On Restricted Edge-Colorings of Bicliques

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Dedicated to Daniel J. Kleitman on the occasion of his 65th birthday.

Abstract

We investigate the minimum and maximum number of colors in edge-colorings of $K_{n,n}$ such that every copy of $K_{p,p}$ receives at least q and at most q' colors. Along the way we improve the bounds on some bipartite Turán numbers.

1 Introduction

Our problem is a generalization of a reinterpretation of the bipartite analogue of the classical Ramsey problem. The *bipartite Ramsey number* $b_k(H)$ is the minimum n such that every k-coloring of $E(K_{n,n})$ yields a monochromatic copy of the bipartite graph H. Like the classical Ramsey numbers, these are hard to determine. Chvátal [6] and Beineke-Schwenk [4] proved that $b_k(K_{p,q}) \le (q-1)k^p + O(k^{p-1})$. Some exact results for the case $p = 2$ appear in [4] (see also [5] for recent progress on an assymetric version of this problem).

An alternative approach is to fix n and ask for the minimum k such that some k-coloring of $E(K_{n,n})$ yields no monochromatic H. More generally, we require that every copy of H receives at least q colors. A further generalization considers edge-colorings of a graph G. The resulting global minimum number of colors has been denoted $r(G, H, q)$; in our problem, $G = K_{n,n}$. The problem was explored by Axenovich, Füredi, and Mubayi [3]; the case where G and H are cliques was studied by Erdős and Gyárfás [9].

We could also ask for the maximum number of colors in a coloring of $E(K_{n,n})$ such that every copy of H receives at most q' colors. Interchanging minimum and maximum and reversing the

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inequality in the constraints makes this problem fundamentally different from classical coloring problems. Such maximization problems were introduced as "anti-Ramsey problems" by Erdős, Simonovits, and Sós [11].

We study a common generalization of the Ramsey and anti-Ramsey problems.

Definition 1.1 *An $(H;q,q')$-coloring of G is a coloring of $E(G)$ such that every copy of H in G receives at least q and at most q' distinct colors on its edges. The minimum and maximum number of colors in an $(H;q,q')$-coloring of G are denoted by $r(G,H;q,q')$ and $R(G,H;q,q')$, respectively.*

Understanding these restricted generalized Ramsey numbers is a huge project; the numbers $b_k(K_{2,p})$ form a small special case. The restricted generalized Ramsey problem is itself a special case of constrained hypergraph coloring. In a rather general model, we color the vertices of a hypergraph, with each edge having a constraint set for the number of colors used on it; we study the global numbers of colors that permit such colorings (see [16]). In our problem, the vertex set of the hypergraph is the edge set of $K_{n,n}$, the edges are the copies of the edge sets of H, and we study only the extreme global feasible numbers of colors.

In this paper we study $(K_{p,p};q,q')$-colorings of $K_{n,n}$, with particular attention to $p=2$ and to the case $(p,q)=(3,1)$. Before listing our results, we mention several known results on the classical Ramsey special case. Here always $q'=e(H)$, and we consider the lower bound on the global number of colors forced by the local lower bound q; thus we use the earlier notation in this discussion.

Axenovich, Füredi, and Mubayi [3] studied $r(K_{n,n},K_{2,p},q)$. The lower bound $r(K_{n,n},K_{2,2},2)\geq(1+o(1))\sqrt{n}$ follows directly from the asymptotic maximum size $n^{3/2}$ of a subgraph of $K_{n,n}$ not containing C_4 (Kővári–Sós–Turán [17]). A result due to Chung and Graham [7] and to Irving [15] implies that this is asymptotically the correct value of $r(K_{n,n},C_4,2)$. A different proof of this appears in [3]. More generally, [3] uses algebraic techniques to prove for $t>0$ that both $r(K_{n,n},K_{2,t+1},2)$ and $r(K_n,K_{2,t+1},2)$ are asymptotic to $\sqrt{n/t}$ as $n\to\infty$.

The asymptotic value of $r(K_{n,n},C_4,3)$ remains open. The bounds $2n/3\leq r(K_{n,n},C_4,3)\leq n+1$ appear in [3]. When n is odd or in $\{4,12,60\}$, the upper bound was improved to n. Determining $\lim r(K_{n,n},C_4,3)/n$ seems difficult.

For each fixed p, [3] determines the smallest $q=q(p)$ such that the growth of $r(K_{n,n},K_{p,p},q)$ reaches various thresholds in terms of n, including linear, quadratic, $n^2-O(n)$, and $n^2-O(1)$. A general upper bound is given using the Local Lemma. These results parallel those of Erdős and Gyárfás [9] for $r(K_n,K_p,q)$. A result of Alon, Rónyai, and Szabó [1] yields $r(K_{n,n},K_{3,3},2)\sim n^{1/3}$, and [3] determines $r(K_{n,n},K_{3,3},8)$ exactly.

Axenovich and Kündgen [2] studied the analogue of our problem for cliques. They determined threshold values of (p, q, q') for various asymptotic behaviors for $R(K_n, K_p; q, q')$ in terms of n.

In an edge-coloring of G, a copy of H is *monochromatic* if its edges all have the same color and *polychromatic* if they all have distinct colors. We say that a vertex is *monochromatic* or *polychromatic* if the set of its incident edges is monochromatic or polychromatic, respectively. A natural problem with nontrivial values of both q and q' considers edge-colorings with no monochromatic copy of H and no polychromatic copy of H. This makes $r(K_{n,n}, C_4; 2, 3)$ and $R(K_{n,n}, C_4; 2, 3)$ particularly interesting. Although we determine the latter exactly (see Table 1), our upper and lower bounds for the former differ by a factor of 3.

We assume throughout that $n \ge p \ge 2$. Note that every $(H; q, q')$-coloring is also an $(H; r, r')$-coloring when $r \le q \le q' \le r'$. Table 1 summarizes the results when $p = 2$. The table suggests the natural question of which values between the minimum and the maximum are achievable; we have not addressed this.

Table 1

$p; q, q'$	$r(K_{n,n}, K_{p,p}; q, q')$	$R(K_{n,n}, K_{p,p}; q, q')$	reference
$2; 1, 1$	1	1	trivial
$2; 1, 2$	1	n	Theorem 3.2
$2; 1, 3$	1	$2n - 1$	Proposition 3.3
$2; 1, 4$	1	n^2	trivial
$2; 2, 4$	$(1 + o(1))\sqrt{n}$	n^2	[17, 7]
$2; 3, 4$	$2n/3 \le r \le n + 1$	n^2	[3]
$2; 4, 4$	n^2	n^2	trivial
$2; 2, 2$	n (for $n \ge 5$)	n	Theorem 6.1
$2; 2, 3$	$n/3 - 11 < r \le n - 3$	$2n - 1$	Thm. 7.3, Prop. 2.4, Cor. 7.1
$2; 3, 3$	(undefined for $n \ge 5$)	(undefined for $n \ge 5$)	Theorem 6.3

The avoidance of monochromatic or polychromatic fixed subgraphs is related to Turán numbers. Given a family $\mathcal{F}$ of graphs, the *Turán number* $\mathrm{ex}(G, \mathcal{F})$ is the maximum number of edges in a subgraph of G containing no member of $\mathcal{F}$; most often $\mathcal{F}$ consists of a single graph. By the pigeonhole principle, every edge-coloring of G with no monochromatic H uses at least $e(G)/\mathrm{ex}(G, H)$ colors. From the other direction, Erdős, Simonovits, and Sós [11] observed that an edge-coloring of G with no polychromatic H uses at most $\mathrm{ex}(G, H)$ colors.

In Section 4, we obtain new bounds on bipartite Turán numbers for the graph $H(r, l)$ obtained from $K_{r,r}$ by deleting the edges of a copy of $K_{l,l}$. For $1 \le l \le r/2$, we prove that $\mathrm{ex}(H_{n,n}, H(r, l)) \le r^{1/t}n^{2-1/t} + (t-1)n$, where $t = r - l$. We also prove that $\mathrm{ex}(K_{n,n}, K_{3,3} - E(P_3)) = (\sqrt{2} + o(1))n^{3/2}$, where P_3 denotes the path with three vertices. These results are of independent interest.

We apply these results in Section 5 to obtain bounds on $R(K_{n,n}, K_{3,3}; 1, q')$ for $6 \leq q' \leq 8$. For $3 \leq q' \leq 5$, we determine the values exactly via direct combinatorial arguments. These results are listed in Table 2.

Table 2

q'	$R(K_{n,n}, K_{3,3}; 1, q')$	reference
1	1	trivial
2	2	trivial
3	n	Theorem 3.2
4	$n+1$	Theorem 5.5
5	$2n-1$	Theorem 3.5
6	$\Omega(n^{4/3}) \leq R \leq O(n^{3/2})$	Theorem 5.4
7	$n^{3/2} + \Omega(n^{4/3}) \leq R \leq (1.61)n^{3/2} + O(n)$	Theorem 5.4
8	$\Omega(n^{3/2}) < R \leq O(n^{5/3})$	Theorem 5.4
9	n^2	trivial

Section 6 discusses "uniform" colorings for the case $p = 2$, where every copy of C_4 must have the same number of colors on it. When this number is 2, the global number of colors can only be n (for $n \geq 5$). When this number is 3, there is no such coloring. In Section 7, we study the edge-colorings of $K_{n,n}$ having no monochromatic C_4 and no polychromatic C_4.

We begin in Section 2 with several explicit constructions, mostly for general p. In Section 3 we prove that one of these constructions is optimal for various of the anti-Ramsey problems. Our results for general p are summarized in Table 3.

Table 3

condition	result	reference
$1 \leq q \leq p$	$R(K_{n,n}, K_{p,p}; q, p) = n$	Theorem 3.2
$q \leq p$	$R(K_{n,n}, K_{p,p}; q, 2p-1) = 2n-1$	Theorem 3.5
$0 \leq s < p$	$R(K_{n,n}, K_{p,p}; p, sp+p-s) \geq sn+n-s$	Proposition 2.1
$1 \leq s < p$	$R(K_{n,n}, K_{p,p}; p, sp) \geq sn$	Proposition 2.2
$2 \leq q \leq p$	$r(K_{n,n}, K_{p,p}; q, 2p-1) \leq \lceil \frac{n}{\lfloor (p-1)/(q-1) \rfloor} \rceil$	Proposition 2.3

2 Constructions

Throughout this paper we use X and Y to designate the partite sets of $K_{n,n}$. Our first construction is optimal for some values of p, q, q'. The subsequent constructions in this section are the best we know for some values of the parameters, but we have not shown them to be optimal anywhere.

Proposition 2.1 *For $0 \le s < p$, there is a $(K_{p,p}; p, sp+p-s)$-coloring of $K_{n,n}$ with $sn+n-s$ colors.*

Proof: For $i \le n-s$, let the ith vertex of X be monochromatic with color i. Let the remaining s vertices of X be polychromatic, using all distinct colors. This uses $sn+n-s$ colors. Since each copy of $K_{p,p}$ has p vertices in X, it has at least p colors on its edges. The number of colors is maximized by using all the polychromatic vertices; such copies of $K_{p,p}$ receive $sp+p-s$ colors. □

When $s = 0$, we obtain a $(K_{p,p}; p, p)$-coloring of $K_{n,n}$ with n colors. When $s = 1$, we obtain a $(K_{p,p}; p, 2p-1)$-coloring of $K_{n,n}$ with $2n-1$ colors. Corollary 7.1, Proposition 3.3, and Theorems 3.2, 3.5, and 6.1 show that these constructions are optimal in various cases. When $s > 1$, we obtain new lower bounds on the maximum number of colors. Our second construction is also good for this purpose when the number of colors on each $K_{p,p}$ is bounded by a multiple of p.

Proposition 2.2 *For $1 \le s \le p$, there is a $(K_{p,p}; p, sp)$-coloring of $K_{n,n}$ with sn colors.*

Proof: Partition Y into s nonempty sets. Partition the set of sn colors into s sets of size n. Let each vertex in Y be polychromatic, with the vertices in the ith block in Y being incident to edges with the ith set of colors. On each copy of $K_{p,p}$, the number of colors used is p times the number of blocks in Y that have contributed vertices. □

Our next construction treats the partite sets symmetrically. We use an equivalent matrix formulation, treating edge-colorings of $K_{n,n}$ as integer n-by-n matrices. To encode a $(K_{p,p}; q, q')$-coloring of $K_{n,n}$, each p-by-p submatrix receives at least q and at most q' labels.

Proposition 2.3 *For $2 \le q \le p$, there is a $(K_{p,p}; q, 2p-1)$-coloring of $K_{n,n}$ with $\lceil n/t \rceil$ colors, where $t = \lfloor \frac{p-1}{q-1} \rfloor$.*

Proof: Let $m = \lceil n/t \rceil$. Partition X into sets $X_1, \ldots, X_m$ and Y into sets $Y_1, \ldots, Y_m$, where each set has size t or $t-1$. Give edge xy the color $\max\{i, j\}$, where $x \in X_i$ and $y \in Y_j$.

Every selection of p vertices from X or from Y contains vertices from at least q distinct blocks of the partition, since $\lceil p/t \rceil \ge p/\frac{p-1}{q-1} > q-1$. Let $x_{i_1}, \ldots, x_{i_q}$ and $y_{j_1}, \ldots, y_{j_q}$ be vertices from distinct blocks of the partitions, with $i_1 < \cdots < i_q$ and $j_1 < \cdots < j_q$. The colors on $\{x_{i_l} y_{j_l} \colon 1 \le l \le q\}$ are distinct, because $a < c$ and $b < d$ imply $\max\{a, b\} < \max\{c, d\}$. Thus every copy of $K_{p,p}$ receives at least q colors.

Given a copy H of $K_{p,p}$, let U be the set of blocks in $\{X_1, \ldots, X_m\} \cup \{Y_1, \ldots, Y_m\}$ that each contain at least one vertex of H. The number of colors on H is less than $|U|$, because each color used is the index of a block in U, and the least such index, j, is used as a color only if both X_j and Y_j are in U. Since $|U| \leq 2p$, at most $2p-1$ colors appear on H. □

The construction of Proposition 2.3 is not helpful when maximizing the number of colors in a $(p, 2p-1)$-coloring, since it uses n colors and Proposition 2.1 with $s = 1$ uses $2n-1$ colors. When $q = p = 2$, Proposition 2.3 also does not use as few colors as Proposition 2.4. Nevertheless, for $2 \leq q < p$ it uses the fewest colors among the constructions we know. Table 3 summarizes the bounds provided by these constructions.

Propositions 2.1 and 2.3 each yield $r(K_{n,n}, C_4; 2, 3) \leq n$. We improve this bound by using a special 3-edge-coloring of $K_{6,6}$. This construction is still far from the lower bound that we later prove in Section 6. We describe the construction in the matrix format.

Proposition 2.4 $r(K_{n,n}, C_4; 2, 3) \leq n - 3$ *for* $n \geq 6$.

Proof: We construct an n-by-n matrix with no monochromatic or polychromatic 2-by-2 submatrix. When $\min\{i, j\} \leq n-6$, put $\min\{i, j\}$ in position (i, j). This leaves a 6-by-6 matrix in the lower right. Using three additional labels, we fill this with the matrix below.

b	c	a	a	a	a
c	a	b	b	b	b
a	b	c	c	c	c
a	b	c	b	c	a
a	b	c	c	a	b
a	b	c	a	b	c

Within this matrix, no color completes a 2-by-2 submatrix, and only three colors are used. If a 2-by-2 submatrix has a row or a column among the first $n-6$, then its first row or column is constant, and its second adds at least one additional color. This considers all cases. □

3 The Anti-Ramsey Case: General Results

In this section, we complete our results for $(K_{p,p}; 1, q')$-colorings when $p = 2$ and prove some results for general p. Always $r(G, H; 1, q') = 1$, and $R(G, H; 1, e(H)) = e(G)$, so for $p = 2$ it remains to determine $R(K_{n,n}, C_4; 1, q')$ for $q' = 2$ and $q' = 3$. When a vertex v is incident to an edge with color i, we say simply that v is incident with color i or that color i *appears at* v. The next lemma is used both here and in Section 6.

Lemma 3.1 *In a $(K_{p,p};1,p)$-coloring of $K_{n,n}$, let S be a set of vertices that is incident with at least p colors. If $|S|<p$, then S is incident with all colors used.*

Proof: Suppose that some edge e has a color not incident with any vertex of S. By the pigeonhole principle, some vertex v in S is incident with at least two colors. Let x be the endpoint of e adjacent to v. Let vw be an edge incident to v whose color is not on vx.

We have now selected three colors on a subgraph induced by two vertices from each partite set. Only two of the colors named are incident with S. Hence we may choose $p-2$ additional colors incident with S, adding one edge for each color added. From each partite set we use at most $p-2$ additional vertices, so the edges with these $p+1$ colors are contained in a copy of $K_{p,p}$. □

In the special case $p=2$, this says that a vertex incident with at least two colors in a $(C_4;1,2)$-coloring of $K_{n,n}$ is incident with all colors used.

Theorem 3.2 *$R(K_{n,n},K_{p,p};q,p)=n$ for $n\ge p\ge 2$ and $1\le q\le p$.*

Proof: The lower bound follows from Proposition 2.1 with $s=0$. To prove the upper bound, consider a coloring with more than n colors. Since $n\ge p$, we may select p edges with distinct colors. If they form a matching, consider an edge joining two of them. If its color agrees with the color of some selected edge, we let it replace that selected edge. Otherwise, we let it replace an arbitrary selected edge.

The result is a set S of size at most $p-1$ in one partite set, say X, that is incident with at least p colors. By Lemma 3.1, S is incident with all colors. Since the number of colors exceeds n and p, there exist two edges $xy,x'y$ with distinct colors where $x,x'\in S$ and $y\in Y$. Choose $p-1$ edges of other colors incident with S. This adds at most $p-1$ vertices of Y, so these edges lie in a copy of $K_{p,p}$ with at least $p+1$ colors. □

We next give a short proof that $R(K_{n,n},K_{p,p};1,2p-1)=2n-1$ when $p=2$. The subsequent generalization also proves this, but the simplicity of this argument for $p=2$ is worth recording. This idea was used by Erdős, Simonovits, and Sós [11] to observe that $R(K_n,C_3;1,2)=n-1$.

Proposition 3.3 *$R(K_{n,n},C_4;1,3)=2n-1$ for $n\ge 2$.*

Proof: The lower bound is provided by Proposition 2.1 with $s=1$. For the upper bound, we prove that every coloring of $E(K_{n,n})$ with at least $2n$ colors has a polychromatic 4-cycle.

Pick an edge of each color, and call the resulting subgraph H. Since H has at least as many edges as vertices, it has a shortest cycle C. Any chord of C in $K_{n,n}$ creates a shorter cycle with distinct colors using one of the paths on C between its endpoints. Thus C is a 4-cycle. □

Lemma 3.4 *When $m, n \geq 3$, every m-by-n matrix using at most $m+n$ distinct labels as entries has a row or column whose deletion eliminates at most one label. When $m \neq n$, there is such a line (row or column) of the shorter length.*

Proof: First consider the case $m = n$. Consider the pairs consisting of a label and a line such that the label appears only in that line. We obtain the desired line unless every line appears in at least two pairs, which yields at least $2(m+n)$ pairs.

On the other hand, every label appears in at most one such pair, unless the label appears only once in the matrix, in which case it appears in two pairs. Thus the number of pairs is at most $2(m+n)$, with equality only if there are $m+n$ labels and each appears only once in the matrix.

Hence we are finished unless there are exactly $m+n$ labels, each appearing exactly once. This requires $m+n = mn$, which we rewrite as $1 = (m-1)(n-1)$, which has only the solution $m = n = 2$ among positive integers.

The case $m \neq n$ is simpler. We may assume that $m > n$ and consider only rows. A label is eliminated by deleting a row only if it is restricted to that row, so every label appears in at most one pair. Hence there are at most $m+n$ pairs. Since $m+n < 2m$, the pigeonhole principle implies that some row deletion eliminates at most one label. □

Theorem 3.5 $R(K_{n,n}, K_{p,p}; q, 2p-1) = 2n-1$ *when $n \geq p \geq 2$ and $q \leq p$.*

Proof: The lower bound is provided by Proposition 2.1 with $s = 1$. For the upper bound, we use the matrix formulation to prove more generally that $R(K_{m,n}, K_{p,p}; 1, 2p-1) \leq m+n-1$ when $m, n \geq p$.

We prove by induction on $m+n$ that an m-by-n matrix containing at least $m+n$ distinct colors has a p-by-p submatrix containing at least $2p$ distinct colors. The claim is trivial when $m = n = p$.

For $m+n > 2p$, we combine colors to obtain an m-by-n matrix with exactly $m+n$ colors. It suffices to prove the claim for this matrix. Using Lemma 3.4, we delete one row or column, retaining at least p rows and p columns and $m+n-1$ colors. By the induction hypothesis, we find within this submatrix a p-by-p submatrix having at least $2p$ colors. □

4 Bipartite Turán numbers

In this section, we prove new bounds for some bipartite Turán numbers. We apply them in the next section to prove bounds on $R(K_{n,n}, K_{3,3}; 1, q')$ for $6 \leq q' \leq 8$. We use $K_{3,3} - P_3$ to mean $K_{3,3} - E(P_3)$, which is well-defined since the automorphism group of $K_{3,3}$ is transitive on copies of P_3. The previous best upper bound on $\mathrm{ex}(K_{n,n}, K_{3,3} - P_3)$ was $O(n^{3/2})$ by a more general result of Erdős [8]. We sharpen this below.

Theorem 4.1

$$\sqrt{2}\, n^{3/2} - O(n^{4/3}) \leq \mathrm{ex}(K_{n,n}, K_{3,3} - P_3) \leq \sqrt{2}\, n^{3/2} + n/2.$$

Proof: The lower bound follows from the lower bound on $\mathrm{ex}(K_{n,n}, K_{2,3})$ due to Füredi [12]. For the upper bound, we proceed by induction on n. The bound is trivial for $n \leq 3$, so we may assume that $n > 3$.

Consider $G \subseteq K_{n,n}$ with $e(G) > \sqrt{2}n^{3/2} + n/2$. We show that G contains $K_{3,3} - P_3$. By the standard counting argument as in [17], G contains two copies of $K_{2,3}$ with the independent 3-sets in opposite partite sets of G. Let x and y be vertices of these two independent 3-sets.

If x or y has degree at least three, then it lies in a copy of $K_{3,3} - P_3$ in G. Hence we may assume that both have degree exactly two. Let $G' = G - \{x, y\}$. We have $G' \subseteq K_{n-1,n-1}$, and it is easy to verify that for $n \geq 4$,

$$e(G') = e(G) - 4 > (\sqrt{2}n^{3/2} + n/2) - 4 > \sqrt{2}(n-1)^{3/2} + (n-1)/2.$$

By the induction hypothesis, G' contains a copy of $K_{3,3} - P_3$. □

For $r > l$, let $H(r, l)$ denote the bipartite graph obtained from $K_{r,r}$ by deleting the edges of a copy of $K_{l,l}$. Our result for $\mathrm{ex}(K_{n,n}, H(r, l))$ is proved by a slight modification of an argument of Füredi and West [14], which in turn sharpened a result of Erdős [8] on $\mathrm{ex}(K_n, H(r, l))$.

In the proof, we extend $\binom{x}{t}$ to nonnegative real x for each nonnegative integer t. We take $\binom{x}{0} = 1$ for all real $x \geq 0$. When $t \geq 1$, we take $\binom{x}{t} = 0$ for $0 \leq x < t - 1$, and for $x \geq t - 1$ we view $\binom{x}{t}$ as the real polynomial $x(x-1)\ldots(x-t+1)/t!$ of degree t in x. The resulting functions are convex, and thus

$$\sum_{i=1}^{m} \binom{x_i}{t} \geq m \binom{\sum x_i/m}{t}. \qquad (1)$$

We will also use the following simple lemma, which was proved and applied by Füredi [13] to a related problem.

Lemma 4.2 ([13]) *If $n, t \geq 1$ are integers and $c, x_0, x_1, \ldots, x_t \geq 0$ are real numbers, then*

$$\sum_{1\leq i\leq n} \binom{x_i}{t} \leq c\binom{x_0}{t} \quad \text{implies} \quad \sum_{1\leq i\leq n} x_i \leq x_0 c^{1/t} n^{1-1/t} + (t-1)n.$$

Theorem 4.3 *Given integers r, l with $1 \leq l < r$, let c be the smallest real number such that $\binom{c}{t} \geq 2\binom{r-1}{t}$, where $t = r - l$. Then*

$$\text{ex}(K_{n,n}, H(r,l)) \leq c^{1/t} n^{2-1/t} + (t-1)n.$$

Proof: Given $K_{n,n}$ with bipartition X, Y, let G be a subgraph that does not contain $H(r,l)$. We bound $e = |E(G)|$. Let $d(x)$ denote the degree of a vertex x, and for $A \subset V(G)$ let $d(A)$ denote the number of common neighbors of A.

Let Z be the number of copies of $K_{t,t}$ in G. We form such a subgraph by choosing a t-set $A \subseteq X$ and choosing t of its common neighbors in Y. Thus $Z = \sum_{A\in\binom{X}{t}} \binom{d(A)}{t}$.

We first find a lower bound on Z. By (1),

$$\sum_{A\in\binom{X}{t}} \binom{d(A)}{t} \geq \binom{n}{t}\binom{\sum d(A)/\binom{n}{t}}{t}.$$

Since $d(A)$ counts the stars with leaf set A, the total $\sum d(A)$ is the number of stars with t edges whose centers lie in Y. These can alternatively be counted by choosing t neighbors for each choice of the central vertex in Y. Applying (1) to the resulting sum yields

$$\sum d(A) = \sum_{y\in Y} \binom{d(y)}{t} \geq n\binom{\sum d(y)/n}{t}.$$

Together, these computations yield

$$Z \geq \binom{n}{t}\binom{n\binom{e/n}{t}/\binom{n}{t}}{t} \tag{2}$$

We next find an upper bound on Z. Let $\mathcal{A} = \{A \in \binom{X}{t} \cup \binom{Y}{t} : d(A) \geq r\}$. Our main observation is that a copy of $K_{t,t}$ with partite sets A, B cannot have both $A, B \in \mathcal{A}$. Such a copy, together with edges to l common neighbors of A and to l common neighbors of B, would form a copy of $H(r,l)$. Consequently, every copy of $K_{t,t}$ has at least one of its partite sets outside $\mathcal{A}$. Hence

$$Z \leq 2\binom{n}{t}\binom{r-1}{t} \leq \binom{n}{t}\binom{c}{t} \tag{3}$$

Comparing (2) and (3) yields

$$n\binom{e/n}{t}/\binom{n}{t} \leq c.$$

With $x_0 = n$ and $x_1 = \cdots = x_n = e/n$, Lemma 4.2 now yields the bound claimed. □

Corollary 4.4 *If α is the positive root of $x^2 - x - 4 = 0$, and $\beta = \sqrt{\alpha} = 1.60048..$, then*

$$\sqrt{2}\, n^{3/2} - O(n^{4/3}) \leq \mathrm{ex}(K_{n,n}, H(3,1)) \leq \beta\, n^{3/2} + n.$$

For $1 \leq l \leq r/2$ and $t = r - l$,

$$\mathrm{ex}(K_{n,n}, H(r,l)) \leq r^{1/t} n^{2-1/t} + (t-1)n.$$

Proof: The upper bounds follow directly from Theorem 4.3 using $2 \leq \binom{\alpha}{2}$ and using $2\binom{r-1}{r-l} \leq \binom{r}{r-l}$ for $1 \leq l \leq r/2$.

For the lower bound, we observe that $K_{2,3} \subseteq H(3,1)$ and use the lower bound for $\mathrm{ex}(K_{n,n}, K_{2,3})$ due to Füredi [12]. □

5 The Anti-Ramsey Case when $p = 3$

In this section, we study the numbers $R(K_{n,n}, K_{3,3}; 1, q')$. For $6 \leq q' \leq 8$, we relate these to Turán numbers via simple observations and then apply the bounds of Section 4. Subsequently, we determine the exact value when $q' = 4$ by combinatorial argument. The values for $q' = 3$ and $q' = 5$ were determined in Theorems 3.2 and 3.5.

The observations we use to relate anti-Ramsey numbers and Turán numbers generalize some notions of Erdős, Simonovits, and Sós [11]. Special cases of some of these were applied by Axenovich and Kündgen [2].

Lemma 5.1 *Let $t = \min_{F \subseteq \mathcal{F}} e(F) - 1$. If every copy in G of each graph in $\mathcal{F}$ belongs to a copy of H in G, then $R(G, H; 1, t) \leq \mathrm{ex}(G, \mathcal{F})$.*

Proof: If an edge-coloring of G has a polychromatic subgraph in $\mathcal{F}$, then a copy of H containing that subgraph has at least $t+1$ colors. Choosing one edge of each color in a coloring with no polychromatic graph in $\mathcal{F}$ thus limits the number of colors to $\mathrm{ex}(G, \mathcal{F})$. □

Lemma 5.2 *If $t = 1 + \mathrm{ex}(H, \mathcal{F})$, then $R(G, H; 1, t) > \mathrm{ex}(G, \mathcal{F})$.*

Proof: Let G' be a subgraph of G having $\mathrm{ex}(G, \mathcal{F})$ edges and containing no graph in $\mathcal{F}$. Color the edges in G' with distinct colors, and let the remaining edges of G have a single additional color. A copy of H with more than t colors has more than $\mathrm{ex}(H, \mathcal{F})$ of them in G'. This forces some subgraph in $\mathcal{F}$ to appear in G', which by construction is impossible. Hence we have constructed an $(H; 1, t)$-coloring of G. □

Corollary 5.3 *If H is edge-transitive, then $R(G,H;1,e(H)-1) > \mathrm{ex}(G,H-e)$, where $H-e$ denotes the graph obtained from H by deleting an edge.*

Proof: $\mathrm{ex}(H,H-e) = e(H)-2$. □

Theorem 5.4

$$\Omega(n^{4/3}) \le \mathrm{ex}(K_{n,n},\{C_4,C_6\}) < R(K_{n,n},K_{3,3};1,6) \le \mathrm{ex}(K_{n,n},K_{3,3}-P_3) \le O(n^{3/2}),$$

$$n^{3/2}+\Omega(n^{4/3}) \le \mathrm{ex}(K_{n,n},C_4) < R(K_{n,n},K_{3,3};1,7) \le \mathrm{ex}(K_{n,n},K_{3,3}-e) \le (1.61)\ n^{3/2}+n,$$

$$\Omega(n^{3/2}) \le \mathrm{ex}(K_{n,n},K_{3,3}-e) < R(K_{n,n},K_{3,3};1,8) \le \mathrm{ex}(K_{n,n},K_{3,3}) \le O(n^{5/3}).$$

Proof: The inner upper bounds follow from Lemma 5.1, since $K_{3,3}-P_3$, $K_{3,3}-e$, and $K_{3,3}$ have 7, 8, and 9 edges, respectively. The first two inner lower bounds follow from Lemma 5.2, since $\mathrm{ex}(K_{3,3},\{C_4,C_6\}) = 5$ (6 edges force a cycle in a 6-vertex graph) and $\mathrm{ex}(K_{3,3},C_4) = 6$ (6 edges force a cycle; if it is not a 4-cycle, then a 7th edge yields a chord of C_6). The third inner lower bound follows from Corollary 5.3, since $K_{3,3}$ is edge-transitive.

The outer bounds primarily use known results. Wenger [18] proved that $\mathrm{ex}(K_{n,n},\{C_4,C_6\}) \ge \Omega(n^{4/3})$. Theorem 4.1 yields $\mathrm{ex}(K_{n,n},K_{3,3}-P_3) \le O(n^{3/2})$. The lower bound on $\mathrm{ex}(K_{n,n},C_4)$ is by Erdős, Rényi, and Sós [10], and the upper bound on $\mathrm{ex}(K_{n,n},K_{3,3})$ is by Kővári, Sós, and Turán [17] (refined by Füredi [13]). The bounds on $\mathrm{ex}(K_{n,n},K_{3,3}-e)$ come from Corollary 4.4. □

With the results from Section 3, only one entry in Table 2 remains to be proved. We obtain a general construction for $R(K_{n,n},K_{p,p};1,p+1) \ge n+1$ and prove optimality when $p=3$.

Theorem 5.5 $R(K_{n,n},K_{3,3};1,4) = n+1$.

Proof: For the lower bound, we use a polychromatic perfect matching and give all other edges a single additional color. Since $K_{p,p}$ has only p vertices in each partite set, each copy of $K_{p,p}$ receives at most $p+1$ colors.

For the upper bound when $p=3$, consider a $(K_{3,3};1,4)$-coloring with the most colors. Let G be a subgraph consisting of one edge of each color. We show first that every component of G is a tree having a partite set with at most two vertices. If G has a 4-cycle, then the $K_{3,3}$ containing it and one edge of another color has five colors. If G contains a tree with three vertices in each partite set (such as a path of five edges from any cycle of length at least 6), then the $K_{3,3}$ encompassing these five edges has at least five colors. Hence G is a forest that satisfies the condition claimed.

We now choose G to maximize the size of a largest component C. Among the partite sets X and Y, let X be the one having at most two vertices of C. Let x be a vertex of X in C, with neighbors $y_1, \ldots, y_s$ in C. If C is not a star, then let x^* be the other vertex of X in C, with neighbors $y_s, \ldots, y_t$ in C, where $t \geq s$.

If G does not have two edges outside C with a common endpoint in Y, then every vertex of Y has degree at most one in G, except that y_s may have degree 2. This limits G to $n+1$ edges.

It thus suffices to prove that no two edges of G outside C have a common endpoint in Y. Suppose that zy and $z'y$ are two such edges. If for all $1 \leq i \leq t$ the color on zy_i appears in C at y_i, then we can replace an edge at each y_i (to x or x^*) with an edge to z to obtain G' having a larger component than C.

Hence some zy_i has different color from the edge at y_i in C. If this color is not on an edge at y_s that cuts C into nontrivial components, then we change G to enlarge C by adding the path y_i, z, y, z' and dropping the possible edge of C that has the same color as $y_i z$.

Hence we may assume that C is not a star (both x and x^* are defined), and that for some $i \neq s$, the color c on zy_i appears at y_s in C. Let $w \in \{x, x^*\}$ be the common neighbor of y_i and y_s in C. Let P be the polychromatic path z', y, z, y_i, w. If c appears at y_s not on $y_s w$ or if w has degree at least 3 in C, then P extends with another edge from w to yield a $K_{3,3}$ having five colors. Hence $y_s w$ has color c and w has degree two in C.

Note that since P has four edges, the choice of G yields $t \geq 3$. Let w' be the vertex of $\{x, x^*\}$ other than w. Since $t \geq 3$, we can choose $y_j \in N_C(w') - N_C(w)$. To avoid enlarging C or adding a fifth color to a $K_{3,3}$ containing P, the color on $y_j z$ must appear in both C and P. The only such color is c, but now z', y, z, y_j, w', y_s is a polychromatic P_6.

By this analysis, there is no pair of edges such as zy and $z'y$, and the claim follows. □

6 Uniform colorings

In this section, we study edge-colorings of $E(K_{n,n})$ in which every copy of C_4 receives exactly q colors. When $q = 1$, the entire $K_{n,n}$ must be monochromatic. When $q = 4$, the entire $K_{n,n}$ must be polychromatic. It remains to consider $q = 2$ and $q = 3$.

Theorem 6.1 *When $n \geq 5$, every $(C_4; 2, 2)$-coloring of $K_{n,n}$ uses exactly n colors, and for $2 \leq n \leq 4$ at most n colors are used.*

Proof: Proposition 2.1 with $s = 0$ provides such a coloring. Since every $(C_4; 2, 2)$-coloring is a $(C_4; 1, 2)$-coloring and $R(K_{n,n}, C_4; 1, 2) = n$, every such coloring uses at most n colors.

We must show that for $n \geq 5$, at least n colors are needed. We argue first that at least three colors must be used. Otherwise, consider the restriction to a copy G of $K_{5,5}$ with partite sets A, B. By the pigeonhole principle, one color is used at least 13 times; we call it *blue*. Let d_i be the number of blue edges in G incident to the ith vertex of A. That vertex is a common neighbor via blue for $\binom{d_i}{2}$ pairs of vertices in B, which must be at most $\binom{5}{2} = 10$ to avoid a monochromatic C_4. Since $\sum d_i \geq 13$, convexity of the quadratic yields

$$11 = \binom{3}{2} + \binom{3}{2} + \binom{3}{2} + \binom{2}{2} + \binom{2}{2} \leq \sum_{i=1}^{5} \binom{d_i}{2} \leq 10,$$

which is impossible.

Lemma 3.1 implies that every color appears at each vertex where at least two colors appear, so every vertex is monochromatic or receives all colors. If two of the latter are adjacent, then since each is incident with at least three colors we have a polychromatic path of three edges. The edge between its endpoints completes a 4-cycle with at least three colors.

Hence in one partite set all the vertices are monochromatic. If two are incident with the same color, then we have a monochromatic C_4. Otherwise, we have n distinct colors used, as claimed. □

Corollary 6.2 *$R(K_{n,n}, C_4; 2, 2) = n$, and* $r(K_{n,n}, C_4; 2, 2) = \begin{cases} n & \text{for } n \geq 5 \\ 2 & \text{for } 2 \leq n \leq 4 \end{cases}$

Proof: The proof of Theorem 6.1 yields these results except for the 2-edge-coloring when $2 \leq n \leq 4$. For this, we use one color on a Hamiltonian cycle in $K_{4,4}$ and the second color on the complementary cycle. Since each color class is an 8-cycle, there is no monochromatic 4-cycle, and only two colors are used. For $n < 4$, we simply delete vertices from this construction. □

Perhaps surprisingly, it is generally impossible to have exactly three colors on every 4-cycle.

Theorem 6.3 *If $n \geq 5$, then $K_{n,n}$ has no $(C_4; 3, 3)$-coloring.*

Proof: We use the matrix formulation. Every 2-by-2 submatrix of our n-by-n matrix must have exactly three labels.

We show first that if $n \geq 4$, then some label is repeated in some row or column. Otherwise, we may assume by symmetry that the first two positions are 0,1 in the first row and 1,2 in the second row. We cannot use 1 again in the first two columns. Since we can't repeat a label in a row, every subsequent row must have both 0 and 2 in the first two columns. With $n \geq 4$, this yields repetitions in the first two columns.

Thus we may assume that the first row starts with 0,0. Each subsequent row must have distinct labels in the first two positions. Furthermore, these pairs in any two of the rows have exactly one common element. The only ways to do this are with three pairs chosen from three elements or with all pairs sharing a common element. The first case limits the matrix to four rows (and is achievable). Hence we may assume that all pairs share a common element. Furthermore, when $n \geq 4$ that common element in the next three pairs must appear in the same column twice.

The argument of the preceding paragraph implies that whenever an element is repeated in a row, there is an element repeated in one of the two columns containing the repetition, and it appears in every row of those two columns other than the original row. Furthermore, the same statement holds with "row" and "column" interchanged.

Thus when we start with a 0,0 repetition, we obtain a 1,1 repetition, and from the 1,1 repetition we obtain a repetition parallel to the original one. This repetition cannot use 0, because it would place three 0s in a 2-by-2 submatrix containing the original repetition. This leaves us with the two cases listed below.

$$\begin{matrix} 0 & 0 & a \\ 1 & 2 & 2 \\ 1 & b & c \end{matrix} \qquad \begin{matrix} 0 & 0 & a & d \\ 1 & e & 2 & 2 \\ 1 & 2 & b & c \end{matrix}$$

In Case 1, the first two rows force $a = 1$. Now the first row and column prevent 0 and 1 from appearing as b or c. Since b, c are different from each other and 0,1, they form a polychromatic submatrix with the first row.

In Case 2, the first two rows force a and d to be 0 and 1. After this the first and third rows force $\{b, c\} \cap \{0, 1\} = \emptyset$. Since the second row forces $b \neq c$, we have a, b, c, d forming a polychromatic submatrix.

We have obtained a contradiction in all cases when $n > 4$. □

Proposition 6.4 *$r(K_{3,3}, C_4; 3, 3) = 3$ and $r(K_{4,4}, C_4; 3, 3) = 5$, and also $R(K_{3,3}, C_4; 3, 3) = R(K_{4,4}, C_4; 3, 3) = 5$.*

Proof: For $n \geq 2$, each C_4 receives three colors, so at least three colors are used. For $n = 3$, this is achievable using a 3-by-3 Latin square. When $n = 3$, a coloring with five colors appears as the upper right 3-by-3 submatrix of the coloring below, and Proposition 3.3 implies that using six colors always yields a polychromatic copy of C_4.

Now let $n = 4$. We require three colors in every 2 by 2 submatrix. The first part of the proof of Theorem 6.3 shows that there must be a repetition in some row or column. If there is a constant row, say 0000, then another row must have four other distinct colors, say 1234. If a sixth color appears elsewhere, then it forces the three other columns to repeat from the row with 1234, and now we have a C_4 with 2 colors. Hence when there is a constant row or column, the total number of colors must be five. Below we exhibit such a coloring.

Suppose that there is no constant row or column. As in Theorem 6.3, the remainder of the two columns containing a row repetition consists of three pairs from a triple or three pairs with a common element. Using these observations, a short case analysis (which we omit here) shows that every $(C_4; 3, 3)$-coloring of $K_{4,4}$ uses exactly five colors. □

0	0	0	0
1	2	3	4
2	3	1	4
3	1	2	4

7 $(C_4; 2, 3)$-colorings

For $(C_4; 2, 3)$-colorings of $K_{n,n}$, the maximum number of colors is given by our earlier results.

Corollary 7.1 $R(K_{n,n}, C_4; 2, 3) = 2n - 1$.

Proof: Proposition 2.1 with $s = 1$ yields a construction with $2n-1$ colors. Since every $(C_4; 2, 3)$-coloring is a $(C_4; 1, 3)$-coloring, the upper bound follows from Proposition 3.3. □

Determining $r(K_{n,n}, C_4; 2, 3)$ is much more difficult. The construction in Proposition 2.4 yields $r(K_{n,n}, C_4; 2, 3) \leq n - 3$ for $n \geq 6$. On the other hand, a $(C_4; 2, 3)$-coloring is also a $(C_4; 2, 4)$-coloring. Since $r(K_{n,n}, C_4; 2, 4) \geq (1 + o(1))\sqrt{n}$, the same lower bound holds also for $r(K_{n,n}, C_4; 2, 3)$.

Nevertheless, the known $(C_4; 2, 4)$-colorings of $K_{n,n}$ with $O(\sqrt{n})$ colors have the property that many C_4's receive all distinct colors; thus they are not $(C_4; 2, 3)$-colorings. Our final result improves the lower bound for $r(K_{n,n}, C_4; 2, 3)$ from about $\sqrt{n}$ to about $n/3$.

Lemma 7.2 *If a $(C_4; 2, 3)$-coloring of $K_{n,n}$ uses at most $(1 - 4/t)n - t + 2$ colors, then for each vertex there cannot be two colors that each appear on at least t edges incident to it.*

Proof: Suppose that the claim fails at $x \in X$; let red and blue be two colors that each appear on at least t edges incident to x. A *blue [red] neighbor* of a vertex is a neighbor of it via an edge with that color. Let A be a set of t blue neighbors of x, and let B be a set of t red neighbors of x.

Our first objective is to find vertices in A and B with few red or blue neighbors. For $y \in A$, let $l(y)$ be the number of blue neighbors of y other than x, and let $m(y)$ be the number of red neighbors of y. To avoid a blue C_4 through x, each vertex of $X - \{x\}$ must have at most one blue neighbor in A; thus $\sum_{y \in A} l(y) \le n-1$. To avoid a red C_4, two vertices in A must have at most one common red neighbor; with $X - \{x\}$ available as red neighbors we have $\sum_{y \in A} m(y) \le n-1+\binom{t}{2}$.

Summing the two inequalities and applying the pigeonhole principle yields a vertex $v \in A$ with $l(v) + m(v) \le \frac{2n-2}{t} + \frac{t-1}{2}$. Applying the analogous argument to B yields a vertex $w \in B$ with at most $\frac{2n-2}{t} + \frac{t-1}{2}$ red or blue neighbors other than x. Including x, fewer than $\frac{4n}{t} + t$ vertices of X have red or blue edges to v or w.

Let z be a vertex of X outside this set. Since the path v, x, w already has edges colored red and blue, and neither zv nor zw has those colors, the edges zv and zw must have the same color to avoid a polychromatic C_4. Furthermore, avoiding a monochromatic C_4 through v and w forces this color to be distinct for distinct choices of z. Thus we must have more than $(1 - 4/t)n - t$ additional colors besides red and blue. □

Theorem 7.3 $r(K_{n,n}, C_4; 2, 3) > n/3 - 11$.

Proof: Consider a $(C_4; 2, 3)$-coloring with fewer colors. Say that a color is *plentiful* at v if it occurs on at least $n/3 + 2$ edges incident to v.

We claim that X contains two vertices with no plentiful color. Otherwise, $n-1 > 2(n/3-11)$ implies that three vertices have the same plentiful color. The $n + 6$ edges from these vertices in this color contain a monochromatic C_4.

Let x, z be two vertices of X with no plentiful color. Some $y \in Y$ has distinct colors on its edges to x and z; otherwise, we have a monochromatic C_4 because there are fewer than n colors. Let blue be the color of yx and red the color of yz.

We claim that fewer than $2n/3+13$ vertices of Y have red or blue edges to x or z. Since x and z have no plentiful color, at each of these vertices the color that appears on the largest number of incident edges appears on fewer than $n/3 + 2$ incident edges. Because the number of colors is at most $n/3 - 4$, Lemma 7.2 with $t = 6$ says that for every vertex there is at most one color that appears on as many as 6 incident edges. Therefore red and blue appear on fewer than $n/3 + 7$ edges in total at x, and similarly for z. Thus fewer than $2n/3 + 13$ vertices of Y have red or blue edges to x or z (y was counted twice).

There remain more than $n/3 - 13$ vertices of Y whose edges to x and z are neither blue nor red. Let w be such a vertex. Since the path x, y, z already has edges colored red and blue, and neither wx nor wy has those colors, the edges wx and wy must have the same color to avoid a polychromatic C_4. Furthermore, avoiding a monochromatic C_4 through x and z forces this color to be distinct for distinct choices of w. Thus we must have more than $n/3 - 13$ additional colors besides red and blue. □

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