Intersecting curves in the plane

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Abstract

We prove that for every family of n pairwise intersecting simple closed planar curves in general position, at least $(4/5)n^2 - O(n)$ points lie on more than one curve. This improves the previous lower bound of $(3/4)n^2 - O(n)$ due to Richter and Thomassen.

1 Crossing numbers and intersecting families

In the context of determining the crossing number of the Cartesian product of two cycles, Richter and Thomassen [3] introduced the problem of determining $i(n)$, the minimum number of intersections in a family of n pairwise intersecting simple closed curves in the plane, where

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no point lies on more than two curves (see also [5] for related results on other surfaces). They observed that $i(5) = 12$, $i(6) = 20$ and that $\lim_{n\to\infty} i(n)/\binom{n}{2}$ exists and lies between 3/2 and 2. In [4] it was shown that this limit is 2 if we impose the restriction that every curve intersects every other curve in at most m points, where m is fixed (but can be arbitrarily large). We prove that $\lim_{n\to\infty} i(n)/\binom{n}{2} \geq 8/5$. Our main result is

Theorem 1. $i(n) \geq (4/5)n^2 - 5n - O(1)$.

While this improvement is admittedly modest, it is the first nontrivial lower bound for $i(n)/\binom{n}{2}$ (the bound from [3] follows from the recurrence $i(n) \geq \lceil \frac{n}{n-2}\ i(n-1) \rceil$ which is obtained by a straightforward counting argument). Moreover, our proof technique relates this geometric problem to a well-known bipartite Turán problem from extremal graph theory. This connection has already proved successful in [4], although our application has a slightly different flavor. We hope that our approach fosters new ideas to attack the conjecture that $\lim i(n)/\binom{n}{2} = 2$

To facilitate the proof, in Section 2 we prove that certain configurations of six or seven curves are absent in an optimal placement of the curves. In Section 3, we prove the main result.

An *intersecting family* $\mathcal{C}$ is a set of simple closed curves in the plane such that every two curves in $\mathcal{C}$ have a point in common. We only consider families of curves in *general position*, i.e., those for which no point lies on three or more curves. The *internal intersections* of a set S of curves are the intersection points of pairs of curves in S. We say that two curves C and C' intersect *tangentially* at the point P if both curves have a common tangent containing P, and C lies in only one region of $\mathbf{R}^2 - C'$ in a neighborhood of P. Given an intersecting family $\mathcal{C}$, let $G_{\mathcal{C}}$ be the graph with vertex set $\mathcal{C}$, with $C \leftrightarrow C'$ whenever C and C' have only a single point in common (which must be a tangential intersection). We will often find it useful to refer to this auxiliary graph.

2 Forbidden Configurations

In this section, we prove the nonexistence of certain configurations of curves. Our arguments throughout implicitly use the Jordan Curve Theorem, which states that every homeomorphic image C of the sphere S^1 in $\mathbf{R}^2$ divides $\mathbf{R}^2 - C$ into exactly two components, each of which has C as its complete boundary.

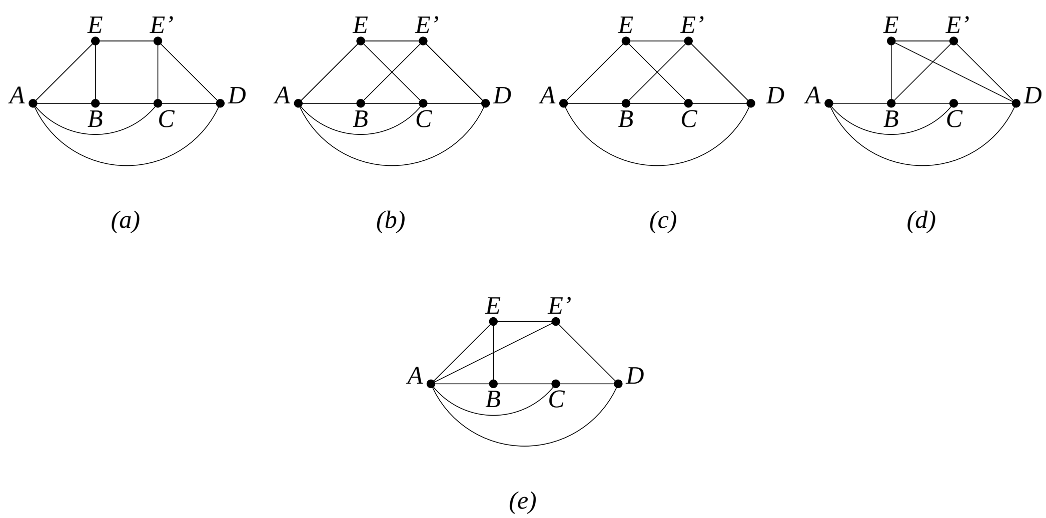

Figure 1

Lemma 2. *Let $\mathcal{S} = \{A, B, C, D, E, E'\}$ be an intersecting family in general position. Suppose that the graphs in Figure 1 represent various possibilities for $G_{\mathcal{S}}$, where nonadjacent vertices imply that the corresponding curves intersect exactly twice and have no tangential intersections.*

(i) If $G_{\mathcal{S}}$ is as shown in Figure 1 (a), (b), (c), or (d), then the curves cannot be drawn in the plane.

(ii) Suppose that $G_{\mathcal{S}}$ is as shown in Figure 1 (e), the curves are drawn in the plane, and $E'' \notin \mathcal{S}$ is a simple closed curve that intersects each curve in $\mathcal{S}$ either exactly once, or in an even number of points. Suppose also that every four curves in $\mathcal{S} \cup \{E''\}$ produce at least 7 internal intersections, and every five curves produce at least 13 internal intersections. Then E'' intersects $\mathcal{S}$ in at least 10 points.

Proof. **(i)** Figure 1(a): We may assume by symmetry that A, B, C, and D are as shown in Figure 2 (a). Since E intersects both A and B exactly once, and intersects both C and D twice, it intersects A in the C, D portion of A (see Figure 2 (b)).

Because E' intersects E, C, and D exactly once and A, B at most twice, E' has a point inside the C, E portion of A, or the E, D portion of A. In the first case, since E' intersects D, it must intersect A four times. In the second case, since E' intersects C, it must intersect A four times.

Figure 1(b) and 1(c): We may assume by symmetry that A, B, C, and D are as shown in Figure 3, where the dotted curve applies to the case of Figure 1(c). There are three possibilities for the position of E; these are shown in Figure 3 (a), (b), and (c). Since E' intersects each of B, D, E only tangentially, and each of the three figures is the same up to a permutation of B, D, E, we may assume that the curves appear as in Figure 3 (a). The curve E' cannot have a point in the region marked R or R', since then it intersects C at least 4 times. Since E' intersects B, it has no point in the B, C portion of A or in the B, E portion of A. Since E' also intersects D, it has no point in the C, D portion of A or in the D, E portion of A. This implies that E' does not intersect A, a contradiction.

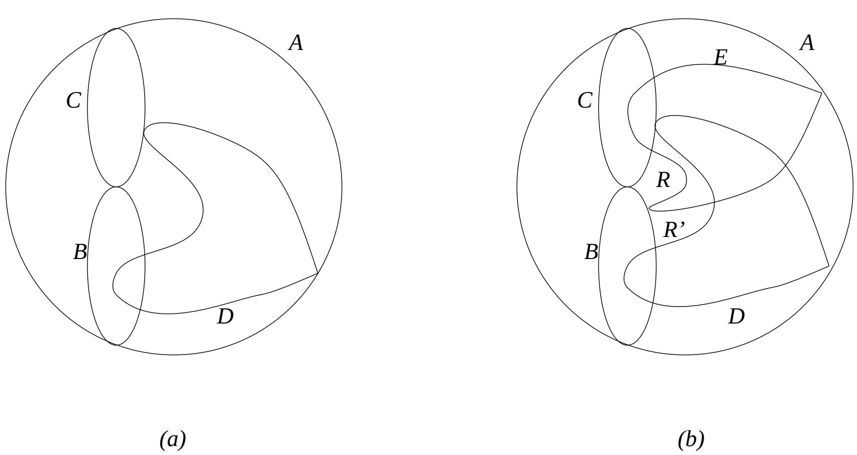

Figure 2

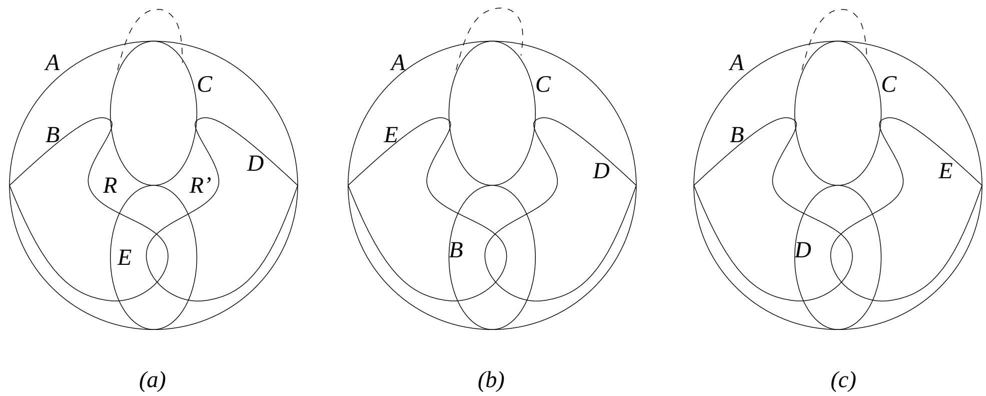

Figure 3

Figure 1(d): We may assume by symmetry that A, B, C, D, are as shown in Figure 4(a). Since E intersects B and D exactly once, and A and C at most twice, E intersects D in the C, A portion of D (see Figure 4(b)). Because the previous statement applies also to E', the curves E and E' cannot intersect only tangentially.

(ii) Figure 1(e): Suppose that E'' intersects $\mathcal{S}$ in at most 9 points. Then E'' intersects each of at least three curves from $\mathcal{S}$ in exactly one point.

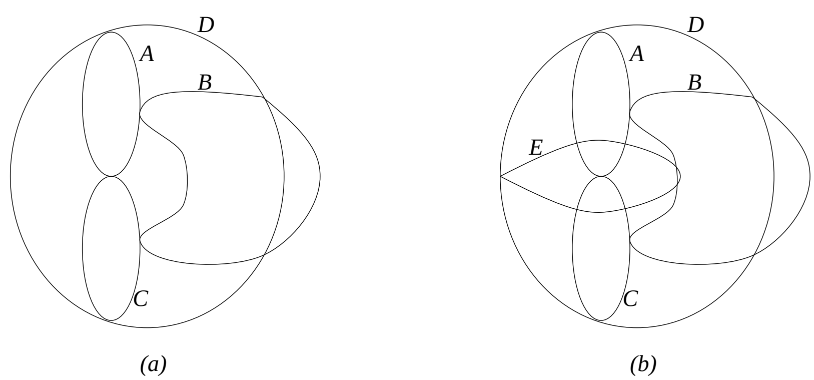

Figure 4

Case 1: E'' intersects A in exactly 1 point. If E'' has precisely 1 intersection point with each of two curves corresponding to adjacent vertices in $\mathcal{S} - A$, then this pair together with A and E'' are four curves that internally intersect in only 6 points. Hence we may assume by symmetry that E' intersects each of B and D in exactly 1 point, and each of C, E, E' in exactly 2 points (otherwise we already have 10 intersection points). In this case, A, B, C, D, E'' are five curves that internally intersect in exactly 12 points.

Case 2: E'' intersects A in more than 1 point. Suppose that E'' has precisely 1 intersection point with each of three curves corresponding to cyclically adjacent vertices in $\mathcal{S} - A$. By symmetry, let this triple be B, C, D. In this case, the five curves A, B, C, D, E'' internally intersect in exactly 12 points *unless* A intersects E'' in at least 4 points. If A and E'' have more than 4 points in common, or E'' intersects a curve in $\mathcal{S} - A$ in at least 2 points, then we have at least 10 intersection points between E'' and $\mathcal{S}$ as required. We may therefore assume that E'' and A intersect in exactly 4 points, and E'' intersects every curve in $\mathcal{S} - A$ only tangentially. It is not hard to see that such a curve system cannot be drawn in the plane.

Hence we may assume that for every triple X, Y, Z of curves corresponding to cyclically adjacent vertices in $\mathcal{S} - A$, at least one of X, Y, Z intersects E'' in at least 2 points. Since E'' intersects three curves in exactly 1 point, we may assume that E'' intersects each of B, C, E' in exactly 1 point. Interchanging the roles of B and D, relabeling E' with E, and relabeling E'' with E', we see that $(\mathcal{S} \cup E'') - E'$ is isomorphic to the curve system depicted in Figure 1 (a), which cannot exist. $\square$

3 The Proof

In this section, we prove Theorem 1. We begin with an observation about optimal placements of intersecting curves.

Lemma 3. *Let $\mathcal{C}$ be an intersecting family of n curves with $i(n)$ intersections. If two curves in $\mathcal{C}$ intersect in more than one point, then they have no tangential intersection. In particular,* *every curve in $\mathcal{C}$ intersects every other curve in $\mathcal{C}$ in exactly one point, or in an even number of points.*

Proof. Suppose that the curves C and C' intersect in $k \geq 2$ points, and P is a tangential intersection point. Then there is a neighborhood $N(P)$ about P which no other curve in $\mathcal{C}$ intersects. Bending C slightly near P destroys the point of tangency and leaves all other intersections intact. But now $\mathcal{C}$ has $i(n) - 1$ points of intersection, a contradiction. The other remark follows easily from this and the Jordan Curve Theorem. □

We need the classical result that an n vertex graph with no subgraph isomorphic to the four-cycle has at most $f(n) \sim n^{3/2}/2$ edges (see, e.g., [1, 2]).

Proof of Theorem 1: Let

$$c_f = \min\{k : n(n-1) - f(n) \geq (4/5)n^2 - 5n - k \text{ for all } n \geq 1\},$$

and let

$$c' = \min\{k' : i(n) \geq (4/5)n^2 - 5n - k' \text{ for } 1 \leq n \leq 6\}.$$

Set $c = \max\{c_f, c'\}$, and $t_n = (4/5)n^2 - 5n - c$ (note that $f(n) \sim n^{3/2}/2$ implies that c_f is well-defined). We will prove that $i(n) \geq t_n$ by induction on n. The cases $n \leq 6$ follow from the definition of c. Let $\mathcal{C}$ be an intersecting family of n curves with $i(n)$ intersections, and let $n \geq 7$. If the associated graph $G_{\mathcal{C}}$ has no four-cycles, then it has at most $f(n)$ edges, which implies that $\mathcal{C}$ has at least $n(n-1) - f(n) \geq t_n$ intersection points as required. Thus we may assume that $\mathcal{H} = ABCD$ is a 4-cycle in $G_{\mathcal{C}}$, with $A \leftrightarrow B \leftrightarrow C \leftrightarrow D \leftrightarrow A$. We may further assume that among all such four-cycles, $\mathcal{H}$ has the fewest internal intersection points.

Suppose that $\mathcal{H}$ has 9 internal intersection points. If every triple of curves has at least 5 internal intersection points, then $i(n) \geq 5\binom{n}{3}/(n-2) = (5/6)n(n-1) \geq t_n$. Otherwise, there is a triple $T = \{X, Y, Z\}$ with at most 4 internal intersection points. Then every curve $W \notin T$ intersects T in at least 5 points. The family $\mathcal{C} - T$ has at least $i(n-3)$ intersection points, hence the induction hypothesis implies that $\mathcal{C}$ has at least $i(n-3) + 5(n-3) \geq$

$t_{n-3} + 5(n-3) \geq t_n$ intersection points. We may therefore assume that $\mathcal{H}$ has at most 8 internal intersection points. The above argument also proves that if $\mathcal{H}$ has 8 internal intersection points, then every triple of curves has at least 4 internal intersection points, i.e., $G_{\mathcal{C}}$ is triangle-free.

Let $E \in \mathcal{C} - \mathcal{H}$ be a curve that has the fewest intersections with $\mathcal{H}$. If E intersects $\mathcal{H}$ in at least 7 points, then every curve in $\mathcal{C} - \mathcal{H}$ also intersects $\mathcal{H}$ in at least 7 points. The family $\mathcal{C} - \mathcal{H}$ has at least $i(n-4)$ intersection points, hence the induction hypothesis implies that $\mathcal{C}$ has at least $i(n-4) + 7(n-4) \geq t_{n-4} + 7(n-4) \geq t_n$ intersection points. We may therefore assume that E intersects $\mathcal{H}$ in at most 6 points.

Case 1: E intersects $\mathcal{H}$ in 4 points.

If any curve from $\mathcal{H}$ intersects another curve from $\mathcal{H}$ in at least two points, then replacing one of the curves from the pair with E gives a 4-cycle with fewer internal intersections than $\mathcal{H}$. On the other hand, if every pair from $\mathcal{H}$ intersects only tangentially, then together with E the curves in $\mathcal{H}$ form a K_5 in $G_{\mathcal{C}}$ which is impossible.

Case 2: E intersects $\mathcal{H}$ in 5 points.

Assume without loss of generality that E intersects A twice and each of B, C, D once. If A intersects any of B, C, D more than once, then replacing A by E yields a 4-cycle with fewer internal intersections than $\mathcal{H}$. Hence we may assume that A intersects each of B, C, D once. It is easy to see that these conditions imply that D must intersect B at least 4 times. But this implies that $\mathcal{H}$ has at least 9 internal intersection points, a contradiction.

Case 3: E intersects $\mathcal{H}$ in 6 points.

We will show that either

(i) every curve $E' \in \mathcal{C} - \mathcal{H} - E$ intersects $\mathcal{H} \cup E$ in at least 8 points, or

(ii) every curve $E'' \in \mathcal{C} - \mathcal{H} - E - E'$ intersects $\mathcal{H} \cup E \cup E'$ in at least 10 points.

In the first case, we apply induction by removing the 5 curves in $\mathcal{H} \cup E$ and obtain at least $i(n-5) + 8(n-5) \geq t_{n-5} + 8(n-5) \geq t_n$ intersection points (note that since $i(6) = 20$, this computation also allows us to assume that every set of five curves from $\mathcal{C}$ has at least 13 internal intersections). In the second case, we apply induction by removing the 6 curves in $\mathcal{H} \cup E \cup E'$ and obtain at least $i(n-6) + 10(n-6) \geq t_{n-6} + 10(n-6) \geq t_n$ intersection points.

Suppose that a curve E' intersects $\mathcal{H} \cup E$ at in at most 7 points. It cannot intersect in fewer than 7 points, since then it intersects $\mathcal{H}$ in at most 5 points, a possibility we have already excluded. So assume that E' intersects $\mathcal{H}$ in 6 points, and E once. In this case, there are at least 7 intersection points among $\mathcal{H}$, since otherwise there are exactly 19 internal intersections among the six curves from $\mathcal{H} \cup E \cup E'$, which contradicts $i(6) = 20$. If E' and E have only a tangential intersection with the curves X, X' from $\mathcal{H}$, then E, E', X, X' together have at most 7 internal intersections. Since $\mathcal{H}$ has at least 7 internal intersections, these four curves have exactly 7 internal intersections, which implies that X and X' intersect twice. We may therefore assume that:

(a) $\mathcal{H}$ yields at least 7, and at most 8, internal intersection points.

(b) E and E' both intersect $\mathcal{H}$ in exactly 6 points, and pairwise intersect only tangentially.

(c) If E and E' intersect the same two curves from $\mathcal{H}$ exactly once, then these curves pairwise intersect in at least 2 points.

(d) Every five curves internally intersect in at least 13 points.

(e) If $\mathcal{H}$ has 8 internal intersection points, then there is no triangle in $G_{\mathcal{C}}$.

(f) No two curves intersect in exactly 3 points (by Lemma 3).

Conditions (a)–(f) imply that

(g) every pair of curves from $H \cup E \cup E'$ intersect in at most 2 points, and

(h) the associated subgraph of $G_{\mathcal{C}}$ corresponding to $\mathcal{H} \cup E \cup E'$ is isomorphic to one of those in Figure 1.

By Lemma 2, no subgraph from Figure 1 (a)–(d) can appear, and if the subgraph in Figure 1(e) is present, then every other curve intersects $\mathcal{H} \cup E \cup E'$ in at least 10 points. □

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