Proof of a Conjecture of Erdős on triangles in set-systems

Dhruv Mubayi * Jacques Verstraëte†

November 11, 2005

Abstract

A triangle is a family of three sets A, B, C such that $A \cap B$, $B \cap C$, $C \cap A$ are each nonempty, and $A \cap B \cap C = \emptyset$. Let $\mathcal{A}$ be a family of r-element subsets of an n-element set, containing no triangle. Our main result implies that for $r \geq 3$ and $n \geq 3r/2$, we have $|\mathcal{A}| \leq \binom{n-1}{r-1}$. This settles a longstanding conjecture of Erdős [7], by improving on earlier results of Bermond, Chvátal, Frankl, and Füredi. We also show that equality holds if and only if $\mathcal{A}$ consists of all r-element subsets containing a fixed element.

Analogous results are obtained for nonuniform families.

1 Introduction.

Throughout this paper, X is an n-element set. For any nonnegative integer r, we write $X^{(r)}$ for the family of all r-element subsets of X. Define $X^{(\leq r)} = \cup_{0 \leq i \leq r} X^{(i)}$ and $X^{(\geq r)} = \cup_{r \leq i \leq n} X^{(i)}$. For $\mathcal{A} \subset X^{(\leq n)}$ and $x \in X$, we let $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$.

A *triangle* is a family of three sets A, B, C such that $A \cap B$, $B \cap C$, $C \cap A$ are each nonempty, and $A \cap B \cap C = \emptyset$. Let $f(r, n)$ denote the maximum size of a family $\mathcal{A} \subset X^{(r)}$ containing no triangle.

*Department of Mathematics, Statistics, & Computer Science, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045; email: mubayi@math.uic.edu

†Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON N2L 3G1, Canada; E-mail: jverstra@math.uwaterloo.ca

2000 Mathematics Subject Classification: 05C35, 05C65, 05D05

Keywords: *Extremal set theory, intersecting family, Baranyai's theorem*

Accepted, Combinatorica

A special case of Turán's theorem (proved by Mantel) implies that $f(2,n) = \lfloor n^2/4 \rfloor$. Motivated by this result, Erdős [7] asked for the determination of $f(r,n)$ for $r > 2$, and conjectured that

$$f(r,n) = \binom{n-1}{r-1} \qquad \text{for} \qquad n \geq 3r/2. \tag{1}$$

(Actually, in [7] it is stated more as a question, and $n \geq 3r/2$ is not explicitly mentioned, but later, e.g. in [3, 10], (1) is referred to as a conjecture of Erdős'.)

This conjecture attracted quite a few researchers. It was proved by Chvátal [3] for $r = 3$. In fact, he proved the more general statement that if $n \geq r+2 \geq 5$, $\mathcal{A} \subset X^{(r)}$, and $|\mathcal{A}| > \binom{n-1}{r-1}$, then $\mathcal{A}$ contains r sets $A_1, \ldots, A_r$ such that every $r-1$ of them have nonempty intersection, but $\cap_i A_i = \emptyset$. This configuration is also called an $(r-1)$-dimensional simplex. Chvátal generalized (1) as follows.

Conjecture 1 (Chvátal) *Let $r \geq d+1 \geq 3$, $n \geq r(d+1)/d$ and $\mathcal{A} \subset X^{(r)}$. If $\mathcal{A}$ contains no d-dimensional simplex, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Equality holds only when $\mathcal{A} = X_x^{(r)}$, for some $x \in X$.*

Recently Csákány and Kahn [6] gave a different proof of the $r = 3$ case of (1) using Homology theory. Frankl [9] settled (1) for $3r/2 \leq n \leq 2r$, and then Bermond and Frankl [2] proved (1) for infinitely many n, r, where $n < r^2$. About five years later, Frankl [10] settled (1) for $n > n_0(r)$, where $n_0(r)$ is an unspecified but exponentially growing function of r. In 1987, Frankl and Füredi [11] proved Conjecture 1 for $n > n_0(r)$. Frankl [10] had earlier verified Conjecture 1 for $(d+1)r/d \leq n < 2r$, using Katona's permutation method. Thus both (1) and Conjecture 1 remained open in the range $2r \leq n < n_0(r)$, where $n_0(r)$ is exponential in r. Also, the uniqueness of the extremal configuration remained open for $3r/2 \leq n < n_0(r)$ in both (1) and Conjecture 1.

Our main result settles (1) for all $n \geq 3r/2$ while also characterizing the extremal examples. A *nontrivial intersecting family* of size $d+1$ is a family of $d+1$ distinct sets $A_1, \ldots, A_{d+1}$ that have pairwise nonempty intersection, but $\cap_i A_i = \emptyset$.

Theorem 2 *Let $r \geq d+1 \geq 3$ and $n \geq (d+1)r/d$. Suppose that $\mathcal{A} \subset X^{(r)}$ contains no nontrivial intersecting family of size $d+1$. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Equality holds if and only if $\mathcal{A} = X_x^{(r)}$ for some $x \in X$.*

Note that the special case $d = 2$ above implies (1). Every d-dimensional simplex is a nontrivial intersecting family of size $d+1$, and in this sense Theorem 2 can be thought of as a solution to a weakening of Conjecture 1.

A hypergraph $\mathcal{F}$ satisfies H_d, the Helly property of order d, if every subfamily of $\mathcal{F}$ with empty intersection contains a subcollection of at most d sets with empty intersection. A related problem is to determine the maximum size of an $\mathcal{F} \in X^{(r)}$ that satisfies H_d. Theorem 2 implies that for $d = 2$, such an $\mathcal{F}$ satisfies $|\mathcal{F}| \leq \binom{n-1}{r-1}$, however, stronger results for this problem were obtained by several authors (see Bollobás and Duchet [4, 5], Tuza [15, 16], and Mulder [13]).

The proof of Theorem 2 actually gives a little more: we may allow $r \leq d \leq \min\{\binom{2r-2}{r-1}, \binom{n-1}{r-1}\}$. Theorem 2 is not valid when $r = 3$ and $d \geq 10$ however, as the next result attests (see Section 4):

Theorem 3 *Let $\mathcal{A} \subset X^{(3)}$ contain no non-trivial intersecting family of size $d+1 \geq 8$. Then*

$$|\mathcal{A}| \leq \left(\left\lfloor \frac{(d+2)}{3} \right\rfloor^{-1} + \left\lfloor \frac{(d+3)}{3} \right\rfloor^{-1} + \left\lfloor \frac{(d+4)}{3} \right\rfloor^{-1} \right)^{-1} \binom{n}{2} \leq \frac{1}{3} \left(\left\lceil \frac{d}{3} \right\rceil + \frac{1}{d+3} \right) \binom{n}{2}.$$

Furthermore, for $d+1 \geq 11$ and infinitely many n, there exists such a family $\mathcal{A}$ with $|\mathcal{A}| \geq (\frac{1}{3}\lceil \frac{d}{3} \rceil - \frac{1}{3})\binom{n}{2}$.

We conjecture that for $r \geq 4$ and n sufficiently large, the phenomenon exhibited by Theorem 3 does not arise:

Conjecture 4 *Let $r \geq 4$, $d \geq 2$, and let $\mathcal{A} \subset X^{(r)}$ contain no non-trivial intersecting family of size $d+1$. Then, provided n is sufficiently large, $|\mathcal{A}| \leq \binom{n-1}{r-1}$ with equality if and only if $\mathcal{A} = X_x^{(r)}$ for some $x \in X$.*

The following table summarizes the above results for $r = 3$:

Size	Lower Bound	Upper Bound
$3 \leq d+1 \leq 7$	$\binom{n-1}{2}$	$\binom{n-1}{2}$
$d+1 = 8$	$\binom{n-1}{2}$	$\binom{n}{2}$
$d+1 = 9$	$\binom{n-1}{2}$	$\frac{12}{11}\binom{n}{2}$
$d+1 = 10$	$\binom{n-1}{2}$	$\frac{6}{5}\binom{n}{2}$
$d+1 \geq 11$	$\frac{1}{3}(\lceil \frac{d}{3} \rceil - 1)\binom{n}{2}$	$\frac{1}{3}(\lceil \frac{d}{3} \rceil + \frac{1}{d+3})\binom{n-1}{2}$

It would be interesting to determine the exact bounds for $d+1 \geq 11$. In the course of the proof of the lower bound in Theorem 3, it is proved that a Steiner $(n, 3, k-1)$-system, when it exists, contains no non-trivial intersecting family of size $3k+1$ whenever $k \geq 2$. We conjecture that this is the extremal family for $r = 3$ and $k \geq 2$:

Conjecture 5 *Let n be sufficiently large and let $k \geq 2$. Let $\mathcal{A} \subset X^{(3)}$ contain no non-trivial intersecting family of size $3k+1$, and suppose there exists a Steiner $(n, 3, k-1)$-system. Then $|\mathcal{A}| \leq \frac{1}{3}(k-1)\binom{n}{2}$, with equality if and only if $\mathcal{A}$ is such a Steiner system.*

Non-uniform families: It is natural to consider these extremal problems for families that are not uniform. Perhaps the most basic statement in this context is the analogue of the Erdős-Ko-Rado Theorem.

$$\textit{If } \mathcal{A} \subset X^{(\leq n)} \textit{ is intersecting, then } |\mathcal{A}| \leq 2^{n-1}.$$

The non-uniform analogue of Erdős' conjecture about triangles in uniform families was asked by Erdős and proved by Milner [7].

Theorem 6 (Milner) *Suppose that $\mathcal{A} \subset X^{(\leq n)}$ is triangle free. Then $|\mathcal{A}| \leq 2^{n-1} + n$.*

Since Milner's proof has not been published, we give our own short proof of this result (see also Lossers [12]). Our proof also yields that equality holds if and only if $\mathcal{A} = X_x^{(\geq 2)} \cup X^{(1)} \cup \{\emptyset\}$ for some $x \in X$; this fact seems not to have been mentioned in the previous literature. We also prove the non-uniform analogue of Theorem 2 (see Section 4).

Theorem 7 *Let $d \geq 2$ and $n > \log_2 d + \log_2 \log_2 d + 2$. Suppose that $\mathcal{A} \subset X^{(\leq n)}$ contains no non-trivial intersecting family of size $d+1$. Then $|\mathcal{A}| \leq 2^{n-1} + n$. Equality holds if and only if $\mathcal{A} = X_x^{(\geq 2)} \cup X^{(1)} \cup \{\emptyset\}$ for some $x \in X$.*

If $n \leq \lfloor \log_2 d \rfloor$, then trivially the bound $|\mathcal{A}| \leq 2^{n-1} + n$ in Theorem 7 does not hold. It can be shown that this remains true for $\lfloor \log_2 d \rfloor + 1$ and $\lfloor \log_2 d \rfloor + 2$. However, once $n > \lfloor \log_2 d \rfloor + \log_2 \log_2 d + 2$, Theorem 7 applies. It would be interesting to determine if the $\log_2 \log_2 d$ term in Theorem 7 can be replaced by an absolute constant.

2 Proof of Theorem 2.

We use the notation $[n] = \{1, \ldots, n\}$ and $[a, b] = \{a, a+1, \ldots, b-1, b\}$. Let $\mathcal{A}$ be a family of r-sets with $|\mathcal{A}| \geq \binom{n-1}{r-1}$, containing no non-trivial intersecting family of size $d+1$. We prove that $\mathcal{A}$ consists of all sets containing a fixed element of X. The proof, for $n \geq (d+1)r/d$, is split into three parts;

Part I	$n < 2r$ and $n = k(n-r) + \ell$ with $k \in [2, d]$ and some $\ell \in [n-r-1]$,
Part II	$n < 2r$ and $n = k(n-r)$ with $k \in [3, d+1]$,
Part III	$n \geq 2r \geq 8$.

Note that for $(d+1)r/d \leq n \leq 2r-1$, there exist $k \in [2, d]$ and $\ell \in [n-r-1]$ such that $n = k(n-r) + l$ or $n = (k+1)(n-r)$. Thus Parts I and II include all these values of n.

Part I uses Katona's permutation method, Part II uses Baranyai's Theorem [1] on partitioning $X^{(r)}$ into matchings, and in Part III we proceed by induction on n. Frankl [9] established the upper bound $|\mathcal{A}| \leq \binom{n-1}{r-1}$ for $(d+1)r/d \leq n \leq 2r-1$; however, it is substantially more difficult to establish the case of equality in Theorem 2, which we achieve in Parts I and II of our proof.

Part I: $n = k(n-r) + \ell$**.**

In this part, we consider the case $n < 2r$ and $n = k(n-r)+\ell$, for some $k \in [2, d]$ and $\ell \in [n-r-1]$. For convenience, let $X = [n]$ and fix a (cyclic) permutation π of X. Let Q_i denote the interval $\{i, i+1, \ldots, i+r-1\}$ (modulo n), and let $\mathcal{A}_\pi$ denote the subfamily of $\mathcal{A}$ consisting of those sets $A \in \mathcal{A}$ such that $\pi(Q_i) = A$ for some i:

$$\mathcal{A}_\pi = \{\pi(Q_i) : \pi(Q_i) \in \mathcal{A}\}.$$

Claim 1. *Let π be any permutation. Then $|\mathcal{A}_\pi| \leq r$ with equality if and only if there exists m such that*

$$\mathcal{A}_\pi = \{\pi(Q_m), \pi(Q_{m+1}), \ldots, \pi(Q_{m+r-1})\}.$$

Proof. It is sufficient to prove Claim 1 for the identity permutation, since we may relabel X. Therefore $\mathcal{A}_\pi = \{Q_i : Q_i \in \mathcal{A}\}$. Without loss of generality, $Q_n \in \mathcal{A}_\pi$. For $j \in [n-r]$, let $P_j = \{i : i \equiv j \pmod{n-r}\} \cap [n]$, together with $\{n\}$ if $j \in [\ell+1, n-r]$. Thus $|P_j| \leq k+1 \leq d+1$. For each $j \in [n-r]$, there is an $i \in P_j$ such that $Q_i \notin \mathcal{A}_\pi$, otherwise $\bigcap_{i \in P_j} Q_i = \emptyset$. Thus $Q_i \notin \mathcal{A}_\pi$ for at least $n-r$ values of i, so $|\mathcal{A}_\pi| \leq r$.

Equality holds only if there is a unique x_j such that $Q_{x_j(n-r)+j} \notin \mathcal{A}_\pi$ for all $j \in [n-r]$. We now show $x_1 \geq x_2 \geq \ldots \geq x_{n-r} \geq x_1 - 1$. Let us illustrate the proof of this fact using Figures 1 and 2 below, where $y_j = x_j(n-r) + j$, and the box (i, j) represents the integer $(i-1)(n-r) + j$:

If $x_j < x_{j+1}$ for some $j \in [\ell]$, then, since $\ell \leq n-r-1$, the intersection of the $k+1$ intervals $Q_{(i-1)(n-r)+j}$, where (i,j) is a shaded box in Figure 1, is empty (this is the only place in Part I where we use $\ell \leq n-r-1$; the case $\ell = n-r-1$ is the content of Part II). This contradiction implies that $x_j \geq x_{j+1}$. In a similar way, $x_j \geq x_{j+1}$ for $j \in [\ell+1, n-r]$, using $Q_n \in \mathcal{A}_\pi$. Finally, if $x_{n-r} < x_1 - 1$, then the intersection of the intervals $Q_{(i-1)(n-r)+j}$, with (i,j) a shaded box in Figure 2, is empty, a contradiction. This proves that $\mathcal{A}_\pi$ has the required form.

Without loss of generality, we assume that for the identity permutation ι, $\mathcal{A}_\iota = \{Q_1, Q_2, \ldots, Q_r\}$.

Claim 2. *For each permutation π, $\mathcal{A}_\pi = \{\pi(Q_1), \pi(Q_2), \ldots, \pi(Q_r)\}$.*

Proof. Each permutation π of $X \backslash \{r\}$ is a product of transpositions. Therefore it suffices to show that if τ is a transposition in which r is a fixed point, then

$$\mathcal{A}_\tau = \{\tau(Q_1), \tau(Q_2), \ldots, \tau(Q_r)\}.$$

Suppose that τ transposes t and $t+1$, with $r \notin \{t, t+1\}$. Then Claim 1 implies that $\mathcal{A}_\tau = \{\tau(Q_m), \tau(Q_{m+1}), \ldots, \tau(Q_{m+r-1})\}$ for some $m \in [n]$. We show below that $m = 1$.

Case 1. $n \notin \{t, t+1\}$: Here $\tau(Q_1) = [r] = Q_1 \in \mathcal{A}$, and $\tau(Q_n) = \{1, \ldots, r-1, n\} = Q_n \notin \mathcal{A}$. Therefore $m = 1$.

Case 2. $t+1=n$: In this case $\tau(Q_i)=Q_i\in\mathcal{A}$ for each $i\in[r]\setminus\{n-r\}$. Consequently $\tau(Q_{n-r})\in\mathcal{A}$ as well, and therefore $m=1$.

Case 3. $t=n$: If $n<2r-1$, then $\tau(Q_r)=Q_r\in\mathcal{A}$ and $\tau(Q_{r+1})=Q_{r+1}\notin\mathcal{A}$. Therefore $m=1$. If $n=2r-1$, then $\tau(Q_i)=Q_i\in\mathcal{A}$ for $i=2,\ldots,r-1$. This leaves the posibilities $m=1,2,n$. However, $\tau(Q_{r+1})=Q_{r+1}\notin\mathcal{A}$, and $\tau(Q_n)=Q_n\notin\mathcal{A}$. Consequently, $\{\tau(Q_1),\tau(Q_r)\}\subset\mathcal{A}$ and $m=1$ again.

We now complete Part I. For each $A\in\mathcal{A}$, there are $\frac{1}{2}r!(n-r)!$ families $\mathcal{A}_\pi$ containing A. The total number of cyclic permutations of X is $(n-1)!/2$. By Claim 1, $|\mathcal{A}_\pi|\le r$ and therefore

$$\tfrac{1}{2}r!(n-r)!|\mathcal{A}|\le\tfrac{1}{2}r(n-1)!.$$

This establishes the upper bound $|\mathcal{A}|\le\binom{n-1}{r-1}$. By Claim 1, equality holds if and only if for every cyclic permutation π of X, we have $\mathcal{A}_\pi=\{\pi(Q_1),\pi(Q_2),\ldots,\pi(Q_r\}$. Set $x=r$. For any $A\subset(X\backslash\{x\})^{(r-1)}$, we may thus choose such a cyclic permutation π so that $\pi(Q_1)=A\cup\{x\}$. Therefore $A\cup\{x\}\in\mathcal{A}$, and $\mathcal{A}=X_x^{(r)}$ is the required family.

Part II: $n=k(n-r)$.

The argument here is different to that of Part I; we use a result of Baranyai [1], stating that the family $X^{(r)}$ may be partitioned into perfect matchings when r divides n. This result is only needed for the characterization of the extremal family $\mathcal{A}$. Recall that $\overline{\mathcal{A}}=\{X\backslash A:A\in\mathcal{A}\}$.

Claim 3. *If $A\in X^{(n-r)}\backslash\overline{\mathcal{A}}$, then $(X\backslash A)^{(n-r)}\subset\overline{\mathcal{A}}$.*

Proof. Pick $A'\in(X\backslash A)^{(n-r)}$. We will show that $A'\in\overline{\mathcal{A}}$. By Baranyai's Theorem, there is a partition of $X^{(n-r)}$ into perfect matchings $\mathcal{M}_1,\ldots,\mathcal{M}_t$ of size k, where $t=\frac{1}{k}\binom{n}{n-r}$. By relabelling X if necessary, we may assume that $\mathcal{M}_1\supset\{A,A'\}$. Since $\overline{\mathcal{A}}$ has no perfect matching, and $n=kr/(k-1)$,

$$|\overline{\mathcal{A}}|\le(k-1)t=\frac{k-1}{k}\binom{n}{n-r}=\frac{k-1}{k}\binom{n}{r}=\frac{k-1}{k}\frac{n}{r}\binom{n-1}{r-1}=\binom{n-1}{r-1}.$$

Therefore $|\mathcal{A}|=|\overline{\mathcal{A}}|=\binom{n-1}{r-1}$, and $|\overline{\mathcal{A}}\cap\mathcal{M}_i|=k-1$ for all i. Since $\mathcal{M}_1\supset\{A,A'\}$ and $A\notin\overline{\mathcal{A}}$, we must have $A'\in\overline{\mathcal{A}}$. Therefore Claim 3 is verified.

We now complete the proof of Theorem 2 for $n = k(n-r)$. Let $\mathcal{B} = X^{(n-r)} \backslash \overline{\mathcal{A}}$. Then $n(\mathcal{B}) = \frac{k}{k-1} r \geq 2(n-r)$ as $k \geq 2$ and $n < 2r$. Furthermore, $\mathcal{B}$ is an intersecting family, by Claim 3, and $|\mathcal{B}| = \binom{n}{n-r} - |\overline{\mathcal{A}}| = \binom{n-1}{n-r-1}$. By Theorem 1, $\mathcal{B} = X_x^{(n-r)}$ for some $x \in X$. This shows that $\mathcal{A} = X_x^{(r)}$, and Part II is complete.

Part III: $n \geq 2r$.

Throughout Part III, we assume $r \geq 4$. Addition of technical details in Claim 3 in the proof below accommodates the case $r = 3$. However, a short proof in this case was presented by weight counting techniques in Frankl and Füredi [11], which we revisit in Section 5.

We need the following notations.

For $\mathcal{A} \subset X^{(r)}$, let $V(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ and $n(\mathcal{A}) = |V(\mathcal{A})|$. For $Y \subset X$, we define $\mathcal{A} - Y = \{A \in \mathcal{A} : A \cap Y = \emptyset\}$. We also write $\overline{\mathcal{A}} = \{X \backslash A : A \in \mathcal{A}\}$. The following five definitions and the associated notations will be used repeatedly throughout the paper:

Sum of Families. The *sum of families* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t$, denoted $\sum_i \mathcal{A}_i$, is the family of all sets in each $\mathcal{A}_i$. Note that $\sum \mathcal{A}_i$ may have repeated sets, even if none of the $\mathcal{A}_i$ have repeated sets.

Trace of a Set. The *trace of a set* Y in $\mathcal{A}$ is defined by $\mathrm{tr}(Y) = \mathrm{tr}_{\mathcal{A}}(Y) = \{A \subset X : A \cup Y \in \mathcal{A}\}$. We define $\mathrm{tr}(\mathcal{A}) = \sum_{x \in X} \mathrm{tr}(x)$.

Degree of a Set. The *edge neighborhood of a set* Y is $\Gamma(Y) = \Gamma_{\mathcal{A}}(Y) = \{A \in \mathcal{A} : A \cap Y \neq \emptyset \text{ and } A \neq Y\}$, and the *degree of* Y is $\deg_{\mathcal{A}}(Y) = |\Gamma_{\mathcal{A}}(Y)|$. If $Y = \{y\}$, then we write y instead of $\{y\}$, and $\deg_{\mathcal{A}}(y) = |\Gamma_{\mathcal{A}}(y)| = |\mathcal{A}_y|$.

The families $\mathcal{S}_x$ and $\mathcal{L}_x$. Let $\mathcal{A}$ be an r-uniform family of sets in X and $x \in X$. Then we define

$$\mathcal{S}_x = \{Y \in \mathrm{tr}(x) : |\mathrm{tr}(Y)| = 1\} \qquad \text{and} \qquad \mathcal{L}_x = \mathrm{tr}(x) \backslash \mathcal{S}_x.$$

We write $\mathcal{S} = \sum_{x \in X} \mathcal{S}_x$ and $\mathcal{L} = \sum_{x \in X} \mathcal{L}_x = \mathrm{tr}(\mathcal{A}) \backslash \mathcal{S}$. Note that if $A \in \mathcal{L}_x$, then there exists $y \neq x$ such that $A \in \mathcal{L}_y$.

Paths and Connectivity. A *path* in $\mathcal{A}$ is a family $\mathcal{P}$ of sets $A_1, A_2, \ldots$ such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Family $\mathcal{A}$ is *connected* if every pair of vertices in $V(\mathcal{A})$ is contained in some path in $\mathcal{A}$. A *component* of $\mathcal{A}$ is a maximal non-empty connected subfamily of $\mathcal{A}$.

We begin with the following simple lemma. Recall that $\mathcal{B} - S = \{A \in \mathcal{B} : A \cap S = \emptyset\}$.

Lemma 8 *Let $\mathcal{B}_0$ be a finite family of sets. Then there exist disjoint sets $S_0, S_1, \ldots, S_{t-1} \in V(\mathcal{B}_0)$, such that the families $\mathcal{B}_i = \mathcal{B}_0 - \cup_{j=0}^{i-1} S_j$ for $i = 1, \ldots, t$ satisfy*

(1) $S_i \in \mathcal{B}_i$ *and* $\deg_{\mathcal{B}_i}(S_i) < d - 1$ *for every* $i < t$,

(2) $\deg_{\mathcal{B}_t}(S) \geq d - 1$ *for every* $S \in \mathcal{B}_t$.

Proof. For $i \geq 0$, if there exists a $T \in \mathcal{B}_i$ with $\deg_{\mathcal{B}_i}(T) < d - 1$, then set $S_i = T$. Form $\mathcal{B}_{i+1}$ and repeat for $i + 1$. If there is no such T, then set $i = t$ and stop. $\square$

For $n \geq 2r - 1$, we proceed by induction on n. The base case $n = 2r - 1$ has been proved in Part I. Suppose that $n \geq 2r$, $|\mathcal{A}| = \binom{n-1}{r-1}$, and $\mathcal{A}$ contains no nontrivial intersecting family of size $d + 1$. We will prove that $\mathcal{A} = X_y^{(r)}$ for some $y \in X$. This implies that if $\mathcal{A}' \subset X^{(r)}$ contains no nontrivial intersecting family of size $d + 1$, then $|\mathcal{A}'| \leq \binom{n-1}{r-1}$, by the following argument: If $|\mathcal{A}'| > \binom{n-1}{r-1}$, then $\mathcal{A}'$ contains an r-set R in addition to $X_y^{(r)}$ by our assumption. Now consider the subfamily consisting of all r-sets of $X_y^{(r)}$ intersecting R as well as R itself. This is clearly a nontrivial intersecting family, and it has size

$$1 + \binom{n-1}{r-1} - \binom{n-1-r}{r-1} > 1 + r > d + 1.$$

Consequently, $|\mathcal{A}'| \leq \binom{n-1}{r-1}$ as claimed.

Our approach is to show that there exists a vertex $x \in X$ with $\deg_{\mathcal{A}}(x) \leq \binom{n-2}{r-2}$. Subsequently, the family $\mathcal{A} - \{x\}$ has size at least $\binom{n-1}{r-1} - \binom{n-2}{r-2} = \binom{n-2}{r-1}$. By induction, equality holds and $\mathcal{A} - \{x\} = X_y^{(r)}$ for some $y \in X$; it is easy to see that every set in $\mathcal{A}$ containing x also contains y and $\mathcal{A}$ is the required family. Let us show that $\deg(x) \leq \binom{n-2}{r-2}$ if $|\mathcal{L}_x|$ is a maximum.

Claim 1. $|\mathcal{L}_x| > \binom{n-3}{r-2}$.

Proof. Note that $r|\mathcal{A}| = \sum_y \deg(y) = \sum_y |\mathcal{S}_y| + \sum_y |\mathcal{L}_y|$. By the choice of x, this is at most $|\mathcal{S}| + n|\mathcal{L}_x|$. Also, $\mathcal{S} \cap \mathcal{L}_x = \emptyset$, so $|\mathcal{S}| \leq \binom{n}{r-1} - |\mathcal{L}_x|$. Consequently

$$(n-1)|\mathcal{L}_x| \geq r|\mathcal{A}| - \binom{n}{r-1} = r\binom{n-1}{r-1} - \binom{n}{r-1} > (n-1)\binom{n-3}{r-2},$$

where the last inequality follows from a short computation and the fact that $r \geq 4$. Dividing by $n - 1$, we obtain $|\mathcal{L}_x| > \binom{n-3}{r-2}$.

Applying Lemma 8 to $\mathcal{L}_x = \mathcal{B}_0$, let $(S_0, S_1, \ldots, S_{t-1})$ be the sets in $V(\mathcal{L}_x)$ satisfying (1) and (2), and let $\mathcal{B}_i$ be as in Lemma 8. Note that $\mathcal{B}_t \neq \emptyset$, since otherwise

$$|\mathcal{L}_x| \leq \sum_{i=0}^{t-1} (\deg_{\mathcal{B}_i}(S_i) + 1) \leq t(d-1) \leq \frac{n-1}{r-1}(r-2) < n - 1, \qquad (2)$$

contradicting Claim 1. Let $\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_s$ be the components of $\mathcal{B}_t$. We let $\mathcal{K}'_i$ denote the union of $\mathcal{K}_i$ and the family of all sets in $\mathcal{S}_x$ intersecting $V(\mathcal{K}_i)$.

Claim 2. *The family $\mathcal{K}'_i$ is an intersecting family.*

Proof. Suppose, for a contradiction, that $\mathcal{K}'_i$ contains disjoint sets A_0, B_0. Since $\mathcal{K}_i$ is connected, $\mathcal{K}_i \cup \{A_0, B_0\}$ is also connected. Choose a path $A_0, A_1, A_2, \ldots, B_0$ in $\mathcal{K}_i \cup \{A_0, B_0\}$ (possibly $A_2 = B_0$). Then $A_1 \in \mathcal{K}_i$. Lemma 8 part (2) implies that $\deg_{\mathcal{K}_i}(A_1) \geq d-1$, hence (if $d \geq 4$) there exist sets $C_1, C_2, \ldots, C_{d-3} \in \mathcal{K}_i \backslash \{A_0, A_2\}$ each of which intersects A_1. By definition of $\mathcal{L}_x$, there exists $y \in X \backslash \{x\}$ such that $A_1 \in \mathcal{L}_y$. Consequently,

$$\{A_0 \cup x, A_1 \cup x, A_2 \cup x, C_1 \cup x, \ldots, C_{d-3} \cup x, A_1 \cup y\}$$

is a non-trivial intersecting family of size $d+1$ in $\mathcal{A}$, since $A_0 \cap A_2 = \emptyset$, a contradiction.

Claim 3. *$\mathcal{L}_x = \mathcal{K}_0$ and $n(\mathcal{L}_x) \geq n-2$.*

Proof. We first show $t = s = 0$, so that $\mathcal{L}_x = \mathcal{K}_0$. For a contradiction, suppose $t > 0$ or $s > 0$. By Claim 2, $\mathcal{K}_i \subset \mathcal{K}'_i$ is an intersecting family of $(r-1)$-sets. Therefore, for $n(\mathcal{K}_i) \geq 2(r-1)$, the Erdős-Ko-Rado theorem shows $|\mathcal{K}_i| \leq \binom{n(\mathcal{K}_i)-1}{r-2} \leq \binom{n(\mathcal{K}_i)}{r-2}$. If $n(\mathcal{K}_i) \leq 2r-3$, then $|\mathcal{K}_i| \leq \binom{n(\mathcal{K}_i)}{r-1}$, and this is at most $\binom{n(\mathcal{K}_i)}{r-2}$. Since $n(\mathcal{K}_i) \geq r-1$ for $i \leq s$, convexity of binomial coefficients yields

$$\sum_{i=0}^{s} |\mathcal{K}_i| \leq \sum_{i=0}^{s} \binom{n(\mathcal{K}_i)}{r-2} = \binom{n(\mathcal{K}_0)}{r-2} + \sum_{i=1}^{s} \binom{n(\mathcal{K}_i)}{r-2} \leq \binom{[\sum_{i=0}^{s} n(\mathcal{K}_i)] - s(r-1)}{r-2} + \sum_{i=1}^{s} \binom{r-1}{r-2}.$$

Recalling that $n(\mathcal{L}_x) = \sum_{i=0}^{s} n(\mathcal{K}_i) + t(r-1)$, we obtain

$$\sum_{i=0}^{s} |\mathcal{K}_i| \leq \binom{n(\mathcal{L}_x) - s(r-1) - t(r-1)}{r-2} + s(r-1).$$

By the argument giving the first two inequalities of (2), and $d-1 \leq r-1$, we have

$$|\mathcal{L}_x| = \sum_{i=0}^{s} |\mathcal{K}_i| + \sum_{i=0}^{t-1} (\deg_{\mathcal{B}_i}(S_i) + 1) \leq \binom{n(\mathcal{L}_x) - (s+t)(r-1)}{r-2} + s(r-1) + t(r-1).$$

If $s + t \geq 1$ then, by convexity of binomial coefficients,

$$|\mathcal{L}_x| \leq \binom{n(\mathcal{L}_x) - (r-1)}{r-2} + (r-1) \leq \binom{n-r}{r-2} + (r-1).$$

As $n \geq 2r$ and $r \geq 4$, this contradicts Claim 1. Thus $s = t = 0$, and $\mathcal{L}_x$ consists of one component, $\mathcal{K}_0$.

We now show that $n(\mathcal{L}_x) \geq n - 2$. By the arguments above, $|\mathcal{K}_0| \leq \binom{n(\mathcal{K}_0)}{r-2}$. Therefore, by Claim 1, $n(\mathcal{K}_0) = n(\mathcal{L}_x) \geq n - 2$. This completes the proof of Claim 3.

We now complete Part III and the proof of Theorem 2, by showing that $\deg(x) \leq \binom{n-2}{r-2}$. By Claim 2, $\mathcal{K}_0'$ is an intersecting family. Since $n(\mathcal{K}_0) \geq n - 2 > n - r + 1$, $\mathrm{tr}(x) = \mathcal{K}_0'$ so $\mathrm{tr}(x)$ is itself an intersecting family of $(r-1)$-sets. As $n - 1 \geq n(\mathcal{K}_0') \geq n(\mathcal{K}_0) \geq n - 2 \geq 2(r-1)$, the Erdős-Ko-Rado theorem implies that

$$\deg(x) = |\mathcal{K}_0'| = |\mathrm{tr}(x)| \leq \binom{n-2}{r-2}.$$

This completes the proof of Theorem 2. □

3 Proof of Theorem 3.

Part III of the proof of Theorem 2 can be extended to the case $r = 3$ and $2 \leq d \leq 6$ by addition of some technical details. However, Chvátal [3] and Frankl and Füredi [11] already settled the case $r = 3$ and $d = 2$ so we do not consider this case here. In fact, from the proof below, it follows that for $2 \leq d \leq 6$ and $n \geq 15$, a family $\mathcal{A} \subset X^{(3)}$ containing no non-trivial intersecting family of size $d + 1$ has at most $\binom{n-1}{2}$ members, with the equality as in Theorem 2.

We now prove Theorem 3, employing the weight counting methods of Frankl and Füredi.

Proof of Theorem 3. Let $\mathcal{A} \subset X^{(3)}$ and suppose $\mathcal{A}$ contains no non-trivial intersecting family of size $d + 1$. Following Frankl and Füredi, the *weight of a set* $A \in \mathcal{A}$ is defined by

$$\omega(A) = \sum_{\{x,y\} \subset A} \frac{1}{|\mathrm{tr}\{x, y\}|}.$$

Then

$$\begin{aligned}\sum_{A \in \mathcal{A}} \omega(A) &= \sum_{A \in \mathcal{A}} \sum_{\{x,y\} \subset A} \frac{1}{|\mathrm{tr}(x, y)|} \\ &\leq \sum_{\{x,y\} \in X} \sum_{\substack{A \in \mathcal{A} \\ \{x,y\} \in A}} \frac{1}{|\mathrm{tr}(x, y)|} \leq \sum_{\{x,y\} \in X} 1 = \binom{n}{2}.\end{aligned}$$

Equality holds if and only if every pair in X is contained in some set in $\mathcal{A}$.

As $\mathcal{A}$ contains no non-trivial intersecting family of size $d+1$, $|\mathrm{tr}\{x,y\}| = 1$ for some $\{x,y\} \in A$ or $\sum_{x,y\in A} |\mathrm{tr}\{x,y\}| \leq d+2$. This implies that for all $A \in \mathcal{A}$,

$$\omega(A) \geq \min\Big\{1 + \tfrac{2}{n-2}, \left\lfloor\tfrac{(d+2)}{3}\right\rfloor^{-1} + \left\lfloor\tfrac{(d+3)}{3}\right\rfloor^{-1} + \left\lfloor\tfrac{(d+4)}{3}\right\rfloor^{-1}\Big\}.$$

For $d \geq 7$, the second term is smaller (in fact, less than 1). Therefore

$$\sum_{A\in\mathcal{A}} \omega(A) \geq \Big(\left\lfloor\tfrac{(d+2)}{3}\right\rfloor^{-1} + \left\lfloor\tfrac{(d+3)}{3}\right\rfloor^{-1} + \left\lfloor\tfrac{(d+4)}{3}\right\rfloor^{-1}\Big)|\mathcal{A}|.$$

Together with $\sum_{A\in\mathcal{A}} \omega(A) \leq \binom{n}{2}$, this gives the upper bound on $|\mathcal{A}|$ in Theorem 3, which is for $d \geq 7$.

For the lower bound in Theorem 3, it suffices to show that every non-trivial intersecting family of size $d+1 \geq 11$ contains a pair in at least $\lceil \frac{d}{3} \rceil$ of its edges. Then a Steiner $(n, 3, \lceil \frac{d}{3} \rceil - 1)$-system, for those n for which such a structure exists, does not contain such an intersecting family.

Lemma 9 *Let $\mathcal{F} \subset X^{(3)}$ be a non-trivial intersecting family with $|\mathcal{F}| \geq 11$. Then there exist distinct elements $x, y \in X$ such that*

$$|\mathrm{tr}_{\mathcal{F}}\{x,y\}| \geq \frac{1}{3}(|\mathcal{F}| - 1).$$

Proof. For $a, b \in X$, we let $d(a,b) = |\mathrm{tr}_{\mathcal{F}}\{a,b\}|$. First suppose that there exist $x, y \in X$ with $d(x,y) \geq 3$. Now let $u, v, w \in \mathrm{tr}_{\mathcal{F}}\{x,y\}$. Throughout the proof, we assume $d(x,y) \leq \lceil \frac{1}{3}(|\mathcal{F}| - 1) \rceil - 1$, otherwise we are done. Let $L_x = \mathrm{tr}_{\mathcal{F}}(x) - \{y\} = \{A \in \mathrm{tr}_{\mathcal{F}}(x) : y \notin A\}$ and let $L_y = \mathrm{tr}_{\mathcal{F}}(y) - \{x\} = \{A \in \mathrm{tr}_{\mathcal{F}}(y) : x \notin A\}$ Since $\mathcal{F}$ is an intersecting family,

$$A \cap B \neq \emptyset \text{ for every } A \in L_x \text{ and } B \in L_y \quad (*).$$

Case 1: *L_x contains a matching of size three.*

In this case, L_x consists of three stars with distinct centers in X. By $(*)$, every pair in L_y intersects all three centers. This implies $L_y = \emptyset$. As $\mathcal{F}$ is a non-trivial intersecting family, there is a triple in $\mathcal{F}$ disjoint from x. Since $L_y = \emptyset$ and $d(x,y) \geq 3$, this triple must be $\{u,v,w\}$ and $d(x,y) = 3$. Since $\mathcal{F}$ is intersecting, the centers of the three stars must also be u, v, w. Now every $F \in \mathcal{F} - \{y\}$ with $F \neq \{u,v,w\}$ contains x. Therefore, assuming $d(x,u) \geq d(x,v) \geq d(x,w)$, we find

$$d(x,u) \geq \frac{1}{3}(|\mathcal{F}|-4)+1 = \frac{1}{3}(|\mathcal{F}|-1).$$

This completes the proof in Case 1.

Case 2: *L_x and L_y contain no matching of size three.*

It is not hard to see by $(*)$ that $|L_x|+|L_y| \leq 2(\lceil\frac{1}{3}(|\mathcal{F}|-1)\rceil-1)+1$, with equality if and only if L_x consists of a pair of stars of size $\lceil(|\mathcal{F}|-1)/3\rceil-1$ with distinct centers a,b and L_y consists of the pair $\{a,b\}$. Then $|\mathcal{F}|-|L_x|-|L_y|-d(x,y) \geq 2$, unless $|\mathcal{F}| = 3k+2$, $k \geq 3$ and L_x and L_y are as described above. By $(*)$, any triple in $\mathcal{F}-\{x,y\}$ contains u,v and w. This shows $|\mathcal{F}-\{x,y\}| = 1$, and therefore $|\mathcal{F}| = 3k+2$, $k \geq 3$. In this case, $\{u,v,w\} \in \mathcal{F}$ and $a,b \in \{u,v,w\}$, since $\mathcal{F}$ is intersecting. Therefore $d(a,x) \geq 4$ (and also $d(b,x) \geq 4$), completing the proof in Case 2.

If every pair $a,b \in X$ has $d(a,b) \leq 2$, then the arguments in Cases 1 and 2 still apply to give a contradiction with $|\mathcal{F}| \geq 11$, since in this case $|\mathcal{F}| = 8$. This completes the proof of the Lemma. $\square$

4 Proof of Theorem 7.

Let $\mathcal{A}$ be a family of subsets of X containing no non-trivial intersecting family of size $d+1$. We prove Theorem 7 by showing that $\mathcal{A}' = \mathcal{A} \cap X^{(\geq 2)}$ has size at most $2^{n-1}-1$, with equality if and only if $\mathcal{A}' = X_x^{(\geq 2)}$ for some $x \in X$. Theorem 7 is proved in two parts. Part I deals with the case $d=2$, by induction on $n \geq 1$. In Part II, we use Part I to prove Theorem 7 for $d \geq 3$.

Part I: $d=2$

Theorem 7 is easily verified for $n \leq 3$. Now let $n \geq 4$ and $w \in X$.

Case 1: For every partition of $X\backslash\{w\}$ into two non-empty sets Y and Z, there exists a set $A \in \mathcal{A}'-\{w\}$ such that $A \cap Y \neq \emptyset$ and $A \cap Z \neq \emptyset$. Then, for each partition of $X\backslash\{w\}$ into sets Y and Z, either $Y \cup \{w\} \notin \mathcal{A}$ or $Z \cup \{w\} \notin \mathcal{A}'$ – otherwise $\mathcal{A}'$ contains a triangle. Therefore $\deg_{\mathcal{A}'}(w) \leq 2^{n-2}$. By induction, $\mathcal{A}'' = \mathcal{A}'-\{w\}$ has size at most $2^{n-2}-1$, with equality if and only if $\mathcal{A}'' = (X\backslash\{w\})_x^{(\geq 2)}$ for some $x \in X-\{w\}$. Thus

$$|\mathcal{A}'| = \deg_{\mathcal{A}'}(w)+|\mathcal{A}''| \leq 2^{n-2}+(2^{n-2}-1) = 2^{n-1}-1.$$

Now suppose that equality holds above. We will show that every set in $\mathcal{A}'$ containing w also contains x. Suppose on the contrary that $w \in S \in \mathcal{A}'$ and $x \notin S$. Among all such S, choose the one of minimum size, call it S_0. Let T be another set containing w. By the choice of S_0, either $T \supset S_0$, or there exist $t \in T - S$, and $s \in S - T$ (possibly $t = x$). In the latter case, $\{x, s, t\}, S, T$ form a triangle (replace $\{x, s, t\}$ by $\{s, t\}$ if $t = x$). We may therefore assume that every set in $\mathcal{A}'$ containing w also contains S_0. Hence $2^{n-2} = \deg_{\mathcal{A}'}(w) \leq 2^{n-|S_0|-1}$ from which we conclude that $S_0 = \{s\}$, and $E \cup \{w\} \in \mathcal{A}'$ for every $E \subset X \backslash \{w, s\}$. Since $|X| \geq 4$, there exist distinct a, b for which $\{w, s, a\}$ and $\{w, s, b\}$ lie in $\mathcal{A}'$. Together with $\{x, a, b\}$ (or just $\{a, b\}$ if $a = x$ or $b = x$) this once again forms a triangle.

Case 2: There exists a partition of $X \backslash \{w\}$ into two nonempty sets Y and Z such that no member of $\mathcal{A}'$ in $X \backslash \{w\}$ contains an element of both Y and Z. By induction, at most $2^{|Y|} - 1$ elements of $\mathcal{A}'$ are contained in $Y \cup \{w\}$, and similarly for Z. The number of sets which contain an element of Y and an element of Z is, by the choice of Y and Z, at most $2^{n-1} - 1 - (2^{|Y|} - 1) - (2^{|Z|} - 1)$. Therefore

$$|\mathcal{A}'| \leq (2^{|Y|} - 1) + (2^{|Z|} - 1) + (2^{n-1} - 1 - (2^{|Y|} - 1) - (2^{|Z|} - 1)) = 2^{n-1} - 1.$$

If equality holds, then by induction there exist $y \in Y \cup \{w\}, z \in Z \cup \{w\}$ with $(Y \cup \{w\})_y^{(\geq 2)} \cup (Z \cup \{w\})_z^{(\geq 2)} \subset \mathcal{A}'$. Since $|X| \geq 4$, we may assume by symmetry that $|Y| \geq 2$. We next show that $y = w$. Observe that for every set $S \subset X \backslash \{w\}$ containing an element of both Y and Z, we have $S \cup \{w\} \in \mathcal{A}'$. If $y \neq w$, then $\{y, a\} \in \mathcal{A}'$ for some $a \in Y \backslash \{y\}$. The set $\{y, a\}$ together with $\{y, w\}$ and $\{w, a, b\}$ for some $b \in Z$ forms a triangle. Consequently $y = w$, and $z = w$ as well unless $Z = \{z\}$. But in this case $(Z \cup \{w\})_z^{(\geq 2)} = (Z \cup \{w\})_w^{(\geq 2)}$, therefore $\mathcal{A}' = X_w^{(\geq 2)}$.

Part II: $d \geq 3$

Define a function f on the positive integers by $f(1) = f(2) = f(3) = 1$, and for $n \geq 4$,

$$f(n) = \max\{0, f(n-3) + d - 2^{n-4}\}. \quad (*)$$

It is easy to see that if $n \geq 4$, and $f(n) \geq 0$, then

$$f(n) = 1 + \left(\left\lceil \frac{n}{3} \right\rceil - 1 \right) d - \sum_{i=0}^{\lfloor (n-4)/3 \rfloor} 2^{n-4-3i}.$$

Set $n_d = \log_2 d + \log_2 \log_2 d + 2$. An easy calculation now shows that $f(n) = 0$ whenever $n \geq n_d$ and $f(n) > f(n-3) + d - 2^{n-4}$ when $n > n_d$.

In this part of the proof, we proceed by induction on $n \geq 1$, with the following hypothesis: Let $\mathcal{A}' \subset X^{(\geq 2)}$ contain no non-trivial intersecting family of size $d+1$. Then $|\mathcal{A}'| \leq 2^{n-1} - 1 + f(n)$.

For $n \leq 3$, the result is true as

$$|\mathcal{A}'| \leq |X^{(\geq 2)}| = 2^n - n - 1 \leq 2^{n-1} - 1 + f(n).$$

Now suppose that $n \geq 4$. By Part I, we may assume $\mathcal{A}'$ contains a triangle $\mathcal{F} = \{F_1, F_2, F_3\}$, otherwise the proof is complete.

Let x, y, z be elements in $F_1 \cap F_2$, $F_2 \cap F_3$ and $F_3 \cap F_1$ respectively. Then at most d sets in $\mathcal{A}'$ intersect $\{x, y, z\}$ in at least two points, otherwise $\mathcal{F}$ together with another $d-2$ of these sets forms a non-trivial intersecting family of size $d+1$. The total number of sets in $\mathcal{A}'$ intersecting $\{x, y, z\}$ is therefore at most $3 \cdot 2^{n-3} + d$. Let $\mathcal{A}'' = \mathcal{A}' - \{x, y, z\}$. Then

$$|\mathcal{A}'| \leq |\mathcal{A}''| + 3 \cdot 2^{n-3} + d.$$

As $\mathcal{A}''$ contains no non-trivial family of size $d+1$, the induction hypothesis shows $|\mathcal{A}''| \leq 2^{n-4} - 1 + f(n-3)$. This gives

$$\begin{aligned} |\mathcal{A}'| &\leq 2^{n-4} - 1 + f(n-3) + 3 \cdot 2^{n-3} + d \\ &= 2^{n-1} - 1 + f(n) - (f(n) - f(n-3) - d + 2^{n-4}) \\ &\leq 2^{n-1} - 1 + f(n), \qquad (**) \end{aligned}$$

where the last inequality follows from $(*)$. By the choice of n_d, we know that $f(n) = 0$ for $n \geq n_d$, so $|\mathcal{A}'| \leq 2^{n-1} - 1$ for $n \geq n_d$, completing the proof of the upper bound in Theorem 7.

Now suppose that $|\mathcal{A}'| = 2^{n-1} - 1$ and $n > n_d$. Then the inequality $(**)$ is strict. This gives the contradiction $|\mathcal{A}'| < 2^{n-1} - 1$. Consequently $\mathcal{A}'$ contains no triangle and Part I of the proof applies to give the case of equality. □

5 Acknowledgments

Most of the work for this paper was done while both authors were Postdoctoral Researchers in the Theory Group at Microsoft Research. The authors also thank a referee for very helpful comments and minor corrections on an earlier draft, and Yi Zhao for carefully reading the proof of Theorem 7.

References

[1] Baranyai, Zs., On the factorization of the complete uniform hypergraph. Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I, pp. 91–108. *Colloq. Math. Soc. Janos Bolyai*, **10** North-Holland, Amsterdam, 1975.

[2] Bermond, J.-C., Frankl, P., On a conjecture of Chvátal on m-intersecting hypergraphs. *Bull. London Math. Soc.* **9** (1977), no. 3, 310–312.

[3] Chvátal, V., An extremal set-intersection theorem. *J. London Math. Soc. (2)* **9** (1974/75), 355–359.

[4] Bollobás, B., Duchet, P., On Helly families of maximal size. J. Combin. Theory Ser. B **35** (1983), no. 3, 290–296.

[5] Bollobás, B., Duchet, P., Helly families of maximal size. J. Combin. Theory Ser. A **26** (1979), no. 2, 197–200.

[6] Csákány, R., Kahn, J., A homological approach to two problems on finite sets. (English. English summary) J. Algebraic Combin. **9** (1999), no. 2, 141–149.

[7] Erdős, P., Topics in combinatorial analysis. *Proc. Second Louisiana Conf. on Comb., Graph Theory and Computing* (R. C. Mullin et al., Eds.) pp 2–20, Louisiana State Univ., Baton Rouge 1971.

[8] Erdős, P., Ko, H., Rado R., Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)* **12** (1961) 313–320.

[9] Frankl, P., On Sperner families satisfying an additional condition. *J. Combinatorial Theory Ser. A* **20** (1976), no. 1, 1–11.

[10] Frankl, P., On a problem of Chvátal and Erdős on hypergraphs containing no generalized simplex. *J. Combin. Theory Ser. A* **30** (1981), no. 2, 169–182.

[11] Frankl, P., Füredi, Z., Exact solution of some Turán-type problems. *J. Combin. Theory Ser. A* **45** (1987), no. 2, 226–262.

[12] O. P. Lossers, P. Erdos and E. Milner: P. 190, Canad. Math Bull 16 (1973), 145-146.

[13] Mulder, H. M., The number of edges in a k-Helly hypergraph. Combinatorial mathematics (Marseille-Luminy, 1981), 497–501, North-Holland Math. Stud., **75**, North-Holland, Amsterdam, 1983.

[14] Ray-Chaudhuri, D., Wilson, R., The existence of resolvable block designs. Survey of combinatorial theory (*Proc. Internat. Sympos.*, Colorado State Univ., Fort Collins, Colo., 1971), pp. 361–375. North-Holland, Amsterdam, 1973.

[15] Tuza, Zs., Helly-type hypergraphs and Sperner families. European J. Combin. **5** (1984), no. 2, 185–187.

[16] Tuza, Zs., Helly property in finite set systems. J. Combin. Theory Ser. A **62** (1993), no. 1, 1–14.