The co-degree density of the Fano plane

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Abstract

It is shown that every n vertex triple system with every pair of vertices lying in at least $(1/2+o(1))n$ triples contains a copy of the Fano plane. The constant $1/2$ is sharp. This is the first triple system for which such a (nonzero) constant is determined.

Let X be a finite set. An r-graph F with vertex set X is a family of r-element subsets of X. These subsets are called edges, and X is written as $V(F)$. When there is no confusion, we denote an edge $\{x,y,z\}$ as xyz. Given a family of r-graphs $\mathcal{F}$, the Turán number $\mathrm{ex}(n,\mathcal{F})$ is the maximum number of edges in an n vertex r-graph containing no member of $\mathcal{F}$. The Turán density of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) := \lim_{n\to\infty} \mathrm{ex}(n,\mathcal{F})/\binom{n}{r}$. If $\mathcal{F}=\{F\}$, we write $\pi(F)$ instead of $\pi(\{\mathcal{F}\})$. For $r>2$, computing $\pi(F)$ when it is nonzero is notoriously hard, even for very simple r-graphs F (see [3] for a survey of results). Determining the Turán density of complete r-graphs is a fundamental question about set-systems. In fact, this is not known in any nontrivial case when $r\geq 3$.

In the past few years, there has been some progress on related problems. The Fano plane is the projective plane $PG(2,2)$ consisting of seven vertices (points) and seven edges (lines). Alternatively, we can write it as $\{yaa', ybb', ycc', abc', ab'c, a'bc, a'b'c'\}$. The starting point of these recent developments was de Caen and Füredi's breakthrough proof [1] of Sós' 1973 conjecture that $\pi(\mathbf{F})=3/4$, where $\mathbf{F}$ is the Fano plane (see also Füredi-Simonovits [4] and Keevash-Sudakov [5] for exact results and further extensions).

The *co-degree* of vertices x,y in a hypergraph G is the number of edges containing both x and y. Define $c(G)$ to be the minimum co-degree over all pairs of vertices in G. A natural problem is to ask what minimum co-degree guarantees a copy of some forbidden configuration. The first question in this area (to the authors knowledge) was posed by S. Abbasi (personal communication with V. Rödl), who (implicitly) asked whether, for an n vertex 3-graph G, the condition $c(G)\geq n/2$ suffices

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to guarantee a copy of K_4^3, the complete 3-graph on four vertices. A construction due to Nagle and Rödl [9] shows that, if true, the constant 1/2 is sharp. This problem remains open, and is now a conjecture of Nagle and Czygrinow [8]. One can ask such questions for other hypergraphs, but the questions seem as difficult as the usual extremal problems, which are well-known to be hard (although only time will tell if this opinion is well-founded).

Co-degree extremal problems for hypergraphs are investigated in [7]. Here we provide the first nontrivial example of a 3-graph whose "co-degree density" is determined.

Theorem 1. *For every $\varepsilon > 0$, there exists n_0 such that if $n > n_0$ and G is an n vertex 3-graph with $c(G) > (1/2+\varepsilon)n$, then G contains a copy of the Fano plane $\mathbf{F}$.*

Consider the n vertex 3-graph G whose vertex set is partitioned into parts of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, and whose edges consist of all triples intersecting both parts. It is well-known that $\mathbf{F}$ is absent from G, and clearly $c(G) = \lfloor n/2 \rfloor$. This shows that Theorem 1 is sharp.

For the proof of the upper bound in Theorem 1, we need some definitions and a result that was earlier proved by Rödl and the current author [6]. In order to make this note self contained, we will repeat the proof in the Appendix.

Let $K_4^{3-} = \{134, 135, 145\}$ be the unique (up to isomorphism) 3-graph with four vertices and three edges. Given an r-graph F, and a vertex $x \in V(F)$, the *cloning* operation adds a new vertex x' whose neighborhood in F is the same as that of x. In other words, $\{x\} \cup S \in F$ if and only if $\{x'\} \cup S \in F$. Say that x' is a *clone* of x.

Definition 2. *Let $F_2 = \{134, 135, 145, 234, 235, 245\}$ be the 3-graph obtained from K_4^{3-} by cloning the unique vertex of degree three. Set $\mathcal{F}_2 = \{F_2\}$, and for $t > 2$, let $\mathcal{F}_t$ be the family of all 3-graphs obtained as follows: let $F \in \mathcal{F}_{t-1}$, add two new vertices, x, y and add any set of t edges of the form axy, where a lies in F.*

Proposition 3. (Proposition 4.2 in [6]) *Let $F \in \mathcal{F}_t$. Then*

1) *F has $2t+1$ vertices*

2) *Every set of $t+2$ vertices in F spans at least one edge*

3) *$\pi(\mathcal{F}_t) \leq 1/2$ for $t \geq 2$.*

Let F' be obtained from F by cloning each vertex of F. Thus F' has twice as many vertices as F. Let $\mathcal{F}'_t$ be the family obtained from $\mathcal{F}_t$ by replacing each member $F \in \mathcal{F}_t$ by F'.

Proof of Theorem 1: Pick $\varepsilon > 0$, and let $t = \lceil 1/\varepsilon \rceil$. A result of Erdős and Simonovits [2] implies that $\pi(\mathcal{F}_t) = \pi(\mathcal{F}'_t)$. Together with Proposition 3, Part 3, this implies the existence of n_0 so that $\mathrm{ex}(n, \mathcal{F}'_t) < (1/2+\varepsilon)\binom{n}{3}$ for $n > n_0$. Now suppose that $n > n_0$, and G is an n vertex 3-graph with $c(G) > (1/2+\varepsilon)n$. We will show that G contains a copy of the Fano plane $\mathbf{F}$.

Since $c(G) > (1/2+\varepsilon)n$, we have $|G| > \binom{n}{2}(1/2+\varepsilon)n/3 > (1/2+\varepsilon)\binom{n}{3}$. By the choice of n_0, we deduce that G contains a copy F' of some member of $\mathcal{F}'_t$. By Proposition 3, Part 1, F' has $2(2t+1)$ vertices.

Let $V(F') = X \cup X'$, where X induces a copy F of some member of $\mathcal{F}_t$ (so $|X| = 2t+1$). For each $x \in X$, let $x' \in X'$ denote the clone of x in F'. Let $C_x = \{z \in V(G) : zxx' \in G\}$. Since $|C_x| \geq c(G) > (1/2+\varepsilon)n$, summing gives $\sum_{x \in X} |C_x| \geq (2t+1)(1/2+\varepsilon)n$. Consequently, there exists $y \in V(G)$ that lies in at least $(1/2+\varepsilon)(2t+1)$ different C_x. Since $t = \lceil 1/\varepsilon \rceil$, we have $(1/2+\varepsilon)(2t+1) \geq t+2$. By Proposition 3, Part 2, there exist a, b, c satisfying $y \in C_x$ for $x \in \{a, b, c\}$ and $abc \in G$. By the definition of $\mathcal{F}'_t$, the set $S = \{a, a'\} \cup \{b, b'\} \cup \{c, c'\}$ induces a complete 3-partite 3-graph with parts $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$. Now $\{y\} \cup S$ contains a copy of $\mathbf{F}$.

□

Concluding Remarks and a Conjecture

• We actually proved a slightly stronger statement than Theorem 1. Define $\mathbf{F}^+$ to be the 3-graph consisting of the complete 3-partite 3-graph with parts A, B, C each of size two, and a new vertex y and three new edges $\{y\} \cup A, \{y\} \cup B, \{y\} \cup C$. Then the proof of Theorem 1 yields the same result for any F satisfying $\mathbf{F} \subset F \subset \mathbf{F}^+$.

• The relationship between the extremal configurations for the co-degree density problem and the usual Turán problem appears to be mysterious. For example, for the complete 3-graph on four vertices, all K_4^3-free constructions G of edge density about 5/9 (which is conjectured to be the correct value of $\pi(K_4^3)$), have $c(G)$ only about 1/3, while it is known [9] that the co-degree density of K_4^3 is at least 1/2. However, for the Fano plane, the extremal examples for both the usual extremal problem and the co-degree problem might be the same.

• Our proof requires $c(G) > (1/2+\varepsilon)n$, however, in light of the exact results for the Turán density of the Fano plane [4, 5], perhaps ε can be made 0.

Conjecture 4. *If n is sufficiently large, and $c(G) \geq n/2$, then G contains a copy of* $\mathbf{F}$.

Appendix

Proof of Proposition 3.

1) F_2 has five vertices, and to form a 3-graph in $\mathcal{F}_t$, we add two vertices to a 3-graph in $\mathcal{F}_{t-1}$.

2) We proceed by induction on t. It is easy to verify it directly for $t = 2$. Suppose that the result holds for $t-1$. Let $F' \in \mathcal{F}_t$ be obtained from $F \in \mathcal{F}_{t-1}$ by adding vertices x, y and a set of t edges involving x, y. Let S be a set of $t+2$ vertices in F'. If $|S \cap V(F)| \geq t+1$, then by induction, the $t+1$ vertices in $S \cap V(F)$ span an edge. Otherwise, $|S \cap V(F)| = t$. But now both $x, y \in S$, and since there are fewer than $2t$ vertices in $V(F)$, we have an edge of the form xya, where $a \in S$.

3) Recall that K_4^{3-} is the unique 3-graph with four vertices and three edges. de Caen proved that $\pi(K_4^{3-}) \leq 1/3$, and it is well known that the same upper bound holds if we clone a vertex. Since

F_2 is formed by cloning a vertex of K_4^{3-}, we obtain $\pi(F_2) \le 1/3$. Because $1/3 < 1/2$, we conclude that there is a constant $c > 2t$ such that $\mathrm{ex}(n, F_2) \le (1/2)\binom{n}{3} + cn^2$ for all n.

We will prove by induction on t that $\mathrm{ex}(n, \mathcal{F}_t) \le (1/2)\binom{n}{3} + cn^2$ for all n. The base case is $t = 2$ above. Assume that the result holds for $t-1$, and let G be a 3-graph with n vertices and at least $(1/2)\binom{n}{3} + cn^2$ edges. We will prove that G contains a copy of some 3-graph in $\mathcal{F}_t$ by induction on n. The result is true for $n = 2t+1$ by the choice of c. Assume that the result holds for $n-1$.

By induction on t, G contains a copy H of a member $F \in \mathcal{F}_{t-1}$. Let L be the link multigraph of $V(H)$ in G. In other words, $V(L) = V(G) - V(H)$, and the multiplicity of uv as an edge of L is the number of $x \in V(H)$ with $uvx \in G$. If L has a pair of vertices with multiplicity at least t, then this pair together with H yields a copy of some $F' \in \mathcal{F}_t$. Hence we may assume that L has multiplicity at most $t-1$. This implies that there is a vertex v in H with

$$\deg_G(v) \le \frac{t-1}{2t-1}\binom{n-2t+1}{2} + (2t-2)(n-2t+1) + \binom{2t-1}{2} < \frac{t-1}{2t-1}\binom{n-1}{2} + 2tn. \quad (1)$$

Removing v leaves a 3-graph G' on $n-1$ vertices with $|G'|$ at least

$$|G| - \frac{t-1}{2t-1}\binom{n-1}{2} - 2tn \ge \frac{1}{2}\binom{n}{3} + cn^2 - \frac{1}{2}\binom{n-1}{2} - 2tn > \frac{1}{2}\binom{n-1}{3} + c(n-1)^2.$$

By induction on n, G' contains a copy of some 3-graph in $\mathcal{F}_t$. □

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