Coloring triple systems with local conditions

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Abstract

We produce an edge-coloring of the complete 3-uniform hypergraph on \( n \) vertices with \( e^{O(\sqrt{\log \log n})} \) colors such that the edges spanned by every set of five vertices receive at least three distinct colors. This answers the first open case of a question of Conlon-Fox-Lee-Sudakov [1] who asked whether such a coloring exists with \((\log n)^{o(1)}\) colors.

1 Introduction

A \( k \)-uniform hypergraph \( H \) (\( k \)-graph for short) with vertex set \( V(H) \) is a collection of \( k \)-element subsets of \( V(H) \). Write \( K^k_n \) for the complete \( k \)-graph with vertex set of size \( n \). A \((p,q)\)-coloring of \( K^k_n \) is an edge-coloring of \( K^k_n \) that gives every copy of \( K^k_p \) at least \( q \) colors. Let \( f_k(n,p,q) \) be the minimum number of colors in a \((p,q)\)-coloring of \( K^k_n \). This paper deals only with \( k = 3 \).

Conlon-Fox-Lee-Sudakov [1] asked whether \( f_3(n,p,p-2) = (\log n)^{o(1)} \) for \( p \geq 3 \) (the case \( p = 4 \) is easy). In this note we answer the first open case with a substantially smaller bound.

Theorem 1.

\[
f_3(n,5,3) = e^{O(\sqrt{\log \log n})}.
\]

The problem of determining \( f_k(n,p,q) \) for fixed \( k, p, q \) has a long history, beginning with its introduction by Erdős and Shelah [3, 4], and subsequent investigation (for graphs) by Erdős and Gyárfás [5]. Studying \( f_k(n,p,q) \) when \( q = 2 \) is equivalent to studying classical Ramsey numbers, and most of the effort on these problems has therefore been for \( q > 2 \). The simplest nontrivial case in this regime is \( f_2(n,4,3) \), which was shown to be \( n^{o(1)} \) in [10] and later \( \Omega(\log n) \) (see [7, 9]). The same upper bound was shown for \( f(n,5,4) \) in [6]. Conlon-Fox-Lee-Sudakov [2] recently extended this construction considerably by proving that \( f_2(n,p,p-1) = n^{o(1)} \) for all fixed \( p \geq 4 \). Their result is sharp in the sense that \( f_2(n,p,p) = \Omega(n^{1/(p-2)}) \).

The first nontrivial hypergraph case is \( f_3(n,4,3) \) and has tight connections to Shelah’s breakthrough proof [12] of primitive recursive bounds for the Hales-Jewett numbers. Answering a question of

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Graham-Rothschild-Spencer [8], Conlon et al. [1] recently proved that $f_3(n, 4, 3) = n^{o(1)}$. They also posed a variety of basic questions about $f_3(n, p, q)$, including the one we address in this note.

Our construction uses an extension of the coloring in [10] together with the stepping up technique of Erdős and Hajnal. The main new observation is contained in Proposition 2 part 3) below. We believe that this specific construction only provides good upper bound for $f_3(n, p, p - 2)$ when $p = 5$. However, it is quite possible that, similar to the situation for graphs, other hypergraph cases will eventually be addressed by using the ideas introduced here to create a more complicated construction that works for larger $p$.

2 The Construction

We begin by defining an edge-coloring $\sigma$ of the complete graph $K_n$ whose vertices are ordered.

Construction of $\sigma$: Given integers $t < m$ and $n = \binom{m}{t}$, let $V(K_n)$ be the set of 0/1 vectors of length $m$ with exactly $t$ 1’s. Write $v = (v(1), \ldots, v(m))$ for a vertex. The vertices are naturally ordered by the integer they represent in binary, so $v < w$ if $v(i) = 0$ and $w(i) = 1$ where $i$ is the first position (minimum integer) in which $v$ and $w$ differ. By considering vertices as characteristic vectors of sets, we may assume that $V(K_n) = \binom{[m]}{t}$ whenever convenient. For each $B \in \binom{[m]}{t}$, let $f_B : 2^B \rightarrow [2^t]$ be a bijection. Given vectors $v < w$ that are characteristic vectors of sets $S \subset T$, let $c_1(vw) = \min\{i : v(i) = 0, w(i) = 1\}$, $c_2(vw) = \min\{j : j > c_1(vw), v(j) = 1, w(j) = 0\}$, $c_3(vw) = f_S(S \cap T)$ and $c_4(vw) = f_T(S \cap T)$. Finally, define

$$\sigma(vw) = (c_1(vw), c_2(vw), c_3(vw), c_4(vw)).$$

If $n$ is not of the form $\binom{m}{t}$, then let $n' \geq n$ be the smallest integer of this form, color $\binom{[n']}{2}$ as described above, and restrict the coloring to $\binom{[n]}{t}$.

It is known [10, 11] that $\sigma$ is both a (3,2) and (4,3)-coloring of $K_n$ (we only need the first and fourth coordinates of the color vector for this) and, for suitable choice of $m$ and $t$, it uses $e^{O(\sqrt{\log n})}$ colors for all $n$. We need the following additional properties.

Proposition 2. The coloring $\sigma$ satisfies the following properties.

1) If $v < w < x$, then $\sigma(vw) \neq \sigma(wx)$.

2) If $v < w < \min\{x, y\}$, and $\sigma(vw) = \sigma(vx)$, then $\sigma(vy) \neq \sigma(wx)$.

3) If $v < w < x < y$ with $\sigma(vw) = \sigma(xy)$, then $\sigma(vx) \neq \sigma(vy)$.

Proof. It suffices to consider the first coordinate $c_1$ of $\sigma$ to prove the first two properties. For 1), observe that $i = c_1(vw)$ implies that $w(i) = 1$, while $i = c_1(wx)$ implies that $w(i) = 0$. For 2), let $i = c_1(vw) = c_1(wx)$ and suppose for contradiction that $i' = c_1(vy) = c_1(wx)$ so that $v(j) = y(j)$ for $j < i'$. Assume first that $i < i'$. Then $y(i) = v(i) = 0$ and $y(k) = v(k)$ for $k < i$. Moreover, $i = c_1(vw)$ implies that $w(i) = 1$ and $v(k) = w(k)$ for $k < i$. So $w(k) = v(k) = y(k)$ for $k < i$ while $w(i) = 1$ and $y(i) = 0$. This implies that $w > y$, a contradiction. Now assume that $i > i'$ ($i = i'$ is impossible since $w(i) = 1$ while $w(i') = 0$). Then $0 = v(i') = x(i') = 1$ due to $c_1(vy) = i', c_1(vx) = i > i'$ and $c_1(wx) = i'$. 

2
We now prove 3) so assume we are given \( v < w < x < y \) with \( c_1(vw) = c_1(xy) = i < j = c_2(vw) = c_2(xy) \). Then \( v(j) = x(j) = 1 \) and \( y(j) = 0 \). Suppose that \( v, w, x, y \) are characteristic vectors of \( V, W, X, Y \) respectively. Then \( c_3(vx) = c_3(VX) = f_V(V \cap X) \) while \( c_3(vy) = c_3(VY) = f_V(V \cap Y) \). If \( c_3(vx) = c_3(vy) \), then \( f_V(V \cap X) = f_V(V \cap Y) \) and since \( f_V \) is a bijection, \( V \cap X = V \cap Y \). But this is impossible as \( j \in (V \cap X) \setminus Y \). 

We are now ready to describe the edge-coloring \( \chi \) of \( K^3_n \) that we will use.

**Construction of \( \chi \):** Given a copy of \( K_n \) on \([n]\) and the edge-coloring \( \sigma \), we produce an edge-coloring \( \chi \) of the 3-graph \( H \) on \( \{0,1\}^n \) as follows. Order the vertices of \( H \) according to the integer that they represent in binary. Given vertices \( x < y \) in \( V(H) \), let \( \gamma_{xy} \) be the first coordinate where \( x \) and \( y \) differ. Given vertices \( x < y < z \), let \( \delta_{xyz} \) equal 1 if \( \gamma_{xy} < \gamma_{yz} \) and -1 otherwise. For an edge \( uvw \) with \( u < v < w \), let  

\[
\chi(uvw) = (\sigma(\gamma_{uv}), \delta_{uvw}).
\]

Since \( \sigma \) is an edge-coloring of \( K_n \) with \( e^{O(\sqrt{\log n})} \) colors, \( \chi \) is an edge-coloring of \( K^3_n \) (\( N = 2^n \)) with \( e^{O((\sqrt{\log \log n})^N)} \) colors as promised. Moreover, extending this construction to all \( N \) is trivial by considering the smallest \( N' \geq N \) which is a power of 2, coloring \( \binom{[N']}{3} \) and restricting to \( \binom{[N]}{3} \). We are left with showing that \( \chi \) is a \((5,3)\)-coloring of \( K^3_N \).

**Proof that \( \chi \) is a \((5,3)\)-coloring:** Suppose, for contradiction, that \( X = \{x_1, \ldots, x_5\} \) where \( x_1 < x_2 < x_3 < x_4 < x_5 \) are five vertices of \( H \) forming a 2-colored \( K^3_5 \). Let \( \gamma_i = \gamma_{x_ix_{i+1}} \). Let \( \gamma = \min \gamma_j \) and assume this minimum is achieved by \( \gamma_p \). Note that this minimum is uniquely achieved, and \( \gamma_i \neq \gamma_{i+1} \) for all \( i \).

**Case 1:** \( p \in \{1, 4\} \). The arguments for both cases are almost identical so we only consider the case \( p = 1 \). By assumption we have \( \gamma_1 < \gamma_2 \). First assume that \( \gamma_3 > \gamma_2 \). If \( \gamma_4 > \gamma_3 \), then the \( K_4 \) on \( \{\gamma_i : i \in [4]\} \) has three colors since \( \sigma \) is a \((4,3)\)-coloring and this gives at least three colors to the edges in \( X \). If \( \gamma_4 < \gamma_3 \) then the \( K_3 \) on \( \{\gamma_i : i \in [3]\} \) has two colors since \( \sigma \) is a \((3,2)\)-coloring and this gives two colors to the edges of \( H \) within \( \{x_i : i \in [4]\} \) with positive \( \delta \)-coordinate. On the other hand \( \delta_{x_3x_4x_5} = -1 \), so we again have three colors on \( X \). We now suppose that \( \gamma_3 < \gamma_2 \). If \( \gamma_4 < \gamma_3 \), then the \( K_3 \) on \( \{\gamma_2, \gamma_3, \gamma_4\} \) has two colors since \( \sigma \) is a \((3,2)\)-coloring and this gives two colors to the edges of \( H \) within \( \{x_2, x_3, x_4, x_5\} \) with negative \( \delta \)-coordinate. On the other hand \( \delta_{x_2x_3x_4} = 1 \), so we again have three colors on \( X \). Finally, we may assume that \( \gamma_1 < \gamma_3 < \min \{\gamma_2, \gamma_4\} \). Now \( \sigma(\gamma_1\gamma_3) \neq \sigma(\gamma_3\gamma_4) \) due to property 1) of \( \sigma \), hence \( \chi(x_1x_3x_4) \neq \chi(x_3x_4x_5) \) and both have positive \( \delta \)-coordinates. But \( \delta_{x_2x_3x_4} = -1 \), so \( \chi(x_2x_3x_4) \) is the third color on \( X \).

**Case 2:** \( p \in \{2, 3\} \). The arguments for both cases are almost identical so we only consider the case \( p = 2 \). We have \( \gamma_3 > \gamma_2 \). If in addition \( \gamma_4 > \gamma_3 \), then we get two colors among \( \{x_2, x_3, x_4, x_5\} \) with positive \( \delta \)-coordinate while \( \delta_{x_1x_2x_3} = -1 \). So we may assume that \( \gamma_2 < \gamma_4 < \gamma_3 \). Now \( \chi(x_2x_3x_4) \) and \( \chi(x_2x_4x_5) \) both have positive \( \delta \)-coordinates while \( \delta_{x_3x_4x_5} = -1 \). Hence we have three colors unless \( \sigma(\gamma_2\gamma_3) = \sigma(\gamma_2\gamma_4) \) which we may assume. Certainly \( \delta_{x_1x_2x_3} = -1 \), so we are done unless \( \sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3) \) which we also assume. If \( \gamma_1 > \gamma_4 \), then \( \sigma(\gamma_2\gamma_4) = \sigma(\gamma_4\gamma_3) \) and hence \( \{\gamma_2, \gamma_4, \gamma_3\} \) is a monochromatic triangle, contradiction. If \( \gamma_1 > \gamma_4 \), then \( \gamma_2 < \gamma_4 < \gamma_3 \) with \( \sigma(\gamma_2\gamma_4) = \sigma(\gamma_2\gamma_3) \) and \( \sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3) \). This contradicts property 2). If \( \gamma_1 < \gamma_4 \), then \( \gamma_2 < \gamma_1 < \gamma_4 < \gamma_3 \) with \( \sigma(\gamma_2\gamma_1) = \sigma(\gamma_4\gamma_3) \) and \( \sigma(\gamma_2\gamma_4) = \sigma(\gamma_2\gamma_3) \). This contradicts property 3) and completes the proof. 

\[ \square \]
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References


