n/(k-1) \rceil + d$ for all $i \in [k-1]$.

The following two results will be key in our proofs.

Theorem 2.5 (Graph removal lemma, e.g. see [4])**.** *Let F be a graph with f vertices. Suppose that G is a graph on n vertices with $N_F(G) = o(n^f)$. Then one can remove $o(n^2)$ edges from G such that the resulting graph is F-free.*

Theorem 2.6 (Erdős-Simonovits stability theorem [8])**.** *Let $k \geq 3$ and F be a k-critical graph. Suppose that G is an F-free graph on n vertices with $t_{k-1}(n) - o(n^2)$ edges. Then G can be made $(k-1)$-partite by removing $o(n^2)$ edges.*

Now we use the results above to obtain a rough structure of a graph with a fixed number of vertices and edges and a fixed F-covering number that contains not many copies of F.

Given a graph G and $v \in V(G)$ we use $N_G(v)$ to denote the neighbors of v in G and let $d_G(v) = |N_G(v)|$. For a partition $V_1 \cup \cdots \cup V_{k-1}$ of $V(G)$ we use $G[V_1, \ldots, V_{k-1}]$ to denote the induced $(k-1)$-partite subgraph of G on $V_1 \cup \cdots \cup V_{k-1}$. We use $B_G(V_1, \ldots, V_{k-1})$ to denote the set of edges in G that are contained inside V_i for some $i \in [k-1]$, i.e. $B_G(V_1, \ldots, V_{k-1}) = G - G[V_1, \ldots, V_{k-1}]$. We use $M_G(V_1, \ldots, V_{k-1})$ to denote the set of pairs which intersect two parts that are not edges in G, i.e. $M_G(V_1, \ldots, V_{k-1}) = K[V_1, \ldots, V_{k-1}] - G[V_1, \ldots, V_{k-1}]$. If it is clear from the context we will use B and M to represent $B_G(V_1, \ldots, V_{k-1})$ and $M_G(V_1, \ldots, V_{k-1})$, respectively.

For a k-critical graph F a potential copy of F in G (with respect to the partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$) is a copy of F in $G \cup M$ that uses exactly one edge of B (so every other edge is between parts).

Lemma 2.7. *Let $s \geq 1, f \geq k \geq 3$ be fixed integers and F be a fixed k-critical graph on f vertices. Then the following holds for sufficiently large n. If G is a graph on n vertices with at least $t_{k-1}(n) + 1$ edges and $N_F(G) \leq (s + 1/2) \cdot c(n, F)$, then G contains a $(k-1)$-partite subgraph H such that $e(H) \geq e(G) - s$.*

Proof. Let $\delta_1, \delta_2, \delta_3, \delta_4, \epsilon, \epsilon_1, \epsilon_2$ be constants such that

$$0 < \delta_1 \ll \delta_2 \ll \delta_3 \ll \delta_4 \ll \epsilon_2 \ll \epsilon_1 \ll \epsilon \ll s^{-1}.$$

Let n be sufficiently large and in particular $n \gg s/\epsilon_2$.

Since $N_F(G) \le (s+1/2)\cdot c(n,F) < 2s\alpha_F n^{f-2} = o(n^f)$, by the Graph removal lemma, we can remove at most $\delta_1 n^2$ edges from G such that the resulting graph G_1 is F-free. Since $e(G_1) \ge e(G) - \delta_1 n^2 > t_{k-1}(n) - \delta_1 n^2$, by the Erdős-Simonovits stability theorem, G_1 contains a $(k-1)$-partite subgraph G_2 such that $e(G_2) \ge t_{k-1}(n) - \delta_2 n^2$.

Now let H be a $(k-1)$-partite subgraph of G with the maximum number of edges. Then by the previous argument, $e(H) \ge e(G_2) \ge t_{k-1}(n) - \delta_2 n^2$. Let $V_1 \cup \cdots \cup V_{k-1}$ be a partition of $V(G)$ such that $H = G[V_1, \ldots, V_{k-1}]$ and let $x_i = |V_i|$ for $i \in [k-1]$. An easy calculation shows that $|x_i - n/(k-1)| \le \delta_3 n$ for all $i \in [k-1]$.

Suppose that $|H| = t_{k-1}(n) - \ell$ for some $\ell \ge 0$. Then $|M| \le \ell$ and $|B| \ge \ell + 1$. For every $e \in B$ let $F(e)$ denote the number of copies of F in G containing the unique edge e from B. Let

$$B_1 = \{e \in B : F(e) > (1-\epsilon)c(n,F)\}$$

and $B_2 = B \setminus B_1$.

Claim 2.8. $|B_1| \ge (1-\epsilon)|B|$.

Proof of Claim 2.8. Suppose that $|B_2| \ge \epsilon |B|$. Let $e \in B_2$ and without loss of generality we may assume that $e \subset V_1$. Then by Lemma 2.3 the number of potential copies of F containing e is

$$c(x_1, \ldots, x_{k-1}, F) \ge c(n,F) - \gamma_F(\delta_3 n) n^{f-3} > (1-\delta_4)c(n,F).$$

At least $\epsilon \cdot c(n,F)/2$ of these potential copies of F have a pair from M, since otherwise

$$F(e) \ge (1-\delta_4)c(n,F) - \frac{\epsilon}{2}c(n,F) > (1-\epsilon)c(n,F),$$

a contradiction. Now suppose that at least $\epsilon \cdot c(n,F)/4$ of these potential copies of F have a pair from M that does not intersect e. For every $e' \in M$ with $e \cap e' = \emptyset$ the number of potential copies of F in G that contains both e and e' is at most n^{f-4}. On the other hand, every potential copy of F contains at most f^2 pairs from M. Therefore,

$$\frac{\epsilon}{4}c(n,F) \ge |M| f^2 n^{f-4},$$

which implies that

$$\delta_2 n^2 \ge |M| \ge \frac{\frac{\epsilon}{4}c(n,F)}{f^2 n^{f-4}} > \frac{\epsilon \alpha_F}{8f^2} n^2,$$

a contradiction. Here we used $|M| \le t_{k-1}(n) - e(H) \le \delta_2 n^2$. Therefore, we may assume that at least $\epsilon \cdot c(n,F)/4$ of these potential copies of F have a pair from M which has nonempty intersection with e. Similarly, since every $e'' \in M$ with $e'' \cap e \ne \emptyset$ is contained in at most n^{f-3} members in $F(e)$ and every potential copy of F contains at most f^2 pairs from M, the number of pairs from M that has nonempty intersection with e is at least

$$\frac{\frac{\epsilon}{4}c(n,F)}{f^2 n^{k-3}} \ge \frac{\epsilon \alpha_F}{8f^2} n.$$

Therefore, there exists $x \in e$ such that $d_M(x) \ge \frac{\epsilon \alpha_F}{16 f^2} n$.

Let $A = \left\{v \in V(G) : d_M(v) \geq \frac{\epsilon \alpha_F}{16f^2} n\right\}$. Since every $e \in B$ contains a vertex in A,

$$\sum_{v \in A} d_{B_2}(v) \geq |B_2| \geq \epsilon|B| \geq \epsilon|M| \geq \frac{\epsilon}{2} \sum_{v \in A} d_M(v) \geq \frac{\epsilon^2 \alpha_F}{32f^2} n|A|.$$

Therefore, there exists $v \in A$ such that $d_{B_2}(v) \geq \frac{\epsilon^2 \alpha_F}{32f^2} n$ and without loss of generality we may assume that $v \in V_1$. Let $V_i' = N_G(v) \cap V_i$ for $i \in [k-1]$. Then by the maximality of H we have $|V_i'| \geq |V_1'| \geq \frac{\epsilon^2 \alpha_F}{32f^2} n$ for all $2 \leq i \leq k-1$. Let $u \in V_1'$. Then by Lemma 2.3, the number of potential copies of F containing uv in the complete $(k-1)$-partite graph $K[V_1', \ldots, V_{k-1}']$ is at least

$$c\left(|V_1'|, \ldots, |V_{k-1}'|, F\right) \geq \frac{1}{2} \alpha_F \left(\frac{\epsilon^2 \alpha_F}{32f^2} n\right)^{k-2} \geq \epsilon_1 n^{k-2}.$$

Summing over all $u \in V_1'$, there are at least

$$\frac{\epsilon^2 \alpha_F}{32f^2} n \times \epsilon_1 n^{f-2} \geq \epsilon_2 n^{f-1} \geq 3s \cdot c(n, F)$$

potential copies of F containing v. By the assumption that $N_F(G) \leq (s+1/2) \cdot c(n,F)$, at least half of these potential copies of F must contain a pair from M, and this pair cannot be incident with v, since v is adjacent to all vertices in $\bigcup_{i=1}^{k-1} V_i'$. Since the number of potential copies of F that contain both v and a pair from M that is disjoint from v is at most n^{f-3} and each potential copy of F contains at most f^2 pairs from M, we obtain

$$\delta_2 n^2 \geq |M| \geq \frac{\epsilon_2 n^{f-1}/2}{f^2 n^{f-3}} \geq \frac{\epsilon_2}{2f^2} n^2,$$

a contradiction. ∎

Claim 2.9. *$|B| \leq s$.*

Proof of Claim 2.9. Suppose that $|B| \geq s+1$. Then by Claim 2.8,

$$\begin{aligned} N_F(G) \geq \sum_{e \in B_1} F(e) \geq \sum_{e \in B_1} (1-\epsilon)c(n,F) &\geq (1-\epsilon)^2 |B| c(n,F) \\ &\geq (1-\epsilon)^2 (s+1) c(n,F) > (s+1/2) \cdot c(n,F), \end{aligned}$$

a contradiction. ∎

Therefore, by Claim 2.9, $e(H) = e(G) - |B| \geq e(G) - s$. This completes the proof of Lemma 2.7. ∎

Now we use Lemma 2.7 to obtain a fine structure for graphs with a fixed F-covering number and not many copies of F.

Lemma 2.10. *Let $f \geq k \geq 3, s > t \geq 1$ be fixed integers and F be a fixed k-critical graph on f vertices. Then the following holds for sufficiently large n. Let G be a graph on n vertices with $t_{k-1}(n) + t$ edges. If $\tau_F(G) = s$ and $N_F(G) \leq (s+1/2) \cdot c(n,F)$, then there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $G - G[V_1, \ldots, V_{k-1}]$ is a matching with s edges.*

Proof. Let H be a $(k-1)$-partite subgraph of G with the maximum number of edges and let $B = G - H$. Since $N_F(G) \le (s+1/2) \cdot c(n, F)$, by Lemma 2.7, $|B| \le s$. So it suffices to show that $|B| \ge s$ and B is a matching.

Let $\tau(B) = \min\{|S| \colon S \subset V(G), e \cap S \ne \emptyset \text{ for all } e \in B\}$. Since every copy of F in G must contain at least one edge in B, $\tau_F(G) \le \tau(B)$. Therefore, $\tau(B) \ge s$. Since $|B| \le s$, the only possibility is that B is a matching of size s. ∎

2.2 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Recall that for $s > t \ge 1$ and $n \in \mathbb{N}$

$$n_{s,t}^{+} = \frac{1}{2}\left(n + R_3(n,s,t)\right) \quad \text{and} \quad n_{s,t}^{-} = \frac{1}{2}\left(n - R_3(n,s,t)\right),$$

where $R_3(n,s,t) = \left(4s - 4t - 4m_{s,t} + n^2 - 4t_2(n)\right)^{1/2}$ and

$$m_{s,t} = \min\left\{m \in \mathbb{N} : \left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2} \in \mathbb{N}\right\}.$$

We will use the following lemma in our proof.

Lemma 2.11. *Let $s > t \ge 1$ and $n \in \mathbb{N}$. Then*

$$n_{s,t}^{+} - n_{s,t}^{-} - m_{s,t} = \begin{cases} 0 & \text{if } n \text{ is even and } s - t = p^2 - 1 \text{ for some } p \in \mathbb{N}, \\ 0 & \text{if } n \text{ is odd and } s - t = p(p+1) - 1 \text{ for some } p \in \mathbb{N}, \\ > 0 & \text{otherwise.} \end{cases}$$

Proof. First, notice that $n_{s,t}^{+} - n_{s,t}^{-} - m_{s,t} = \left(4s - 4t - 4m_{s,t} + n^2 - 4t_2(n)\right)^{1/2} - m_{s,t}$.

If n is even, then $n^2 - 4t_2(n) = 0$. Let $p \in \mathbb{N}$ be the largest integer such that $s - t = p^2 + q$ for some $q \in \mathbb{N}$. Note that $q \le 2p$ since otherwise we would have $p^2 + q \ge (p+1)^2$, a contradiction. Then $m_{s,t} = q$ and hence

$$\left(4s - 4t - 4m_{s,t} + n^2 - 4t_2(n)\right)^{1/2} - m_{s,t} = 2p - m_{s,t} \ge 0$$

and equality holds iff $q = 2p$.

If n is odd, then $n^2 - 4t_2(n) = 1$. Let $p \in \mathbb{N}$ be the largest integer such that $s - t = p(p+1) + q$ for some $q \in \mathbb{N}$. Note that $q \le 2p + 1$ since otherwise we would have $p(p+1) + q \ge (p+1)(p+2)$, a contradiction. Then $m_{s,t} = q$ and hence

$$\left(4s - 4t - 4m_{s,t} + n^2 - 4t_2(n)\right)^{1/2} - m_{s,t} = 2p + 1 - m_{s,t} \ge 0$$

and equality holds iff $q = 2p + 1$. ∎

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $s > t \ge 1$ be fixed and let n be sufficiently large. Let G be a graph on n vertices with $t_2(n) + t$ edges and $\tau_3(G) = s$. Since $s \cdot n_{s,t}^{-} - m_{s,t} < (s+1/2) \cdot c(n, K_3)$, we may assume that $N_3(G) \le (s+1/2) \cdot c(n, K_3)$. So, by Lemma 2.10, there exists a partition $V(G) = V_1 \cup V_2$ such that $B := G - G[V_1, V_2]$ is a matching of size s.

Let $x = |V_1|$ and $y = |V_2|$ and note that $x + y = n$. Without loss of generality we may assume that $x \geq y$. Let $H = G[V_1, V_2]$, $M = K[V_1, V_2] - H$, and $m = |M|$. Since $G - B = H = K[V_1, V_2] - M$, we obtain $t_2(n) + t - s = xy - m = (n-y)y - m$. Therefore, $m \in M_{s,t}$ and

$$y = \frac{1}{2}\left(n - \left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2}\right).$$

Let $s_i = |B \cap \binom{V_i}{2}|$ for $i = 1, 2$ and note that $s_1 + s_2 = s$. It is easy to see that the number of potential copies of K_3 is $s_1 y + s_2 x$. We will consider two cases: either $s_i = s$ for some $i \in \{1, 2\}$ or $s_1 \geq 1$ and $s_2 \geq 1$.

Case 1: $s_i = s$ for some $i \in \{1, 2\}$.

We may assume that $s_2 = 0$ and the case $s_1 = 0$ can be solved using a similar argument. Notice that for every $e \in M$ there is at most one potential copy of K_3 containing e. Therefore,

$$N_3(G) \geq sy - m = \frac{sn}{2} - \frac{s}{2}\left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2} - m =: f(m).$$

Then

$$\frac{\mathrm{d}f(m)}{\mathrm{d}m} = \frac{s}{\left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2}} - 1.$$

First let us assume that $s \geq 3$. Then

$$s^2 \geq 4s - 4 + 1 \geq 4s - 4t - 4m + n^2 - 4t_2(n).$$

Therefore, $\frac{\mathrm{d}f(m)}{\mathrm{d}m} > 0$ for all $m > 0$, which implies that $f(m)$ is increasing in m. Therefore, for $s \geq 3$

$$N_3(G) \geq \frac{sn}{2} - \frac{s}{2}\left(4s - 4t - 4m_{s,t} + n^2 - 4t_2(n)\right)^{1/2} - m_{s,t} = s \cdot n_{s,t}^- - m_{s,t}.$$

For the case $s = 2$, one could easily check that the minimum of $f(m)$ is uniquely attained at $m = m_{s,t}$. Therefore, if $s_i = s$ for some $i \in \{1, 2\}$, then $N_3(G) \geq s \cdot n_{s,t}^- - m_{s,t}$ for all $s > t \geq 1$.

If $N_3(G) = s \cdot n_{s,t}^- - m_{s,t}$, then the argument above shows that we must have $|V_1| = n - n_{s,t}^- = n_{s,t}^+$ and $|V_2| = n_{s,t}^-$, all edges in B are contained in V_1, all pairs in M must be contained in one potential copy of K_3, and no two pairs in the same potential copy. Therefore, $G \cong BM_{s,t}(n)$.

Case 2: $s_1 \geq 1$ and $s_2 \geq 1$.

Notice that for every $e \in M$ there are at most two potential copies of K_3 containing e. Since $x \geq y$, this gives

$$\begin{aligned}
N_3(G) \geq s_1 y + s_2 x - 2m &\geq (s-1)y + x - 2m \\
&= (s-2)y + n - 2m \\
&= \frac{sn}{2} - \frac{s-2}{2}\left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2} - 2m =: g(m).
\end{aligned}$$

Let us first assume that $s \geq 20$. Since

$$\frac{\mathrm{d}g(m)}{\mathrm{d}m} = \frac{s-2}{\left(4s - 4t - 4m + n^2 - 4t_2(n)\right)^{1/2}} - 2.$$

and

$$s-2 > 2\,(4s-4+1)^{1/2} \geq 2\left(4s-4t-4m+n^2-4t_2(n)\right)^{1/2},$$

$\frac{\mathrm{d}g(m)}{\mathrm{d}m} > 0$ for $m > 0$. Therefore, by Lemma 2.11,

$$N_3(G) \geq g(m_{s,t}) = f(m_{s,t}) + \left(4s-4t-4m_{s,t}+n^2-4t_2(n)\right)^{1/2} - m_{s,t} \geq f(m_{s,t}),$$

and equality holds iff for some $p \in \mathbb{N}$

$$s-t = \begin{cases} p^2-1, & \text{if } n \equiv 0 \mod 2, \\ p(p+1)-1, & \text{if } n \equiv 1 \mod 2. \end{cases} \tag{$\star$}$$

For $s \leq 19$ a computer-aided calculation shows that $f(m_{s,t}) \leq \min_m\{g(m)\}$ always holds[1]. Moreover, the minimum of $g(m)$ is uniquely achieved at $m = m_{s,t}$ except for when $(s,t) \in \{(2,1),(3,1),(4,1)\}$ and n even, or $(s,t) \in \{(3,2),(4,1),(5,1),(6,1)\}$ and n odd.

If $N_3(G) = s\cdot n^-_{s,t} - m_{s,t}$, then the argument above shows that ($\star$) holds, $|V_1| = n - n^-_{s,t} = n^+_{s,t}$ and $|V_2| = n^-_{s,t}$, exactly one edge $e \in B$ is contained in V_2, all other edges in B are contained in V_1, and all pairs in M must be contained in two potential copies of K_3. Therefore, $G \cong BS_{s,t}(n)$.

For $(s,t) \in \{(2,1),(3,1),(4,1)\}$ and n even, or $(s,t) \in \{(3,2),(4,1),(5,1),(6,1)\}$ and n odd, our bound $s \cdot n^-_{s,t} - m_{s,t}$ in Theorem 1.4 is also tight, but there are more constructions that achieve this bound. One could easily recover all these constructions using our calculation file. ∎

2.3 Proof of Theorem 1.5

In this section we prove Theorem 1.5. Recall that for $n, k \in \mathbb{N}$, $q_{n,k} = \lfloor n/(k-1) \rfloor$ and $r_{n,k} = n - (k-1)q_{n,k}$.

Proof of Theorem 1.5. Let $s \geq 2, k \geq 4$ be fixed integers and n be sufficiently large. Let $q = q_{n,k}$ and $r = r_{n,k}$. Let G be a graph on n vertices with $t_{k-1}(n)+s-1$ edges and $\tau_k(G) = s$. Notice that $N_k(n,s) = (1+o(1))s\left(\frac{n}{k-1}\right)^{k-2}$ while $c(n,K_k) = (1+o(1))\left(\frac{n}{k-1}\right)^{k-2}$, so $N_k(n,s) < (s+1/2)\cdot c(n,K_k)$. Therefore, we may assume that $N_k(G) \leq (s+1/2)\cdot c(n,K_k)$. So by Lemma 2.10, there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $B := G - G[V_1,\ldots,V_{k-1}]$ is a matching of size s.

Let $x_i = |V_i|$ for $i \in [k-1]$ and without loss of generality we may assume that $x_1 \geq \cdots \geq x_{k-1}$. Let $H = G[V_1,\ldots,V_{k-1}]$, $M = K[V_1,\ldots,V_{k-1}] - H$, and $m = |M|$. Since $t_{k-1}(n) - 1 = |H| = |K[V_1,\ldots,V_{k-1}]| - m$, we obtain $m \in \{0,1\}$ and

$$\sum_{1\leq i<j\leq k-1} x_i x_j = t_{k-1}(n) - 1 + m.$$

Suppose that $m = 1$. Then $\sum_{1\leq i<j\leq k-1} x_i x_j = t_{k-1}(n)$, so $x_1 = \cdots = x_r = q+1$ and $x_{r+1} = \cdots = x_{k-1} = q$.

[1] A simple Mathematica worksheet verifying this fact can be found at the web page `http://homepages.math.uic.edu/~mubayi/papers/ErdosRademacher.pdf`.

Let $s_i = |B \cap \binom{V_i}{2}|$ for $i \in [k-1]$ and $S = \{i \in [k-1] : s_i \geq 1\}$.

Case 1: $|S| = 1$.

Let $i_0 \in [k-1]$ such that $s_{i_0} = s$. Then there are $s \cdot \prod_{i \neq i_0} x_i$ potential copies of K_k. Let $uv \in M$. If uv has empty intersection with all edges in B, then there are at most $s \cdot n^{k-4} = o(n^{k-3})$ potential copies of K_k containing uv. If uv has nonempty intersection with some $e \in B$, then every potential copy of K_k that contains uv must contain e as well. So in this case there are at most $\left(\prod_{i \notin \{i_0\}} x_i\right)/x_{k-1}$ potential copies of K_k containing uv. Therefore,

$$\begin{aligned}
N_k(G) &\geq s \cdot \prod_{i \notin \{i_0\}} x_i - \frac{1}{x_{k-1}} \prod_{i \notin \{i_0\}} x_i \\
&\geq \left(s - \frac{1}{x_{k-1}}\right) \prod_{i=2}^{k-1} x_i = \begin{cases} \left(s - \frac{1}{q}\right) q^{k-2}, & \text{if } r \leq 1, \\ \left(s - \frac{1}{q}\right) (q+1)^{r-1} q^{k-r-1}, & \text{if } 2 \leq r \leq k-2, \end{cases} \\
&\geq N_k(n, s),
\end{aligned}$$

and equality holds only if $r = 1$.

Case 2: $|S| \geq 2$.

The number of potential copies of K_k is $\sum_{i=1}^{k-1} \left(s_i \cdot \prod_{j \neq i} x_j\right)$. Suppose that the pair in M has nonempty intersection with V_{i_0} and V_{i_1} for some $i_0, i_1 \in [k-1]$. If $s_{i_0} = 0$, then there are at most $\left(\prod_{i \neq i_0} x_i\right)/x_{k-1}$ potential copies of K_k containing the pair in M. If both $s_{i_0} \geq 1$ and $s_{i_1} \geq 1$, then there are most $2 \prod_{i \neq i_0, i_1} x_i$ potential copies of K_k containing the pair in M. Therefore,

$$\begin{aligned}
N_k(G) \geq \sum_{i=1}^{k-1} \left(s_i \cdot \prod_{j \neq i} x_j\right) - 2 \prod_{i \neq i_0, i_1} x_i &= \left(\sum_{i=1}^{k-1} \frac{s_i}{x_i} - \frac{2}{x_{i_0} x_{i_1}}\right) \prod_{j=1}^{k-1} x_j \\
&\geq \left(\frac{s-2}{x_1} + \frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2}{x_{i_0} x_{i_1}}\right) \prod_{j=1}^{k-1} x_j
\end{aligned}$$

Since

$$\frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2}{x_{i_0} x_{i_1}} = \frac{1}{2} - 2\left(\frac{1}{2} - \frac{1}{x_{i_0}}\right)\left(\frac{1}{2} - \frac{1}{x_{i_1}}\right)$$

is decreasing in x_{i_0} and x_{i_1},

$$\frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2}{x_{i_0} x_{i_1}} \geq \frac{1}{x_1} + \frac{1}{x_2} - \frac{2}{x_1 x_2}.$$

Therefore,

$$\begin{aligned}
N_k(G) &\geq \left(\frac{s-1}{x_1} + \frac{1}{x_2} - \frac{2}{x_1 x_2}\right) \prod_{j=1}^{k-1} x_j = \begin{cases} \left(s - \frac{2}{q}\right) q^{k-2}, & \text{if } r = 0, \\ \left(s - \frac{1}{q}\right) q^{k-2}, & \text{if } r = 1, \\ \left(s - \frac{2}{q+1}\right) (q+1)^{r-1} q^{k-r-1}, & \text{if } 2 \leq r \leq k-2. \end{cases} \\
&\geq N_k(n, s).
\end{aligned}$$

Note that if $s \geq 3$, then the first inequality above is strict since there are copies of K_k in G containing at least two edges in B.

Now we may assume that $m = 0$. Then every $e \in B$ is contained in at least $\prod_{i=2}^{k-1} x_i$ copies of K_k and hence

$$N_k(G) \geq s \cdot \prod_{i=2}^{k-1} x_i.$$

So we just need to find the minimum of $\prod_{i=2}^{k-1} x_i$ subject to the constraint that $\prod_{i=1}^{k-1} x_i = t_{k-1}(n) - 1$.

If $r = 0$, then $x_1 = q+1, x_2 = \cdots = x_{k-2} = q$, and $x_{k-1} = q-1$. Therefore, $\prod_{i=2}^{k-1} x_i = q^{k-3}(q-1)$.

If $r = 1$, then $x_1 = x_2 = q+1, x_3 = \cdots = x_{k-2} = q$, and $x_{k-1} = q-1$. Therefore, $\prod_{i=2}^{k-1} x_i = q^{k-4}(q+1)(q-1)$.

If $r \geq 2$, then

$$\begin{aligned} \text{either} \quad & x_1 = \cdots = x_{r+1} = q+1, x_{r+2} = \cdots = x_{k-2} = q, x_{k-1} = q-1 \\ \text{or} \quad & x_1 = q+2, x_2 = \cdots = x_{r-1} = q+1, x_r = \cdots = x_{k-1} = q. \end{aligned}$$

The later one gives a smaller $\prod_{i=2}^{k-1} x_i$, which is $(q+1)^{r-2}p^{k-r}$.

Therefore, for the case $m = 0$

$$\begin{aligned} N_k(G) &\geq \begin{cases} s \cdot q^{k-3}(q-1), & \text{if } r = 0, \\ s \cdot q^{k-4}(q+1)(q-1), & \text{if } r = 1, \\ s \cdot (q+1)^{r-2}q^{k-r}, & \text{if } 2 \leq r \leq k-2. \end{cases} \\ &\geq N_k(n, s), \end{aligned}$$

and equality only if $r \neq 1$. ∎

2.4 Proof of Theorem 1.6

In this section we prove Theorem 1.6. Recall that for $n, k \in \mathbb{N}$, $q_{n,k} = \lfloor n/(k-1) \rfloor$ and $r_{n,k} = n - (k-1)q_{n,k}$. For $s > t \geq 1, k \geq 3$,

$$R_k(n, s, t) = \left(\frac{2(k-1)(s-t)}{k-2} + \frac{(k-1-r_{n,k})\, r_{n,k}}{k-2} \right)^{1/2},$$

$n^+_{k,s,t} = \frac{n+(k-2)R_k(n,s,t)}{k-1}$, and $n^-_{k,s,t} = \frac{n-R_k(n,s,t)}{k-1}$.

Proof of Theorem 1.6. Let $k \geq 4$, $s > t \geq 2$ be fixed integers and n be sufficiently large. Suppose that $s > 2R_k(n, s, t)$. Let $q = q_{n,k}$, $r = r_{n,k}$, and $R = R_k(n, s, t)$. Let G be a graph on n vertices with $t_{k-1}(n) + t$ edges and $\tau_k(G) = s$. Since $s \cdot \left(n^-_{k,s,t}\right) < (s+1/2) \cdot c(n, K_k)$, we may assume that $N_k(G) \leq (s+1/2) \cdot c(n, K_k)$. So by Lemma 2.10, there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $B := G - G[V_1, \ldots, V_{k-1}]$ is a matching of size s.

Let $x_i = |V_i|$ for $i \in [k-1]$ and without loss of generality we may assume that $x_1 \geq \cdots \geq x_{k-1}$. Let $H = G[V_1, \ldots, V_{k-1}]$, $M = K[V_1, \ldots, V_{k-1}] - H$, and $m = |M|$. Since $t_{k-1}(n) + t - s = |H| = |K[V_1, \ldots, V_{k-1}]| - m$,

$$\sum_{1 \leq i < j \leq k-1} x_i x_j = t_{k-1}(n) + t - s + m,$$

which is equivalent to

$$\sum_{i=1}^{k-1} x_i^2 = n^2 - 2t_{k-1}(n) + 2s - 2t - 2m.$$

Let $s_i = |B \cap \binom{V_i}{2}|$ for $i \in [k-1]$ and $S = \{i \in [k-1] : s_i \geq 1\}$.

Case 1: $|S| = 1$.

Without loss of generality we may assume that $s_1 = s$ since the other cases can be solved using a similar argument. Notice that there are $s \cdot \prod_{i=2}^{k-1} x_i$ potential copies of K_k, and for every $e \in M$ there are at most $\prod_{i=2}^{k-2} x_i$ potential copies of K_k containing e. Therefore,

$$N_k(G) \geq s \cdot \prod_{i=2}^{k-1} x_i - m \cdot \prod_{i=2}^{k-2} x_i = \left(s - \frac{m}{x_{k-1}}\right) \cdot \prod_{i=2}^{k-1} x_i.$$

Fix $0 \leq m \leq s - t$. Let $\mathbb{R}_{\geq 0}$ be the collection of all nonnegative real numbers. Define

$$C_m(\mathbb{N}) = \left\{(x_1, \ldots, x_{k-1}) \in \mathbb{N}^{k-1} : \sum_{i=1}^{k-1} x_i = n, \sum_{i=1}^{k-1} x_i^2 = n^2 - 2t_{k-1}(n) + 2s - 2t - 2m\right\},$$

and

$$C_m(\mathbb{R}) = \left\{(x_1, \ldots, x_{k-1}) \in \mathbb{R}_{\geq 0}^{k-1} : \sum_{i=1}^{k-1} x_i = n, \sum_{i=1}^{k-1} x_i^2 = n^2 - 2t_{k-1}(n) + 2s - 2t - 2m\right\}.$$

Note that $C_m(\mathbb{N}) \subset C_m(\mathbb{R})$. In order to get a lower bound for $N_k(G)$ we need to solve the following optimization problem.

$$\mathrm{OPT}_m^{\mathrm{A}} : \begin{cases} \text{Minimize} & \left(s - \frac{m}{x_{k-1}}\right) \cdot \prod_{i=2}^{k-1} x_i \\ \text{subject to} & (x_1, \ldots, x_{k-1}) \in C_m(\mathbb{N}). \end{cases}$$

However, it is not easy to get an optimal solution for $\mathrm{OPT}_m^{\mathrm{A}}$. So we are going to consider the following two auxiliary optimization problems. Let

$$\mathrm{OPT}_m^{\mathrm{B}} : \begin{cases} \text{Minimize} & \left(s - \frac{m}{x_{k-1}}\right) \cdot \prod_{i=2}^{k-1} x_i \\ \text{subject to} & (x_1, \ldots, x_{k-1}) \in C_m(\mathbb{R}), \end{cases}$$

and

$$\mathrm{OPT}_m^{\mathrm{C}} : \begin{cases} \text{Minimize} & \prod_{i=2}^{k-1} x_i \\ \text{subject to} & (x_1, \ldots, x_{k-1}) \in C_m(\mathbb{R}). \end{cases}$$

Let $\mathrm{opt}_m^{\mathrm{a}}$, $\mathrm{opt}_m^{\mathrm{b}}$, and $\mathrm{opt}_m^{\mathrm{c}}$ denote the optimal value of the optimization problems $\mathrm{OPT}_m^{\mathrm{A}}$, $\mathrm{OPT}_m^{\mathrm{B}}$, $\mathrm{OPT}_m^{\mathrm{C}}$, respectively. It is easy to see that $\mathrm{opt}_m^{\mathrm{a}} \geq \mathrm{opt}_m^{\mathrm{b}}$. Moreover, if $\mathrm{OPT}_m^{\mathrm{B}}$ has an optimal solution $x_1, \ldots, x_{k-1}$ such that $x_i \in \mathbb{N}$, then $\mathrm{opt}_m^{\mathrm{a}} = \mathrm{opt}_m^{\mathrm{b}}$. Our goal is to find $\mathrm{opt}_m^{\mathrm{b}}$ and it will be a lower bound for $N_k(G)$.

Claim 2.12. *There exists a constant $C' > 0$ such that*

$$\left(s - \frac{k-1}{n}m\right) \cdot \mathrm{opt}_m^{\mathrm{c}} - C'n^{k-4} < \mathrm{opt}_{\mathrm{m}}^{\mathrm{b}} \leq \left(s - \frac{k-1}{n}m\right) \cdot \mathrm{opt}_m^{\mathrm{c}} + C'n^{k-4}.$$

Proof of Claim 2.12. We abuse notation by assuming that $x_1, \ldots, x_{k-1}$ is an optimal solution of $\mathrm{OPT}^{\mathrm{B}}_m$. Since $\sum_{1\le i<j\le k-1} x_i x_j = t_{k-1}(n) + t - s + m > t_{k-1}(n) - s$, by Lemma 2.4, $n/(k-1) - s \le x_i \le n/(k-1) + s$ for all $i \in [k-1]$. Therefore,

$$\begin{aligned}
\mathrm{opt}^{\mathrm{b}}_m = \left(s - \frac{m}{x_{k-1}}\right) \cdot \prod_{i=2}^{k-1} x_i &\ge \left(s - \frac{m}{n/(k-1) - s}\right) \cdot \prod_{i=2}^{k-1} x_i \\
&= \left(s - \frac{(k-1)m}{n}\right) \cdot \prod_{i=2}^{k-1} x_i - \frac{(k-1)^2 sm}{n(n-ks+s)} \cdot \prod_{i=2}^{k-1} x_i \\
&> \left(s - \frac{(k-1)m}{n}\right) \cdot \prod_{i=2}^{k-1} x_i - C' n^{k-4} \\
&\ge \left(s - \frac{(k-1)m}{n}\right) \cdot \mathrm{opt}^{\mathrm{c}}_m - C' n^{k-4},
\end{aligned}$$

where C' is a constant depending only on k, s, m.

Now let $x'_1, \ldots, x'_{k-1}$ be an optimal solution of $\mathrm{OPT}^{\mathrm{C}}_m$. Then similarly we have

$$\begin{aligned}
\left(s - \frac{(k-1)m}{n}\right) \cdot \mathrm{opt}^{\mathrm{c}}_m &= \left(s - \frac{m}{n/(k-1)}\right) \cdot \prod_{i=2}^{k-1} x'_i \\
&\ge \left(s - \frac{m}{x'_{k-1} - s}\right) \cdot \prod_{i=2}^{k-1} x'_i \\
&= \left(s - \frac{m}{x'_{k-1}}\right) \cdot \prod_{i=2}^{k-1} x'_i - \frac{sm}{x'_{k-1}(x'_{k-1}+s)} \prod_{i=2}^{k-1} x'_i \ge \mathrm{opt}^{\mathrm{b}}_m - C' n^{k-4}.
\end{aligned}$$

∎

Claim 2.12 shows that $\mathrm{opt}^{\mathrm{b}}_m = \left(s - \frac{k-1}{n} m\right) \cdot \mathrm{opt}^{\mathrm{c}}_m \pm C' n^{k-4}$. So we could view $\left(s - \frac{k-1}{n} m\right) \cdot \mathrm{opt}^{\mathrm{c}}_m$ as a "trajectory" (in other words, the "expected" value) for $\mathrm{opt}^{\mathrm{b}}_m$, and this will be useful later for us to show that $\mathrm{opt}^{\mathrm{b}}_{m-1} \le \mathrm{opt}^{\mathrm{b}}_m$.

Let us solve the optimization problem $\mathrm{OPT}^{\mathrm{C}}_m$ first. We use the Lagrangian multiplier method. Let

$$\mathcal{L}(\vec{x}, \lambda, \mu) = \prod_{i=2}^{k-1} x_i + \lambda \left(\sum_{i=1}^{k-1} x_i - n\right) + \mu \left(\sum_{i=1}^{k-1} x_i^2 - \left(n^2 - 2t_{k-1}(n) + 2s - 2t - 2m\right)\right).$$

Again, we abuse notation here by assuming that $(x_1, \ldots, x_{k-1}) \in C_m(\mathbb{R})$ is an optimal solution of $\mathrm{OPT}^{\mathrm{C}}_m$. Then by the Lagrangian multiplier method,

$$\begin{cases}
\frac{\partial \mathcal{L}}{\partial x_1} & = \lambda + 2\mu x_1 = 0 \Rightarrow x_1 = -\frac{\lambda}{2\mu}, \\
\frac{\partial \mathcal{L}}{\partial x_j} & = \frac{\prod_{i=2}^{k-1} x_i}{x_j} + \lambda + 2\mu x_j = 0, \\
\frac{\partial \mathcal{L}}{\partial \lambda} & = \sum_{i=1}^{k-1} x_i - n = 0, \\
\frac{\partial \mathcal{L}}{\partial \mu} & = \sum_{i=1}^{k-1} x_i^2 - \left(n^2 - 2t_{k-1}(n) + 2s - 2t - 2m\right) = 0.
\end{cases}$$

Note that $x_1 \neq 0$ so it is an interior point and hence we can apply the Lagrangian multiplier here.

Let $\pi = \prod_{i=2}^{k-1} x_i$. Note that the equation

$$\frac{\pi}{x} + \lambda + 2\mu x = 0$$

has only two solutions

$$x' = \frac{-\lambda + \sqrt{\lambda^2 - 8\mu\pi}}{4\mu} \quad \text{and} \quad x'' = \frac{-\lambda - \sqrt{\lambda^2 - 8\mu\pi}}{4\mu}.$$

Therefore, referring to $\partial\mathcal{L}/\partial x_j$, for every $2 \leq j \leq k-1$ either $x_j = x'$ or $x_j = x''$.

Before we state the next claim let us recall from the beginning of Case 1 that $s_1 = s$.

Claim 2.13. $x_1 \geq x_2 = \cdots = x_{k-1}$.

Proof of Claim 2.13. First we show that $x_1 \geq x_i$ for all $2 \leq i \leq k-1$. Suppose to the contrary that there exists some $i \in [k-1] \setminus \{1\}$ such that $x_i > x_1$, and without loss of generality we may assume that $x_2 > x_1$. Then let $x'_i = x_i$ for $3 \leq i \leq k-1$, $x'_1 = x_2$, and $x'_2 = x_1$. It is clear that $(x'_1, \dots, x'_{k-1}) \in C_m(\mathbb{R})$, but $\prod_{i=2}^{k-1} x'_i < \prod_{i=2}^{k-1} x_i$, which contradicts our assumption that $(x_1, \dots, x_{k-1})$ is an optimal solution of $\mathrm{OPT}_m^{\mathrm{C}}$. Therefore, $x_1 \geq x_i$ for all $2 \leq i \leq k-1$.

Now we show that $x_2 = \cdots = x_{k-1}$. Suppose that $x_{i_1} \neq x_{i_2}$ for some $2 \leq i_1 < i_2 \leq k-1$. Then $\{x_{i_1}, x_{i_2}\} = \{x', x''\}$, which implies that $x_{i_1} + x_{i_2} = -\lambda/(2\mu) = x_1$. Since $\sum_{1 \leq i < j \leq k-1} x_i x_j = t_{k-1}(n) + t - s + m > t_{k-1}(n) - s$, by Lemma 2.4, $|x_i - n/(k-1)| < s$ for all $i \in [k-1]$. Therefore,

$$x_{i_1} + x_{i_2} > 2 \times \frac{n}{k-1} - 2s > \frac{n}{k-1} + s > x_1,$$

a contradiction. Therefore, $x_2 = \cdots = x_{k-1}$. ∎

By Lemma 2.2,

$$n^2 - 2t_{k-1}(n) = \frac{n^2}{k-1} + \frac{(k-1-r)r}{k-1}.$$

Let $x = x_1$, $y = x_2 = \cdots = x_{k-1}$. Since $(x_1, \dots, x_{k-1}) \in C_m(\mathbb{R})$,

$$\begin{cases} x + (k-2)y = n, \\ x^2 + (k-2)y^2 = \frac{n^2}{k-1} + \frac{(k-1-r)r}{k-1} + 2s - 2t - 2m, \\ x_i \geq 0, \forall i \in [k-1], \end{cases}$$

which implies

$$\begin{cases} x = \frac{n}{k-1} + (k-2)\Delta_m \\ y = \frac{n}{k-1} - \Delta_m, \end{cases}$$

where

$$\Delta_m := \frac{(2(k-1)(k-2)(s-t-m) + (k-2)(k-1-r)r)^{1/2}}{(k-1)(k-2)}.$$

Therefore,

$$\mathrm{opt}_m^{\mathrm{c}} = y^{k-2} = \left(\frac{n}{k-1} - \Delta_m\right)^{k-2}.$$

Now we are going to use $\mathrm{opt}_m^{\mathrm{c}}$ to describe the behavior of $\mathrm{opt}_m^{\mathrm{b}}$.

Claim 2.14. *The value* $\mathrm{opt}_m^{\mathrm{b}}$ *is strictly increasing in* m. *In particular,* $\mathrm{opt}_0^{\mathrm{b}} < \mathrm{opt}_m^{\mathrm{b}}$ *for all* $m > 0$.

Proof of Claim 2.14. Since

$$\mathrm{opt}_m^{\mathrm{c}} = \left(\frac{n}{k-1} - \Delta_m\right)^{k-2} = \left(\frac{n}{k-1}\right)^{k-2} - (k-2)\Delta_m \left(\frac{n}{k-1}\right)^{k-3} + \Theta(n^{k-4}),$$

by Claim 2.12, there exists a constant $C' > 0$ such that

$$\begin{aligned}\mathrm{opt}_m^{\mathrm{b}} &= \left(s - \frac{k-1}{n}m\right) \cdot \mathrm{opt}_m^{\mathrm{c}} \pm C'n^{k-4} \\ &= s\left(\frac{n}{k-1}\right)^{k-2} - (m + s(k-2)\Delta_m)\left(\frac{n}{k-1}\right)^{k-3} \pm C''n^{k-4},\end{aligned}$$

where $C'' > C'$ is a constant depending only on s, k, m. Therefore,

$$\mathrm{opt}_{m-1}^{\mathrm{b}} - \mathrm{opt}_m^{\mathrm{b}} = (1 - s(k-2)(\Delta_{m-1} - \Delta_m))\left(\frac{n}{k-1}\right)^{k-3} \pm 2C''n^{k-4}.$$

Now view Δ_m as a function of the variable m. Then it is easy to see that Δ_m is concave down, i.e. $\mathrm{d}^2\Delta_m/\mathrm{d}m^2 < 0$ for $0 \le m \le s - t$. Therefore,

$$\begin{aligned}s(k-2)(\Delta_{m-1} - \Delta_m) &\ge s(k-2)(-1) \cdot \left.\frac{\mathrm{d}\Delta_m}{\mathrm{d}m}\right|_{m=0} \\ &= \frac{s(k-2)}{(2(k-1)(k-2)(s-t) + (k-2)(k-1-r)r)^{1/2}}.\end{aligned}$$

Since

$$s > 2R = 2\frac{(2(k-1)(k-2)(s-t) + (k-2)(k-1-r)r)^{1/2}}{k-2},$$

we obtain $s(k-2)(\Delta_{m-1} - \Delta_m) > 2$. Therefore,

$$1 - s(k-2)(\Delta_{m-1} - \Delta_m) < -1, \qquad (*)$$

and hence $\mathrm{opt}_{m-1}^{\mathrm{b}} - \mathrm{opt}_m^{\mathrm{b}} < -(n/(k-1))^{k-3} + \Theta(n^{k-4}) < 0$. ∎

Therefore,

$$N_k(G) \ge \mathrm{opt}_m^{\mathrm{a}} \ge \mathrm{opt}_m^{\mathrm{b}} \ge \mathrm{opt}_0^{\mathrm{b}} = s \cdot \mathrm{opt}_0^{\mathrm{c}} = s \cdot \left(\frac{n}{k-1} - \Delta_0\right)^{k-2} = s \cdot \left(n_{k,s,t}^{-}\right)^{k-2}.$$

Here we used that fact that $\Delta_0 = R/(k-1)$.

Case 2: $|S| \ge 2$.

The number of potential copies of K_k is $\sum_{i=1}^{k-1}\left(s_i \cdot \prod_{j\ne i} x_j\right)$. Suppose that $uv \in M$ satisfies $u \in V_{i_0}$ and $v \in V_{i_1}$ for some $i_0, i_1 \in [k-1]$. Similar to the proof of Theorem 1.6 we may assume that $s_{i_0} \ge 1$ and $s_{i_1} \ge 1$. Then there are at most $\prod_{i \ne i_0, i_1} x_i$ potential copies of K_k containing uv. Therefore,

$$N_k(G) \ge \sum_{i=1}^{k-1}\left(s_i \cdot \prod_{j\ne i} x_j\right) - 2\sum_{uv\in M} \prod_{\substack{i\ne i_0,i_1 \\ u\in V_{i_0}, v\in V_{i_1}}} x_i = \left(\sum_{i=1}^{k-1}\frac{s_i}{x_i} - \sum_{\substack{uv\in M \\ u\in V_{i_0}, v\in V_{i_1}}} \frac{2}{x_{i_0}x_{i_1}}\right)\prod_{i=1}^{k-1} x_i.$$

We abuse notation by assuming that $x_{i_0}x_{i_1} = \min\{x_ix_j : \exists uv \in M \text{ such that } u \in V_i, v \in V_j\}$. Then

$$N_k(G) \geq \left(\sum_{i=1}^{k-1} \frac{s_i}{x_i} - \frac{2m}{x_{i_0}x_{i_1}}\right) \prod_{i=1}^{k-1} x_i = \left(\frac{s-2}{x_1} + \frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2m}{x_{i_0}x_{i_1}}\right) \prod_{i=1}^{k-1} x_i.$$

Since

$$\frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2m}{x_{i_0}x_{i_1}} = \frac{1}{2m} - 2m\left(\frac{1}{2m} - \frac{1}{x_{i_0}}\right)\left(\frac{1}{2m} - \frac{1}{x_{i_1}}\right),$$

is decreasing in x_{i_0} and x_{i_1},

$$\frac{1}{x_{i_0}} + \frac{1}{x_{i_1}} - \frac{2m}{x_{i_0}x_{i_1}} \geq \frac{1}{x_1} + \frac{1}{x_2} - \frac{2m}{x_1x_2}.$$

Therefore,

$$\begin{aligned} N_k(G) &\geq \left(\frac{s-1}{x_1} + \frac{1}{x_2} - \frac{2m}{x_1x_2}\right) \prod_{i=1}^{k-1} x_i \\ &= \left(\frac{s}{x_1} + \frac{x_1 - x_2}{x_1x_2} - \frac{2m}{x_1x_2}\right) \prod_{i=1}^{k-1} x_i = \left(s + \frac{x_1 - x_2}{x_2} - \frac{2m}{x_2}\right) \prod_{i=2}^{k-1} x_i. \end{aligned}$$

Therefore, in order to get a lower bound for $N_k(G)$ we need solve the following optimization problem.

$$\mathrm{OPT}^{\mathrm{D}}_m : \begin{cases} \text{Minimize} & \left(s + \frac{x_1 - x_2}{x_2} - \frac{2m}{x_2}\right) \prod_{i=2}^{k-1} x_i \\ \text{subject to} & (x_1, \ldots, x_{k-1}) \in C_m(\mathbb{N}). \end{cases}$$

Similarly, we are going to consider the following auxiliary optimization problem.

$$\mathrm{OPT}^{\mathrm{E}}_m : \begin{cases} \text{Minimize} & \left(s + \frac{x_1 - x_2}{x_2} - \frac{2m}{x_2}\right) \prod_{i=2}^{k-1} x_i \\ \text{subject to} & (x_1, \ldots, x_{k-1}) \in C_m(\mathbb{R}). \end{cases}$$

Theoretically, one could solve $\mathrm{OPT}^{\mathrm{E}}_m$ exactly using the Lagrange multiplier method. However, the optimal solution of $\mathrm{OPT}^{\mathrm{E}}_m$ is very complicated. So we are going to compare $\mathrm{OPT}^{\mathrm{E}}_m$ with $\mathrm{OPT}^{\mathrm{C}}_m$.

Let $\mathrm{opt}^{\mathrm{d}}_m$ and $\mathrm{opt}^{\mathrm{e}}_m$ denote the optimal values of the optimization problems $\mathrm{OPT}^{\mathrm{D}}_m$ and $\mathrm{OPT}^{\mathrm{E}}_m$, respectively. It is easy to see that $\mathrm{opt}^{\mathrm{d}}_m \geq \mathrm{opt}^{\mathrm{e}}_m$. The following claim is very similar to Claim 2.12, and can be proved in a similar fashion so so we omit the proof.

Claim 2.15. *There exists a constant $\hat{C} > 0$ such that*

$$\left(s - \frac{2(k-1)m}{n}\right) \cdot \mathrm{opt}^{\mathrm{c}}_m - \hat{C}n^{k-4} < \mathrm{opt}^{\mathrm{e}}_m \leq \left(s - \frac{2(k-1)m}{n}\right) \cdot \mathrm{opt}^{\mathrm{c}}_m + \hat{C}n^{k-4}.$$

Claim 2.16. *The value $\mathrm{opt}^{\mathrm{e}}_m$ is strictly increasing in m. In particular, $\mathrm{opt}^{\mathrm{e}}_0 < \mathrm{opt}^{\mathrm{e}}_m$ for all $m > 0$.*

Proof of Claim 2.16. The proof is basically the same as the proof for Claim 2.14. The only difference is that $s > 2R$ implies that there exists $\varepsilon > 0$ such that $s(k-2)\left(\Delta_{m-1} - \Delta_m\right) > 2 + \varepsilon$. Therefore, $(*)$ now becomes

$$2 - s(k-2)\left(\Delta_{m-1} - \Delta_m\right) < -\varepsilon,$$

which implies that

$$\begin{aligned}\mathrm{opt}^{\mathrm{b}}_{m-1} - \mathrm{opt}^{\mathrm{b}}_{m} &= \left(2 - s(k-2)\left(\Delta_{m-1} - \Delta_m\right)\right)\left(\frac{n}{k-1}\right)^{k-3} \pm 2C' n^{k-4} \\ &< -\varepsilon\left(n/(k-1)\right)^{k-3} + \Theta(n^{k-4}) < 0.\end{aligned}$$

∎

Therefore, if $s > 2R$, then

$$N_k(G) \geq \mathrm{opt}^{\mathrm{d}}_{m} \geq \mathrm{opt}^{\mathrm{e}}_{m} \geq \mathrm{opt}^{\mathrm{e}}_{0} \geq s \cdot \mathrm{opt}^{\mathrm{c}}_{0} = s \cdot \left(\frac{n}{k-1} - \Delta_0\right)^{k-2} = s \cdot \left(n^{-}_{k,s,t}\right)^{k-2}.$$

Note that we may assume that $s - t \geq 2$ since the case $s - t = 1$ has been solved by Theorem 1.5. Therefore, there exists copies of K_k in G that contains at least two edges in B, which implies that the first inequality above is strict. ∎

2.5 Proof of Theorem 1.9

In this section we prove Theorem 1.9. We need the following lemma.

Lemma 2.17 ([7])**.** *Fix $k \geq 3$ and a k-critical graph F with f vertices. Then there are positive constants α_F and β_F such that if n is sufficiently large, then $|c(n,F) - \alpha_F n^{f-2}| < \beta_F n^{f-3}$.*

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Let $s > t \geq 1, k \geq 3$ be fixed integers and let F be a k-critical graph on f vertices. Let n be sufficiently large. Let G be a graph on n vertices with $t_{k-1}(n) + t$ edges and $\tau_F(G) = s$. We may assume that $N_F(G) \leq s \cdot c(n,F)$, since otherwise we are done.

By Lemma 2.10, there exists a partition $V(G) = V_1 \cup \cdots \cup V_{k-1}$ such that $B := G - G[V_1, \ldots, V_{k-1}]$ is a matching of size s. Let $x_i = |V_i|$ for $i \in [k-1]$ and without loss of generality we may assume that $x_1 \geq \cdots \geq x_{k-1}$. Let $H = G[V_1, \ldots, V_{k-1}]$, $M = K[V_1, \ldots, V_{k-1}] - H$, and $m = |M|$. Since $t_{k-1}(n) - t = |H| = |K[V_1, \ldots, V_{k-1}]| - m$,

$$\sum_{1 \leq i < j \leq k-1} x_i x_j = t_{k-1}(n) + t - s + m.$$

Therefore, by Lemma 2.4, $n/(k-1) - s < x_i < n/(k-1) + s$ for all $i \in [k-1]$. Let

$$c_{min} = \min\{c(x_{\sigma(1)}, \ldots, x_{\sigma(k-1)}) : \sigma \in S_{k-1}\},$$

where S_{k-1} is the collection of all permutations of $[k-1]$. By Lemma 2.3, $c_{min} \geq c(n,F) - \gamma_F s n^{f-3}$ for some constant γ_F. Note the the number of potential copies of K_k is at least $s \cdot c_{min}$. Since every $e \in M$ is contained in at most n^{f-3} potential copies of K_k,

$$N_F(G) \geq s \cdot c_{min} - m n^{f-3} \geq s \cdot c(n,F) - C n^{f-3}$$

for some constant C. This completes the proof of Theorem 1.9. ∎

3 Concluding remarks

We proved several bounds on the number of copies of K_k (and also for k-critical graphs F) in a graph G on n vertices with $t_{k-1}(n)+t$ edges and $\tau_k(G)=s$. In our proof we need s and t to be fixed. Using the same method we are able to show that the same conclusions as in Theorems 1.4, 1.5, 1.6, and 1.9 hold for all $s > t \geq 1$ (for Theorem 1.6 we still need $s > 2R_k(n,s,t)$) as long as $s(s-t)^{1/2} < \xi n$ for some small constant $\xi > 0$. In particular, if $s-t < C$ for some constant C, then the conclusions hold for all $s < \xi' n$ for some small constant $\xi' > 0$. The proofs are more involved and tedious, so we chose to omit them here.

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