The Turán number of $F_{3,3}$

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Abstract

Let $F_{3,3}$ be the 3-graph on 6 vertices, labelled $abcxyz$, and 10 edges, one of which is abc, and the other 9 of which are all triples that contain 1 vertex from abc and 2 vertices from xyz. We show that for all $n \geq 6$, the maximum number of edges in an $F_{3,3}$-free 3-graph on n vertices is $\binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$. This sharpens results of Zhou [9] and of the second author and Rödl [7].

1 Introduction

The *Turán number* $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free r-graph on n vertices.[1] It is a long-standing open problem in Extremal Combinatorics to understand these numbers for general r-graphs F. For ordinary graphs ($r = 2$) the picture is fairly complete, although there are still many open problems, such as determining the order of magnitude for Turán numbers of bipartite graphs. However, for $r \geq 3$ there are very few known results. Having solved the problem for the complete graph $F = K_t$, Turán [8] posed the natural question of determining $\mathrm{ex}(n, F)$ when $F = K_t^r$ is a complete r-graph on t vertices. To date, no case with $t > r > 2$ of this question has been solved, even asymptotically. Despite the lack of progress on the Turán problem for complete hypergraphs, there are certain hypergraphs for which the problem has been solved asymptotically, or even exactly; we refer the reader to the survey [5]. While it would be more satisfactory to have a general theory, we may hope that this will develop out of the methods discovered in solving isolated examples.

The contribution of this paper is a short complete solution to the Turán problem for the following 3-graph. Let $F_{3,3}$ be the 3-graph on 6 vertices, labelled $abcxyz$, and 10 edges, one of which is abc, and the other 9 of which are all triples that contain 1 vertex from abc and 2 vertices from xyz. A lower bound for $\mathrm{ex}(n, F_{3,3})$ is given by the following construction. Let $B(n)$ denote the balanced complete bipartite 3-graph, which is obtained by partitioning a set of n vertices into parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, and taking as edges all triples that are not contained within either part. Let

$$b(n) = \binom{n}{3} - \binom{\lfloor n/2 \rfloor}{3} - \binom{\lceil n/2 \rceil}{3}$$

denote the number of edges in $B(n)$. Since $F_{3,3}$ is not 2-colourable, $B(n)$ is $F_{3,3}$-free. We prove the following result.

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[1] An *r-graph* (or *r-uniform hypergraph*) G consists of a vertex set and an edge set, each edge being some r-set of vertices. We say G is *F-free* if it does not have a (not necessarily induced) subgraph isomorphic to F.

Theorem 1.1. *For all $n \geq 6$, we have* $\mathrm{ex}(n, F_{3,3}) = b(n)$.

The Turán problem for $F_{3,3}$ was previously studied by the second author and Rödl [7], who obtained the asymptotic result $\mathrm{ex}(n, F_{3,3}) = (1 + o(1))b(n)$. Another related problem is the following result of Zhou [9]. Say that two vertices x, y in a 3-graph G are *t-connected* if there are vertices a, b, c such that every triple with 2 vertices from abc and 1 from xy is an edge. Say that xyz is a *t-triple* if xyz is an edge and each pair in xyz is t-connected. A 3-graph is a t-triple if it contains a t-triple xyz. For example, K_5^3 is a t-triple (this is the motivation for the definition). The result of [9] is that the unique largest 3-graph on n vertices with no t-triple is complete bipartite. Note that $F_{3,3}$ is a t-triple, so Theorem 1.1 strengthens Zhou's extremal result (but not the classification of the extremal example; see Section 4).

Our proof uses the link multigraph method introduced by de Caen and Füredi [2]. There are now a few examples where this method has been used to obtain asymptotic results, or exact results for n sufficiently large. We used it in [6] to obtain an exact result for cancellative 3-graphs for all n, and an exact result for the configuration $F_5 = \{123, 124, 345\}$ for $n \geq 33$. However, until recently there were no known applications to an exact result for all n with a single forbidden hypergraph. The result in this paper gives such an application; another was given very recently by Goldwasser [4], who obtained an exact result for F_5 for all n.

In the next section we describe the link multigraph construction and state a lemma of Bondy and Tuza that applies to such multigraphs. We use this to prove Theorem 1.1 in Section 3. The final section contains some concluding remarks about the characterization of equality.

2 A multigraph lemma

The proof of Theorem 1.1 will use the following construction of a multigraph from a 3-graph G. Suppose S is a set of vertices in G. The 'link multigraph' of S has vertex set $X = V \setminus S$ and edge set $M = \sum_{a \in S} G(a)[X]$. Here we write $G(a) = \{xy : axy \in G\}$, denote the restriction to X by $[X]$, and use summation to denote multiset union. Thus we obtain a multigraph M in which each pair of vertices has multiplicity between 0 and $|S|$. Furthermore, we may regard each edge of M as being 'coloured' by a vertex in S (an edge may have several colours). We write $w(xy)$ for the multiplicity of the pair xy in M.

Now suppose M is any multigraph on n vertices (not necessarily as above). Write $w(xy)$ for the multiplicity of a pair xy in M, and write $e(M)$ for the sum of $w(xy)$ over all (unordered) pairs of vertices in M. For any $S \subseteq V(M)$ let $i(S)$ denote the sum of $w(xy)$ over all pairs of vertices that contain at least one vertex of S. If $S = \{x\}$ consists of a single vertex then $i(S) = d(x)$ is the weighted degree of x. Define

$$m(n) = \begin{cases} \frac{3n^2}{2} - n & \text{if } n \text{ is even,} \\ \frac{3n^2 - 1}{2} - n & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 2.1. (Bondy-Tuza [1]) *Suppose M is a multigraph on n vertices with $0 \leq w(xy) \leq 4$ for every pair xy and $w(xy) + w(xz) + w(yz) \leq 10$ for every triple xyz. Then $e(M) \leq m(n)$.*

Note the following two examples where equality holds in Lemma 2.1. (There are other similar examples; all cases where equality holds were characterized in [1] and [3].)

1. Define a multigraph $M_1(n)$ on n vertices as follows. Let $A \cup B$ be a balanced partition of the vertex set. Let crossing pairs have multiplicity 4 and pairs inside each part have multiplicity 2. If n is even then $e(M_1(n)) = 2\binom{n}{2} + 2(n/2)^2 = 3n^2/2 - n$. If n is odd then $e(M_1(n)) = 2\binom{n}{2} + 2\frac{n^2-1}{4} = \frac{3n^2-1}{2} - n$.

2. Define a multigraph $M_2(n)$ on n vertices as follows. Let all pairs have multiplicity 3 except for a maximum size matching of multiplicity 4. If n is even then $e(M_2(n)) = 3\binom{n}{2} + n/2 = 3n^2/2 - n$. If n is odd then $e(M_2(n)) = 3\binom{n}{2} + \frac{n-1}{2} = \frac{3n^2-1}{2} - n$.

The following two calculations will also be useful. Note that $M_2(n-1)$ can be obtained from $M_2(n)$ by deleting a vertex, which can be any vertex if n is even, but must be the vertex not incident to an edge of multiplicity 4 when n is odd. Then $m(n) - m(n-1)$ is equal to the number of edges removed, which is $3(n-2) + 4 = 3(n-1) + 1$ when n is even, or $3(n-1)$ when n is odd.

Next consider a copy of $B(n)$ with parts A and B. Construct a copy of $B(n-4)$ by removing vertices wx from A and yz from B. Then we have $b(n) - b(n-4) = i(wxyz)$, where similarly to our multigraph notation, we let $i(S)$ denote the number of edges that contain at least one vertex of S. We can count $i(wxyz)$ as follows. There are 4 edges of $B(n)$ contained in $wxyz$. Next consider edges with 2 vertices in $wxyz$. The 4 crossing pairs wy, wz, xy, xz form an edge with each of the $n-4$ vertices of $V \setminus \{w, x, y, z\}$, so contribute $4(n-4)$ edges. The pairs wx and yz form an edge with any of the vertices in the other part, so contribute $n-4$ edges (this holds whether n is even or odd). Finally, note that the link multigraph of $wxyz$ in $B(n)$ is precisely $M_1(n-4)$. Thus the number of edges with 1 vertex in $wxyz$ is $m(n-4)$. Then we have

$$b(n) - b(n-4) = m(n-4) + 5(n-4) + 4.$$

3 Proof of Theorem 1.1

We start with the lower bound. We have already described the construction $B(n)$, which is 2-colourable. It is easy to check that $F_{3,3}$ is not 2-colourable (we leave the details to the reader), hence $B(n)$ is $F_{3,3}$-free. This shows that $\mathrm{ex}(n, F_{3,3}) \geq b(n)$ for all $n \geq 6$.

The main task in the proof is to establish the upper bound. We prove the following statement by induction on n:

- Suppose G is an $F_{3,3}$-free 3-graph on $n \geq 1$ vertices. Then $e(G) \leq b(n)$, unless $n = 5$, in which case $e(G) \leq 10$.

Note that this statement is trivial for $n \leq 5$, as $B(n)$ is complete for $n \leq 4$ so $b(n) = \binom{n}{3}$ for these values of n, and for $n = 5$ the statement allows $e(K_5^3) = 10$ edges. Furthermore, the bound holds for $n = 6$, as $B(6)$ is obtained by deleting 2 edges from K_6^3, and it is clear that if one only deletes one edge from K_6^3 then there is a copy of $F_{3,3}$. Moreover, $B(6)$ is the unique $F_{3,3}$-free 3-graph on 6 vertices with $\binom{6}{3} - 2 = 18$ edges. To see this, we exhibit appropriate edges in the complement of $F_{3,3}$ as defined above: xab and yac are not in $F_{3,3}$ and intersect in 1 vertex, whereas xab and xac are not in $F_{3,3}$ and intersect in 2 vertices.

Now suppose for a contradiction that $n \geq 7$ with $n \neq 9$ and G is an $F_{3,3}$-free 3-graph on n vertices with $e(G) = b(n) + 1$. (We will need to modify the argument in the case $n = 9$.) We start by finding a copy of K_4^3 in G. For this we use the following averaging argument. Given a 3-graph H on m vertices, let $d(H) = e(H)\binom{m}{3}^{-1}$ denote the density of H. A simple calculation shows that $d(G)$ is the average of $d(G \setminus v)$ over all vertices v of G. Note that deleting a vertex from $B(n)$ leaves a complete bipartite 3-graph on $n-1$ vertices; it is not necessarily balanced, but certainly has at most $b(n-1)$ edges. It follows that $d(B(n)) \le d(B(n-1))$, i.e. $d(B(n))$ is non-increasing in n. Since $d(B(n)) \to 3/4$ as $n \to \infty$ we have $d(B(n)) \ge 3/4$ for all n. Since $e(G) > b(n)$ we have $d(G) > 3/4$. Averaging again, we see that there is a set $abcd$ of 4 vertices where $d(G[abcd]) > 3/4$. This implies that all 4 triples in $abcd$ are edges of G, as desired.

Note that $G \setminus \{a, b, c, d\}$ is an $F_{3,3}$-free 3-graph on $n-4$ vertices with $e(G) - i(abcd)$ edges. By induction this is at most $b(n-4)$ (since $n \ne 9$), so we obtain $i(abcd) \ge b(n) - b(n-4) + 1$. Now we count the edges incident to $abcd$ according to the number of vertices of $abcd$ they contain. There are 4 such edges contained in $abcd$. To estimate edges with one vertex in $abcd$ let M be the link multigraph of $abcd$ in G. Note that there is no triangle xyz in M such that each pair xy, xz, yz is coloured by the same set of 3 colours from $abcd$: this would give a copy of $F_{3,3}$. This implies that $w(xy) + w(xz) + w(yz) \le 10$ for every triple xyz in M. Thus we can apply Lemma 2.1 to get $e(M) \le m(n-4)$. We conclude that the number of edges with 2 vertices in $abcd$ is at least $b(n) - b(n-4) + 1 - 4 - m(n-4) = 5(n-4) + 1$. It follows that there is some $e \in V(G) \setminus \{a, b, c, d\}$ such that all 6 pairs from $abcd$ form an edge with e. Thus $abcde$ forms a copy of K_5^3 in G.

For each $x \in abcde$ we have $i(abcde \setminus x) \ge b(n) - b(n-4) + 1$, so

$$\Sigma := \sum_{x \in abcde} i(abcde \setminus x) \ge 5(b(n) - b(n-4) + 1) = 5(m(n-4) + 5(n-3)).$$

We can also count Σ according to the intersection of edges with $abcde$. Edges with at least 2 vertices in $abcde$ are counted 5 times, and edges with 1 vertex in $abcde$ are counted 4 times. By Lemma 2.1, for each $x \in abcde$ the link multigraph of $abcde \setminus x$ restricted to $V(G) \setminus \{a, b, c, d, e\}$ has at most $m(n-5)$ edges. By averaging, there are at most $\frac{5}{4}m(n-5)$ edges with 1 vertex in $abcde$. Since these are counted 4 times they contribute at most $5m(n-5)$ to Σ. Also, $abcde$ is complete, so we have 10 edges inside $abcde$.

Writing Z for the number of edges with 2 vertices in $abcde$, we obtain

$$5(m(n-4) + 5(n-3)) = 5(b(n) - b(n-4) + 1) \le \Sigma \le 5(10 + Z + m(n-5)),$$

so $Z \ge m(n-4) - m(n-5) + 5(n-5)$. Recall that $m(n-4) - m(n-5)$ is $3(n-5) + 1$ when n is even, or $3(n-5)$ when n is odd. Thus $Z \ge 8(n-5)$. It follows that there is some $f \in V(G) \setminus \{a, b, c, d, e\}$ such that at least 8 pairs from $abcde$ form an edge with f. Thus $abcdef$ is obtained from K_6^3 by deleting at most 2 edges, and if 2 edges are deleted they cannot be disjoint, as they both contain f. As noted above, this implies that $abcdef$ contains $F_{3,3}$, so we have a contradiction.

It remains to prove the bound for $n = 9$. Again we start by choosing $abcd$ as a copy of K_4^3 in G. Since $b(5) = \binom{5}{3} - 1$, we obtain $i(abcd) \ge b(n) - b(n-4)$. Then the same calculation as above shows that there are at least $5(n-4)$ edges with 2 vertices in $abcd$. Furthermore, equality can only hold if deleting $abcd$ leaves $\binom{5}{3}$ edges, i.e. a copy of K_5^3. If equality does not hold then we can find a copy of K_5^3 as above, so either way we have a copy of K_5^3. Let X be the vertex set of this K_5^3. Now note that for any Y spanning a copy of K_4^3 we either have $i(Y) \ge b(n) - b(n-4) + 1$ or $G \setminus Y$ spans K_5^3. Also, there cannot be 3 vertices x_1, x_2, x_3 in X such that $G \setminus (X \setminus x_i)$ spans K_5^3 for $1 \le i \le 3$, as then x_1, x_2, x_3 together with any 3 vertices of $G \setminus X$ spans a copy of $F_{3,3}$. Now we can modify the second calculation above to get $3(b(n) - b(n-4) + 1) + 2(b(n) - b(n-4)) \leq 5(10 + Z + m(n-5))$, so $Z \geq 8(n-5) - 2 = 30 > 7(n-5)$. It follows that there is some $f \in V(G) \setminus \{a, b, c, d, e\}$ such that at least 8 pairs from *abcde* form an edge with f. As above, this creates a copy of $F_{3,3}$, so we have a contradiction. This proves the theorem. $\square$

4 Concluding remarks

The obvious unanswered question from this paper is to characterise the extremal examples for the problem: is it true that for $n \geq 6$, equality can only be achieved by $B(n)$? In the original version of this paper, we had proved Lemma 2; however, a referee pointed out that it had been proved earlier by Bondy and Tuza [1] (and generalised by Füredi and Kündgen [3]). Indeed, both papers also characterize the extremal examples in Lemma 2, and using this characterization, one can extend our proof to show that equality holds for $n \geq 6$ only for $B(n)$. We recently learned that Goldwasser and Hansen have independently characterised the extremal example using the same proof, so we omit the details in this paper.

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References

[1] A. Bondy and Z. Tuza, A weighted generalization of Turán's theorem, *J. Graph Theory* **25** (1997), 267–275.

[2] D. de Caen and Z. Füredi, The maximum size of 3-uniform hypergraphs not containing a Fano plane, *J. Combin. Theory Ser. B* **78** (2000), 274-276.

[3] Z. Füredi and A. Kündgen, Turán problems for integer-weighted graphs, *J. Graph Theory* **40** (2002), 195–225.

[4] J. Goldwasser, On the Turán Number of $\{123, 124, 345\}$, manuscript.

[5] P. Keevash, Hypergraph Turán Problems, to appear in Surveys in Combinatorics 2011.

[6] P. Keevash and D. Mubayi, Stability theorems for cancellative hypergraphs, *J. Combin. Theory Ser. B* **92** (2004), 163–175.

[7] D. Mubayi and V. Rödl, On the Turán number of triple systems, *J. Combin. Theory Ser. A* **100** (2002), 136–152.

[8] P. Turán, Research problem, *Közl MTA Mat. Kutató Int.* **6** (1961), 417–423.

[9] B. Zhou, A Turán-type problem on 3-graphs, *Ars Combin.* **31** (1991), 177–181.