Coloring triple systems with local conditions

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Abstract

We produce an edge-coloring of the complete 3-uniform hypergraph on n vertices with $e^{O(\sqrt{\log\log n})}$ colors such that the edges spanned by every set of five vertices receive at least three distinct colors. This answers the first open case of a question of Conlon-Fox-Lee-Sudakov [1] who asked whether such a coloring exists with $(\log n)^{o(1)}$ colors.

1 Introduction

A k-uniform hypergraph H (k-graph for short) with vertex set $V(H)$ is a collection of k-element subsets of $V(H)$. Write K_n^k for the complete k-graph with vertex set of size n. A (p,q)-coloring of K_n^k is an edge-coloring of K_n^k that gives every copy of K_p^k at least q colors. Let $f_k(n,p,q)$ be the minimum number of colors in a (p,q)-coloring of K_n^k. This paper deals only with $k=3$.

Conlon-Fox-Lee-Sudakov [1] asked whether $f_3(n,p,p-2) = (\log n)^{o(1)}$ for $p \geq 3$ (the case $p=4$ is easy). In this note we answer the first open case with a substantially smaller bound.

Theorem 1.

$$f_3(n,5,3) = e^{O(\sqrt{\log\log n})}.$$

The problem of determining $f_k(n,p,q)$ for fixed k,p,q has a long history, beginning with its introduction by Erdős and Shelah [3, 4], and subsequent investigation (for graphs) by Erdős and Gyárfás [5]. Studying $f_k(n,p,q)$ when $q=2$ is equivalent to studying classical Ramsey numbers, and most of the effort on these problems has therefore been for $q>2$. The simplest nontrivial case in this regime is $f_2(n,4,3)$, which was shown to be $n^{o(1)}$ in [10] and later $\Omega(\log n)$ (see [7, 9]). The same upper bound was shown for $f(n,5,4)$ in [6]. Conlon-Fox-Lee-Sudakov [2] recently extended this construction considerably by proving that $f_2(n,p,p-1) = n^{o(1)}$ for all fixed $p \geq 4$. Their result is sharp in the sense that $f_2(n,p,p) = \Omega(n^{1/(p-2)})$.

The first nontrivial hypergraph case is $f_3(n,4,3)$ and has tight connections to Shelah's breakthrough proof [12] of primitive recursive bounds for the Hales-Jewett numbers. Answering a question of

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Graham-Rothschild-Spencer [8], Conlon et al. [1] recently proved that $f_3(n,4,3) = n^{o(1)}$. They also posed a variety of basic questions about $f_3(n,p,q)$, including the one we address in this note.

Our construction uses an extension of the coloring in [10] together with the stepping up technique of Erdős and Hajnal. The main new observation is contained in Proposition 2 part 3) below. We believe that this specific construction only provides good upper bound for $f_3(n,p,p-2)$ when $p = 5$. However, it is quite possible that, similar to the situation for graphs, other hypergraph cases will eventually be addressed by using the ideas introduced here to create a more complicated construction that works for larger p.

2 The Construction

We begin by defining an edge-coloring σ of the complete graph K_n whose vertices are ordered.

Construction of σ: Given integers $t < m$ and $n = \binom{m}{t}$, let $V(K_n)$ be the set of 0/1 vectors of length m with exactly t 1's. Write $v = (v(1), \dots, v(m))$ for a vertex. The vertices are naturally ordered by the integer they represent in binary, so $v < w$ iff $v(i) = 0$ and $w(i) = 1$ where i is the first position (minimum integer) in which v and w differ. By considering vertices as characteristic vectors of sets, we may assume that $V(K_n) = \binom{[m]}{t}$ whenever convenient. For each $B \in \binom{[m]}{t}$, let $f_B : 2^B \to [2^t]$ be a bijection. Given vectors $v < w$ that are characteristic vectors of sets $S < T$, let $c_1(vw) = \min\{i : v(i) = 0, w(i) = 1\}$, $c_2(vw) = \min\{j : j > c_1(vw), v(j) = 1, w(j) = 0\}$, $c_3(vw) = f_S(S \cap T)$ and $c_4(vw) = f_T(S \cap T)$. Finally, define

$$\sigma(vw) = (c_1(vw), c_2(vw), c_3(vw), c_4(vw)).$$

If n is not of the form $\binom{m}{t}$, then let $n' \geq n$ be the smallest integer of this form, color $\binom{[n']}{2}$ as described above, and restrict the coloring to $\binom{[n]}{2}$. □

It is known [10, 11] that σ is both a $(3,2)$ and $(4,3)$-coloring of K_n (we only need the first and fourth coordinates of the color vector for this) and, for suitable choice of m and t, it uses $e^{O(\sqrt{\log n})}$ colors for all n. We need the following additional properties.

Proposition 2. *The coloring σ satisfies the following properties.*

1) If $v < w < x$, then $\sigma(vw) \neq \sigma(wx)$.

2) If $v < w < \min\{x, y\}$, and $\sigma(vw) = \sigma(vx)$, then $\sigma(vy) \neq \sigma(wx)$.

3) If $v < w < x < y$ with $\sigma(vw) = \sigma(xy)$, then $\sigma(vx) \neq \sigma(vy)$.

Proof. It suffices to consider the first coordinate c_1 of σ to prove the first two properties. For 1), observe that $i = c_1(vw)$ implies that $w(i) = 1$, while $i = c_1(wx)$ implies that $w(i) = 0$. For 2), let $i = c_1(vw) = c_1(vx)$ and suppose for contradiction that $i' = c_1(vy) = c_1(wx)$ so that $v(j) = y(j)$ for $j < i'$. Assume first that $i < i'$. Then $y(i) = v(i) = 0$ and $y(k) = v(k)$ for $k < i$. Moreover, $i = c_1(vw)$ implies that $w(i) = 1$ and $v(k) = w(k)$ for $k < i$. So $w(k) = v(k) = y(k)$ for $k < i$ while $w(i) = 1$ and $y(i) = 0$. This implies that $w > y$, a contradiction. Now assume that $i > i'$ ($i = i'$ is impossible since $w(i) = 1$ while $w(i') = 0$). Then $0 = v(i') = x(i') = 1$ due to $c_1(vy) = i', c_1(vx) = i > i'$ and $c_1(wx) = i'$.

We now prove 3) so assume we are given $v < w < x < y$ with $c_1(vw) = c_1(xy) = i < j = c_2(vw) = c_2(xy)$. Then $v(j) = x(j) = 1$ and $y(j) = 0$. Suppose that v, w, x, y are characteristic vectors of V, W, X, Y respectively. Then $c_3(vx) = c_3(VX) = f_V(V \cap X)$ while $c_3(vy) = c_3(VY) = f_V(V \cap Y)$. If $c_3(vx) = c_3(vy)$, then $f_V(V \cap X) = f_V(V \cap Y)$ and since f_V is a bijection, $V \cap X = V \cap Y$. But this is impossible as $j \in (V \cap X) \setminus Y$. $\square$

We are now ready to describe the edge-coloring χ of K_n^3 that we will use.

Construction of χ: Given a copy of K_n on $[n]$ and the edge-coloring σ, we produce an edge-coloring χ of the 3-graph H on $\{0,1\}^n$ as follows. Order the vertices of H according to the integer that they represent in binary. Given vertices $x < y$ in $V(H)$, let γ_{xy} be the first coordinate where x and y differ. Given vertices $x < y < z$, let δ_{xyz} equal 1 if $\gamma_{xy} < \gamma_{yz}$ and -1 otherwise. For an edge uvw with $u < v < w$, let

$$\chi(uvw) = (\sigma(\gamma_{uv}\gamma_{vw}), \delta_{uvw}). \qquad \square$$

Since σ is an edge-coloring of K_n with $e^{O(\sqrt{\log n})}$ colors, χ is an edge-coloring of K_N^3 ($N = 2^n$) with $e^{O(\sqrt{\log\log N})}$ colors as promised. Moreover, extending this construction to all N is trivial by considering the smallest $N' \geq N$ which is a power of 2, coloring $\binom{[N']}{3}$ and restricting to $\binom{[N]}{3}$. We are left with showing that χ is a $(5,3)$-coloring of K_N^3.

Proof that χ is a $(5,3)$-coloring: Suppose, for contradiction, that $X = \{x_1, \ldots, x_5\}$ where $x_1 < x_2 < x_3 < x_4 < x_5$ are five vertices of H forming a 2-colored K_5^3. Let $\gamma_i = \gamma_{x_i x_{i+1}}$. Let $\gamma = \min \gamma_j$ and assume this minimum is achieved by γ_p. Note that this minimum is uniquely achieved, and $\gamma_i \neq \gamma_{i+1}$ for all i.

Case 1: $p \in \{1, 4\}$. The arguments for both cases are almost identical so we only consider the case $p = 1$. By assumption we have $\gamma_1 < \gamma_2$. First assume that $\gamma_3 > \gamma_2$. If $\gamma_4 > \gamma_3$, then the K_4 on $\{\gamma_i : i \in [4]\}$ has three colors since σ is a $(4,3)$-coloring and this gives at least three colors to the edges in X. If $\gamma_4 < \gamma_3$ then the K_3 on $\{\gamma_i : i \in [3]\}$ has two colors since σ is a $(3,2)$-coloring and this gives two colors to the edges of H within $\{x_i : i \in [4]\}$ with positive δ-coordinate. On the other hand $\delta_{x_3x_4x_5} = -1$, so we again have three colors on X. We now suppose that $\gamma_3 < \gamma_2$. If $\gamma_4 < \gamma_3$, then the K_3 on $\{\gamma_2, \gamma_3, \gamma_4\}$ has two colors since σ is a $(3,2)$-coloring and this gives two colors to the edges of H within $\{x_2, x_3, x_4, x_5\}$ with negative δ-coordinate. On the other hand $\delta_{x_1x_2x_3} = 1$, so we again have three colors on X. Finally, we may assume that $\gamma_1 < \gamma_3 < \min\{\gamma_2, \gamma_4\}$. Now $\sigma(\gamma_1\gamma_3) \neq \sigma(\gamma_3\gamma_4)$ due to property 1) of σ, hence $\chi(x_1x_3x_4) \neq \chi(x_3x_4x_5)$ and both have positive δ-coordinates. But $\delta_{x_2x_3x_4} = -1$, so $\chi(x_2x_3x_4)$ is the third color on X.

Case 2: $p \in \{2, 3\}$. The arguments for both cases are almost identical so we only consider the case $p = 2$. We have $\gamma_3 > \gamma_2$. If in addition $\gamma_4 > \gamma_3$, then we get two colors among $\{x_2, x_3, x_4, x_5\}$ with positive δ-coordinate while $\delta_{x_1x_2x_3} = -1$. So we may assume that $\gamma_2 < \gamma_4 < \gamma_3$. Now $\chi(x_2x_3x_4)$ and $\chi(x_2x_4x_5)$ both have positive δ coordinates while $\delta_{x_3x_4x_5} = -1$. Hence we have three colors unless $\sigma(\gamma_2\gamma_3) = \sigma(\gamma_2\gamma_4)$ which we may assume. Certainly $\delta_{x_1x_2x_3} = -1$, so we are done unless $\sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3)$ which we also assume. If $\gamma_1 = \gamma_4$, then $\sigma(\gamma_2\gamma_4) = \sigma(\gamma_4\gamma_3)$ and hence $\{\gamma_2, \gamma_4, \gamma_3\}$ is a monochromatic triangle, contradiction. If $\gamma_1 > \gamma_4$, then $\gamma_2 < \gamma_4 < \min\{\gamma_1, \gamma_3\}$ with $\sigma(\gamma_2\gamma_4) = \sigma(\gamma_2\gamma_3)$ and $\sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3)$. This contradicts property 2). If $\gamma_1 < \gamma_4$, then $\gamma_2 < \gamma_1 < \gamma_4 < \gamma_3$ with $\sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3)$ and $\sigma(\gamma_2\gamma_4) = \sigma(\gamma_2\gamma_3)$. This contradicts property 3) and completes the proof. $\square$

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