Co-degree density of hypergraphs

Dhruv Mubayi *
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607

Yi Zhao †
Department of Mathematics and Statistics
Georgia State University
Atlanta, GA 30303

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Abstract

For an r-graph H, let $\mathcal{C}(H) = \min_S d(S)$, where the minimum is taken over all $(r-1)$-sets of vertices of H, and $d(S)$ is the number of vertices v such that $S \cup \{v\}$ is an edge of H. Given a family $\mathcal{F}$ of r-graphs, the co-degree Turán number co-ex$(n, \mathcal{F})$ is the maximum of $\mathcal{C}(H)$ among all r-graphs H which contain no member of $\mathcal{F}$ as a subhypergraph. Define the co-degree density of a family $\mathcal{F}$ to be

$$\gamma(\mathcal{F}) = \limsup_{n\to\infty} \frac{\text{co-ex}(n, \mathcal{F})}{n}.$$

When $r \geq 3$, non-zero values of $\gamma(\mathcal{F})$ are known for very few finite r-graphs families $\mathcal{F}$. Nevertheless, our main result implies that the possible values of $\gamma(\mathcal{F})$ form a dense set in $[0, 1)$. The corresponding problem in terms of the classical Turán density is an old question of Erdős (the jump constant conjecture), which was partially answered by Frankl and Rödl [14]. We also prove the existence, by explicit construction, of finite $\mathcal{F}$ satisfying $0 < \gamma(\mathcal{F}) < \min_{F\in\mathcal{F}} \gamma(F)$. This is parallel to recent results on the Turán density by Balogh [1], and by the first author and Pikhurko [23].

1. Introduction

All hypergraphs discussed in this paper are finite and have no multiple edges. An r-graph H (for $r \geq 2$) is a hypergraph whose edges all have size r. Write $V(H)$ and $E(H)$ for the vertex

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set and edge set of H, respectively, with $e(H) = |E(H)|$. The notation H_n indicates that $|V(H_n)| = n$.

1.1. The Turán problem and co-degree problem

Given a family $\mathcal{F}$ of r-graphs, the r-graph G_n is $\mathcal{F}$-free if it contains no member of $\mathcal{F}$ as a (not necessarily induced) subhypergraph. The *Turán number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum of $e(G_n)$ over all $\mathcal{F}$-free r-graphs G_n. The limit $\lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$ exists by a standard averaging argument of Katona-Nemetz-Simonovits [18], and is often called the *Turán density* and written $\pi(\mathcal{F})$.

Classical extremal graph theory began with Turán's theorem, which determines $\mathrm{ex}(n, F)$ (and therefore $\pi(F)$) when F is a complete graph. The celebrated Erdős-Simonovits-Stone theorem (**ESS**) [10, 12] generalizes Turán's theorem. It states that for any graph F with chromatic number $\chi(F)$, and any $\varepsilon > 0$, there exists $N > 0$ such that every graph G_n with $n > N$ and $e(G) \geq (1 + \varepsilon)(1 - \frac{1}{\chi(F)-1})\binom{n}{2}$ contains a copy of F. It is easy to see that **ESS** implies that $\pi(\mathcal{F}) = \min_{F\in\mathcal{F}} 1 - \frac{1}{\chi(F)-1}$ for any graph family $\mathcal{F}$.

Although the Turán problems study the largest size of graphs not containing certain subgraphs, Turán's theorem and **ESS** can also be viewed as theorems on the largest possible minimum degree of such graphs. For example, letting $\delta(G)$ denote minimum degree in a graph G, **ESS** is equivalent to the following statement:

For any graph F and $\varepsilon > 0$, there exists $N > 0$ such that any graph G_n with $n > N$ and $\delta(G_n) \geq (1 + \varepsilon)(1 - \frac{1}{\chi(F)-1})n$ contains a copy of F.

To see this, we first note that an n-vertex graph with minimum degree cn has at least $c\binom{n}{2}$ edges. On the other hand, given a graph G_n with $(c+\varepsilon)\binom{n}{2}$ edges (fixed $c, \varepsilon > 0$ and large n), we can delete its vertices of small degrees obtaining a subgraph G' on $m \geq \varepsilon^{1/2}n$ vertices with minimum degree at least cm (see, *e.g.*, [2] p. 121 for details).

In this paper we investigate a corresponding extremal problem on hypergraphs. We must first clarify how to define *degree* in hypergraphs. If we consider the usual degree $d(v)$ of a vertex v, defined as the number of edges containing v, then (as indicated above) the minimum degree problem is again essentially equivalent to the Turán problem, which is well-studied and known to be extremely hard (see, *e.g.*, [16] for a survey). Therefore we consider another generalization of degree to hypergraphs, called *co-degree*. Given an r-graph G and a set $S \subset V(G)$ with $|S| = r - 1$, we denote by $N(S)$ or $N_G(S)$ the set of $v \in V(G)$ such that $S \cup \{v\} \in E(G)$. The *co-degree* of S is $d(S) = d_G(S) = |N(S)|$. When $S = \{v_1, \dots, v_{r-1}\}$, we abuse notation by writing $N(v_1, \dots, v_{r-1})$ and $d(v_1, \dots, v_{r-1})$. Let $\mathcal{C}(G) = \min\{d(S) : S \subset V(G), |S| = r - 1\}$ denote the minimum co-degree in G, and let $c(G) = \mathcal{C}(G)/|V(G)|$.

Co-degree in hypergraphs seems to be the natural extension of degree in graphs for many problems. Two examples are the recent results of Kühn-Osthus [20] and Rödl-Ruciński-Szemerédi [26] who extended Dirac's theorem on Hamilton cycles to 3-graphs, and results by the same sets of authors [21, 25] on the minimum co-degree threshold guaranteeing a perfect matching in r-graphs.

The purpose of this paper is to show that, for hypergraphs, the co-degree extremal problem exhibits some different phenomena than the classical Turán problem. Since co-degree reduces to degree when the uniformity $r = 2$, our results on co-degree (for all $r \geq 2$) will also reveal some similarities in the graph and hypergraph cases.

Definition 1.1. *Let $\mathcal{F}$ be a family of r-graphs. The co-degree Turán number* $\text{co-ex}(n, \mathcal{F})$ *is the maximum of $\mathcal{C}(G_n)$ over all $\mathcal{F}$-free r-graphs G_n. The co-degree density of $\mathcal{F}$ is*

$$\gamma(\mathcal{F}) := \limsup_{n\to\infty} \frac{\text{co-ex}(n, \mathcal{F})}{n}.$$

Remark. Strictly speaking, one should divide by $n - (r - 1)$ instead of n in the definition of $\gamma(\mathcal{F})$. However, since r is fixed and $n \to \infty$, this will not change any of our results on $\gamma(\mathcal{F})$, and so we prefer the technically simpler version above.

The argument in [18] shows that $\text{ex}(n, \mathcal{F})/\binom{n}{r}$ is non-increasing in n, and therefore one obtains that $\pi(\mathcal{F}) = \lim_{n\to\infty} \text{ex}(n, \mathcal{F})/\binom{n}{r}$ exists. Although we could not prove that $\text{co-ex}(n, \mathcal{F})/n$ (or $\text{co-ex}(n, \mathcal{F})/(n - r + 1)$) is non-increasing, we do prove that $\lim_{n\to\infty} \text{co-ex}(n, \mathcal{F})/n$ exists.

Proposition 1.2. $\gamma(\mathcal{F}) = \lim\limits_{n\to\infty} \text{co-ex}(n, \mathcal{F})/n$ *for all r-graph families $\mathcal{F}$.*

1.2. Comparing γ and π

It is easy to see that $\gamma(\mathcal{F}) = \pi(\mathcal{F})$ for every graph family $\mathcal{F}$. The situation for r-graphs when $r \geq 3$ is more complicated. There exists an r-graph F for which $\pi(F)$ and $\gamma(F)$ differ almost by 1. For example, fix $r = 3$ and $k \geq 3$, and let F be the 3-graph obtained from a complete graph on k vertices by enlarging each edge with a new (distinct) vertex. Then a simple greedy procedure shows that $\text{co-ex}(n, F) \leq \binom{k}{2} + k - 2$ and thus $\gamma(F) = 0$. On the other hand, let G_n be the 3-graph whose vertices are equally partitioned into $k - 1$ sets and whose edges are the triples intersecting each partition set in at most one vertex. Clearly G_n does not contain F, and since $e(G_n) \geq \frac{(k-2)(k-3)}{(k-1)^2}\binom{n}{3}$, we conclude that $\pi(F) \to 1$ as $k \to \infty$.

In the opposite direction, for every even $r \geq 4$, there is an r-graph whose π and γ values are the same. Let $T^{(2k)}$ be the $2k$-graph obtained by letting P_1, P_2, P_3 be pairwise disjoint sets of size k and taking as edges the three sets $P_i \cup P_j$ with $i \neq j$. Frankl [13] determined that $\pi(T^{(2k)}) = 1/2$ (see also [19, 28]). Since the extremal configuration given in [13] has minimum co-degree $n/2 - o(n)$, we conclude that $\gamma(T^{(2k)}) = 1/2 = \pi(T^{(2k)})$.

There are a few 3-graphs whose γ values are known or even conjectured. The *only* known nontrivial examples are the Fano plane $\mathbf{F}$ and some hypergraphs closely resembling $\mathbf{F}$. The first author recently [22] proved that $\gamma(\mathbf{F}) = 1/2$, in contrast to a well-known result of de Caen and Füredi [6] that $\pi(\mathbf{F}) = 3/4$. Let K_4^3 denote the complete 3-graph on 4 vertices. It was conjectured by Czygrinow and Nagle [7] that $\gamma(K_4^3) = 1/2$ while the famous Turán Conjecture [29] claims that $\pi(K_4^3) = 5/9$.

As far as we know, all 3-graphs G_n providing lower bounds for π satisfy that $e(G_n)/\binom{n}{3} > \mathcal{C}(G_n)/n + \alpha$ for some fixed $\alpha > 0$ and all large n. For example, the well-known construction of Turán forbidding K_4^3 has about $\frac{5}{9}\binom{n}{3}$ edges but its minimum co-degree is only $\frac{n}{3}$. Hence it is an interesting problem to determine if $0 < \gamma(F) = \pi(F)$ for *any* 3-graph F.

1.3. Our results

One fundamental result in extremal hypergraph theory is the so called supersaturation phenomenon, discovered by Brown, Erdős and Simonovits [11]. An indication of its usefulness is that when applied to graphs, it is essentially equivalent to **ESS**.

As Proposition 1.4 below shows, the supersaturation phenomenon also holds for γ.

Definition 1.3. *Let ℓ, n be positive integers and let F be an r-graph on $[h]$. The* blow-up *$F(\ell)$ is the h-partite r-graph (V, E) with $V = V_1 \cup V_2 \cup \cdots \cup V_h$, every $|V_i| = \ell$ and $E = \{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} : v_{i_j} \in V_{i_j}, \{i_1, i_2, \ldots, i_r\} \in E(F)\}$.*

For example, blowing up one r-set creates a complete r-partite r-graph $K_r^r(\ell)$.

Proposition 1.4. (Supersaturation) *Let F be an r-graph on f vertices. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and N such that every r-graph G_n with $n > N$ and $c(G_n) > \gamma(F) + \varepsilon$ contains $\delta\binom{n}{f}$ copies of F. Consequently, for every positive integer ℓ, $\gamma(F) = \gamma(F(\ell))$.*

For each $r \geq 2$, let

$$\Pi_r = \{\pi(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-graphs}\}.$$

Then **ESS** implies that $\Pi_2 = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{k-1}{k}, \ldots\}$. The well-ordered property of Π_2 leads one to the following definition [5, 14, 27] (although there are several equivalent formulations): a real number $0 \leq a < 1$ is called a *jump* for r if there exists $\delta > 0$, such that no family of r-graphs $\mathcal{F}$ satisfies $\pi(\mathcal{F}) \in (a, a+\delta)$. The set Π_2 shows that every real number in $[0, 1)$ is a jump for $r = 2$. Erdős conjectured [9] that this is also the case for $r \geq 3$ and offered \$1000 for its solution. By supersaturation we have $\pi(K_r^r(\ell)) = 0$. This, together with $\lim_{l\to\infty} e(K_r^r(\ell))/\binom{\ell r}{r} = r!/r^r$ implies that no $\mathcal{F}$ satisfies $\pi(\mathcal{F}) \in (0, r!/r^r)$. Thus every $\alpha \in [0, r!/r^r)$ is a jump for $r \geq 3$. A striking result of Frankl and Rödl [14] showed that $1 - 1/\ell^{r-1}$ is not a jump for $r \geq 3$ and $\ell > 2r$, thus disproving Erdős' conjecture. However, one may still ask whether other numbers in $[r!/r^r, 1)$ are jumps for $r \geq 3$. For example, whether $2/9$ is a jump for $r = 3$ is a well-known open problem (Erdős actually considered this as the main part of his original conjecture). A recent result of Frankl, et al. [15] showed that $\frac{5r!}{2r^r}$ is not a jump for $r \geq 3$ and described an infinite sequence of non-jumps for $r = 3$.

The analogous problem for multigraphs with edge-multiplicity at most q was first considered by Brown, Erdős and Simonovits. They conjectured [3] that all numbers in $[0, q)$ are jumps and verified [4] it for $q = 2$ (**ESS** confirms the $q = 1$ case). Later Rödl and Sidorenko [27] disproved their conjecture by finding infinitely many non-jumps in $[3, q)$ for $q \geq 4$.

In this paper we consider the same problem for γ. For $r \geq 2$, let

$$\Gamma_r = \{\gamma(\mathcal{F}) : \mathcal{F} \text{ is a family of } r\text{-graphs}\}.$$

Note that $\Gamma_r \subseteq [0, 1)$ because $\gamma(\mathcal{F}) < 1$ for every family $\mathcal{F}$ of finite r-graphs. Since $\gamma(\mathcal{F}) = \pi(\mathcal{F})$ for all graph families $\mathcal{F}$, we have $\Gamma_2 = \{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{k-1}{k}, \ldots\}$. However, Γ_r behaves differently for $r \geq 3$.

Definition 1.5. *Fix $r \geq 2$. A real number $0 \leq \alpha < 1$ is called a γ-jump (or jump if the density is clear from the context) for r if there exists $\delta = \delta(\alpha) > 0$, such that every (infinite or finite) family of r-graphs $\mathcal{F}$ satisfies $\gamma(\mathcal{F}) \notin (\alpha, \alpha + \delta)$.*

Theorem 1.6 below completely answers the corresponding jump question for γ. The constructions proving Theorem 1.6 are different from the ones in [15, 14]. One key step in our proof is that 0 is a not a γ-jump, which again suggests that γ is fundamentally different than π (recall that 0 is indeed a jump in terms of π).

Theorem 1.6. *Fix $r \geq 3$. Then no $\alpha \in [0, 1)$ is a γ-jump. In particular, Γ_r is dense in $[0, 1)$.*

We believe that Theorem 1.6 can be strengthened to show that $\Gamma_r = [0, 1)$ for each $r \geq 3$. The missing step for the following conjecture is a compactness property for γ. Note that, in particular, Conjecture 1.7 clearly implies that $\Gamma_r = [0, 1)$.

Conjecture 1.7. *Fix $r \geq 3$. For every $0 \leq \alpha < 1$ there exists an infinite family $\mathcal{F}$ of r-graphs such that $\gamma(\mathcal{F}) = \alpha$ and all finite families $\mathcal{F}' \subset \mathcal{F}$ satisfying $\gamma(\mathcal{F}') > \alpha$.*

A family $\mathcal{F}$ of r-graphs is called *non-principal* [1, 23] if its Turán density is strictly less than the density of each member. When $r = 2$, **ESS** implies that no family is non-principal because $\pi(\mathcal{F}) = \min_{F \in \mathcal{F}} 1 - \frac{1}{\chi(F)-1} = \min_{F \in \mathcal{F}} \pi(F)$. Motivated by exploring the difference between graphs and hypergraphs, the first author and Rödl [24] conjectured that non-principal families exists for $r \geq 3$. Balogh [1] proved this conjecture by constructing a non-principal 3-graph family with finitely many members. The first author and Pikhurko [23] extended this result by constructing, for each $r \geq 3$, a non-principal r-graph family of size two. One might suspect that a similar result holds for γ. Our final theorem shows this to be the case. Its proof is similar but more complicated than the corresponding statement for π.

Theorem 1.8. *Fix $r \geq 3$. Then there is a finite family $\mathcal{F}$ of r-graphs such that $0 < \gamma(\mathcal{F}) < \min_{F \in \mathcal{F}} \gamma(F)$.*

The rest of the paper is organized as follows. We prove Propositions 1.2 and 1.4 in Section 2, Theorem 1.6 in Section 3 and Theorem 1.8 in Section 4. In the last section we give some concluding remarks and open problems.

2. Supersaturation

Our goal in this section is to prove Propositions 1.2 and 1.4. Our main tool (Lemma 2.1 below) is a useful technical result used in this section and in the proof of Theorem 1.8.

Let $a, \lambda > 0$ with $a + \lambda < 1$. Suppose that $S \subseteq [n] = \{1, \ldots, n\}$ and $|S| \geq (a + \lambda)n$. Then a result on the hypergeometric distribution (see, e.g., [17] page 29) says that

$$\left|\left\{M \in \binom{[n]}{m} : |M \cap S| \leq am\right\}\right| \leq \binom{n}{m} e^{-\frac{\lambda^2 m}{3(a+\lambda)}} \leq \binom{n}{m} e^{-\lambda^2 m/3}. \tag{1}$$

For a hypergraph H and a subset $S \subset V(H)$, we denote by $H[S]$ the subhypergraph of H induced by the set S. For positive integers $r < n$, let $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{r}$ be the family of all subsets of $[n]$ of size r.

Lemma 2.1. *Fix $r \geq 2$. Given $\varepsilon, \alpha > 0$ with $\alpha + \varepsilon < 1$, let $M(\varepsilon)$ be the smallest integer such that every $m > M(\varepsilon)$ satisfies $m \geq \frac{2(r-1)}{\varepsilon}$ and $\binom{m}{r-1} e^{-\varepsilon^2(m-r+1)/12} \leq \frac{1}{2}$. If $n \geq m \geq M(\varepsilon)$ and G is an r-graph on $[n]$ with $c(G) \geq \alpha + \varepsilon$, then the number of m-sets S satisfying $c(G[S]) > \alpha$ is at least $\frac{1}{2}\binom{n}{m}$. In particular, every r-graph H_n ($n \geq m$) contains a subhypergraph H'_m with $c(H'_m) > c(H_n) - \varepsilon$.*

Proof. Given an $(r-1)$-set T of $[n]$, we call an m-set S of $[n]$ *bad for T* if $T \subset S$ and $|N(T) \cap S| \leq \alpha m$. We call an m-set S *bad* if it is bad for some T. Let Φ denote the number of bad m-sets, and let Φ_T be the number of m-sets that are bad for T. We need to show that $\Phi \leq \frac{1}{2}\binom{n}{m}$. Clearly

$$\Phi \leq \sum_{T \in \binom{[n]}{r-1}} \Phi_T = \sum_{T \in \binom{[n]}{r-1}} \left|\left\{S' \in \binom{[n] \setminus T}{m-(r-1)} : |N(T) \cap S'| \leq \alpha m\right\}\right|.$$

Now $\alpha + \varepsilon < 1$ and $m \geq \frac{2(r-1)}{\varepsilon}$ imply that the summand above is upper bounded by

$$\left|\left\{S' \in \binom{[n] \setminus T}{m-r+1} : |N(T) \cap S'| \leq \left(\alpha + \frac{\varepsilon}{2}\right)(m-r+1)\right\}\right|.$$

Applying (1) with $a = \alpha + \varepsilon/2$ and $\lambda = \varepsilon/2$ yields

$$\Phi_T \leq \binom{n-r+1}{m-r+1} e^{-(\varepsilon/2)^2(m-r+1)/3}.$$

Finally, we apply the hypothesis $\binom{m}{r-1} e^{-\varepsilon^2(m-r+1)/12} \leq \frac{1}{2}$ to obtain

$$\Phi \leq \binom{n}{r-1}\binom{n-r+1}{m-r+1} e^{-(\varepsilon/2)^2(m-r+1)/3} = \binom{n}{m}\binom{m}{r-1} e^{-\varepsilon^2(m-r+1)/12} \leq \frac{1}{2}\binom{n}{m}. \qquad \square$$

Its immediate consequence, Corollary 2.2, is needed for Proposition 1.2 and in Section 3.1. Call a hypergraph *nontrivial* if it contains at least one edge, and a family of hypergraphs nontrivial it contains at least one nontrivial member.

Corollary 2.2. *For any $0 < \varepsilon < 1$, define $M(\varepsilon)$ as in Lemma 2.1. Then for every $n > m \geq M(\varepsilon)$ and every non-trivial family $\mathcal{F}$ of r-graphs,*

$$\frac{\text{co-ex}(n, \mathcal{F})}{n} - \frac{\text{co-ex}(m, \mathcal{F})}{m} < \varepsilon.$$

Proof. Since $\mathcal{F}$ is nontrivial, there is an $\mathcal{F}$-free r-graph H_n with $\mathcal{C}(H_n) = \text{co-ex}(n, \mathcal{F})$. By Lemma 2.1, H_n contains a subhypergraph H'_m with $c(H'_m) > c(H_n) - \varepsilon$. Since H'_m is $\mathcal{F}$-free, $c(H'_m) \leq \text{co-ex}(m, \mathcal{F})/m$ and the desired inequality follows. □

Proof of Proposition 1.2. Let $a_n = \text{co-ex}(n, \mathcal{F})/n$. Corollary 2.2 says that for every $n > m \geq M(\varepsilon)$, we have $a_n - a_m < \varepsilon$. Since $a_n \geq 0$ for every n, it is easy to see that $\lim_{n\to\infty} a_n$ exists and equals to $\liminf_{n\to\infty} a_n$. □

Proof of Proposition 1.4. The proof follows the arguments of Erdős and Simonovits for π, with a suitable application of Lemma 2.1. We sketch the main steps below.

Let $\alpha = \gamma(F)$ and $f = |V(F)|$. For each positive n, let G_n be an r-graph with $\mathcal{C}(G_n) > (\alpha+\varepsilon)n$. By Lemma 2.1, there exists an integer m, such that for $n \geq m$ at least $\frac{1}{2}\binom{n}{m}$ induced subgraphs of G_n on m vertices have minimum co-degree at least $(\alpha + \varepsilon/2)m$. Since $\gamma(F) = \alpha$ and m is sufficiently large, each of these subgraphs contains a copy of F. Consider an f-uniform graph G' on $V(G_n)$ whose edges are f-sets S in which $G[S] \supseteq F$. Then

$$e(G') \geq \frac{1}{2} \frac{\binom{n}{m}}{\binom{n-f}{m-f}} = \frac{1}{2\binom{m}{f}} \binom{n}{f} = \delta \binom{n}{f}.$$

A result of Erdős [8] implies that for each L, there is a sufficiently large n such that $\mathcal{K} = K_f^f(L) \subseteq G'$. Furthermore, each edge $e = (v_1, v_2, \ldots, v_f) \in E(\mathcal{K})$ corresponds to an embedding of F and the mapping of $[f] = V(F)$ to $v_1, v_2, \ldots, v_f$ is regarded as a permutation ρ_e of $[f]$. A result in Ramsey theory says that if L is large enough, then we can always find $\mathcal{K}' = K_f^f(\ell) \subseteq \mathcal{K}$ such that all ℓ^f edges in $\mathcal{K}'$ follow the same permutation. This implies that for n sufficiently large the induced subgraph $G[V(\mathcal{K}')]$ contains a copy of $F(\ell)$. Therefore $\gamma(F) = \gamma(F(\ell))$. □

3. Jumps

Unless stated otherwise, when we say *jump* we mean γ-jump. We begin by giving three equivalent definitions for jumps.

Proposition 3.1. *Fix $r \geq 2$. Let $0 \leq \alpha < 1$, $0 < \delta \leq 1 - \alpha$. The following statements are equivalent.*

S1*: Every family of r-graphs $\mathcal{F}$ satisfies $\gamma(\mathcal{F}) \notin (\alpha, \alpha + \delta)$.*

S2*: Every finite family of r-graphs $\mathcal{F}$ satisfies $\gamma(\mathcal{F}) \notin (\alpha, \alpha + \delta)$.*

S3: *For every $\varepsilon > 0$ and every $M \geq r-1$, there exists an integer N such that, for every r-graph G_n with $n > N$ and $\mathcal{C}(G_n) \geq (\alpha + \varepsilon)n$, we can find a subhypergraph $G'_m \subseteq G_n$ with $\mathcal{C}(G'_m) \geq (\alpha + \delta - \varepsilon)m$ for some $m > M$ (Note that the order of quantifiers above is $\forall \varepsilon, M \quad \exists N \quad \forall n > N \quad \exists m > M$).*

Remark. In terms of π, a slightly stronger statement than **S3** was stated in the abstract of [14]. There the factor $\alpha+\delta-\varepsilon$ was replaced by $\alpha+\delta$, and the quantification $\forall M \geq r-1, \exists m > M, G'_m$ was replaced by $\forall M \geq r-1, \exists G'_M$. The stronger statement was valid in that context because of the monotonicity of $\text{ex}(n, \mathcal{F})/\binom{n}{r}$. As mentioned in the introduction, we could not prove that co-ex$(n, \mathcal{F})/n$ is monotone, hence we have the different but essentially equivalent statement **S3**.

In Section 3.1 we prove Proposition 3.1. The proof of Theorem 1.6 is then divided into two cases: $\alpha = 0$ (Section 3.2) and $0 < \alpha < 1$ (Section 3.3).

Let us briefly compare our proof with those on π-jumps [14, 15]. Fix a density of r-graphs G_n, either the normalized co-degree $c(G_n)$ or the edge density $e(G_n)/\binom{n}{3}$. All of these proofs show that $\alpha \in [0, 1)$ is not a jump in terms of this density by definition **S3** or its equivalent form. Roughly speaking, for every $\delta > 0$, we construct a sequence of r-graphs $\{G_n\}$ ($n = n(i) \to \infty$ as $i \to \infty$) such that

1. the density of G_n is slightly greater than α,
2. any reasonably large subgraph of G_n has density less than $\alpha + \delta$.

To satisfy the first property above, one can obtain G_n from any r-graph of density α by adding some extra edges. Hence the main task is to verify the second property for the choice of G_n. For π, this is only known when G_n has the structure as described in [14, 15]. When $r = 3$, the essential part of this structure is a 3-graph H_m with vertex set $V = \cup_{i=0}^{\ell-1} V_i$, where $\ell \geq 3$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. Its edge set consists of all triples of the vertices from three different V_i's and all $\{a, b, c\}$ with $a \in V_i$, $b, c \in V_j$ for $j = i+1, \ldots, i+t \pmod{\ell}$ for some fixed $t < \ell$. Note that their actual G_n is a blow-up of H^*_m, which is H_m plus some extra edges. It is easy to see that the edge density, $|E(H_m)|/\binom{m}{3}$, is about $1 - \frac{3}{\ell} + \frac{3t+2}{\ell^2}$. For appropriate choices of ℓ and t, we obtain all known non-π-jumps for $r = 3$. In contrast, our construction for γ is more general: we construct G_n satisfying the above two properties for all rational $\alpha \in [0, 1)$. This, of course, is due to the nature of co-degree conditions; it does not suggest any new construction for non-π-jumps.

3.1. Proof of Proposition 3.1

We need the following so-called *Continuity property* (which holds for π as shown in [5, 27]).

Lemma 3.2. *Let $\mathcal{F}$ be a family of r-graphs. For every $\varepsilon > 0$, there exists a finite family $\mathcal{F}' \subseteq \mathcal{F}$ with $\gamma(\mathcal{F}) \leq \gamma(\mathcal{F}') \leq \gamma(\mathcal{F}) + \varepsilon$.*

Proof. Trivially $\gamma(\mathcal{F}) \le \gamma(\mathcal{F}')$ for any $\mathcal{F}' \subseteq \mathcal{F}$, so we only need to show that $\gamma(\mathcal{F}') \le \gamma(\mathcal{F}) + \varepsilon$ for some finite family $\mathcal{F}' \subseteq \mathcal{F}$. Set $\gamma = \gamma(\mathcal{F})$ and choose $m = m(\varepsilon)$ such that

1. $\frac{\text{co-ex}(m,\mathcal{F})}{m} < \gamma + \frac{\varepsilon}{2}$ and

2. $m > M(\frac{\varepsilon}{2})$, where $M(\varepsilon)$ is defined as in Corollary 2.2.

Let $\mathcal{F}'$ be the set of members of $\mathcal{F}$ on at most m vertices. Then $\text{co-ex}(m, \mathcal{F}) = \text{co-ex}(m, \mathcal{F}')$. Now we apply Corollary 2.2 to derive that for every $n > m$,

$$\frac{\text{co-ex}(n, \mathcal{F}')}{n} < \frac{\text{co-ex}(m, \mathcal{F}')}{m} + \frac{\varepsilon}{2} = \frac{\text{co-ex}(m, \mathcal{F})}{m} + \frac{\varepsilon}{2} < \gamma + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \gamma + \varepsilon.$$

Therefore $\gamma(\mathcal{F}') = \lim_{n\to\infty} \frac{\text{co-ex}(n,\mathcal{F}')}{n} \le \gamma + \varepsilon$. □

Proof of Proposition 3.1.

Trivially **S1** ⇒ **S2**. We will show **S2** ⇒ **S1**, **S3** ⇒ **S1** and **S1** ⇒ **S3**.

S2 ⇒ **S1**. Assume that there exists $\delta > 0$, such that no finite family of r-graphs $\mathcal{F}$ satisfies $\gamma(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Suppose that **S1** does not hold, i.e., there exists a family of r-graphs $\mathcal{F}$ satisfying $\gamma(\mathcal{F}) \in (\alpha, \alpha+\delta)$. Let $0 < \varepsilon < \alpha+\delta-\gamma(\mathcal{F})$. We apply Lemma 3.2 to obtain a finite family $\mathcal{F}' \subseteq \mathcal{F}$ with $\gamma(\mathcal{F}') \le \gamma(\mathcal{F}) + \varepsilon < \alpha + \delta$, a contradiction.

S3 ⇒ **S1**. Suppose that **S3** holds. We will show that no family of r-graphs $\mathcal{F}$ satisfies $\gamma(\mathcal{F}) \in (\alpha, \alpha + \delta)$. Suppose instead, that there exist a family $\mathcal{F}$ satisfying $\gamma(\mathcal{F}) = \alpha + b$ for some $0 < b < \delta$. Set $\varepsilon_0 = \min\{\frac{b}{2}, \frac{\delta-b}{2}\}$. Then there exists $N_1 = N_1(\varepsilon)$ so that the following two statements hold:

D1: For every $n > N_1$, there exists an $\mathcal{F}$-free hypergraph H_n with $c(H_n) \ge \alpha + \varepsilon_0$.

D2: Every hypergraph G_m with $m > N_1$ and $c(G_m) \ge \alpha + b + \varepsilon_0$ contain a member of $\mathcal{F}$.

By **S3** (with $\varepsilon = \varepsilon_0$, $M = N_1$), we may find an n such that the $\mathcal{F}$-free hypergraph H_n in D1 contains an m-vertex subhypergraph G'_m with $c(G'_m) \ge \alpha + \delta - \varepsilon_0 > \alpha + b + \varepsilon_0$ for some $m > N_1$. This contradicts D2 because $G_m \subset H_n$ is $\mathcal{F}$-free.

S1 ⇒ **S3**. Suppose that **S1** holds but **S3** does not. If **S3** is false for $\varepsilon > 0$, then it is also false for $\varepsilon' < \varepsilon$. Consequently, we may assume that there exist $0 < \varepsilon < \delta$, $M \ge r - 1$, and a sequence of hypergraphs H_{n_i} ($n_i \to \infty$ as $i \to \infty$) such that

P1: $c(H_{n_i}) \ge \alpha + \varepsilon$,

P2: $c(H'_m) < \alpha + \delta - \varepsilon$ for every subhypergraph $H'_m \subseteq H_{n_i}$ with $m > M$.

Let $\mathcal{F} = \{G : G \not\subseteq H_{n_i} \text{ for any } i\}$. Note that $\mathcal{F}$ is nonempty because P2 implies that $K_m \not\subseteq H_{n_i}$ for every i, thus $K_m \in \mathcal{F}$ (for every $m > M$). Then $\text{co-ex}(n_i, \mathcal{F}) \ge (\alpha+\varepsilon)n_i$ for every i because H_{n_i} is $\mathcal{F}$-free. Thus $\gamma(\mathcal{F}) \ge \alpha + \varepsilon$. From **S1**, we know that $\gamma(\mathcal{F}) \ge \alpha + \delta$. Hence for every natural number $m > M$, there exists an $\mathcal{F}$-free hypergraph G_m with $c(G_m) \ge \alpha + \delta - \varepsilon$. Since

G_m is $\mathcal{F}$-free, there exists some n_i such that $G_m \subseteq H_{n_i}$ (otherwise G_m itself is a member of $\mathcal{F}$). But this clearly contradicts P2. $\square$

3.2. Proof of Theorem 1.6 for $\alpha = 0$

Using Statement **S3** in Proposition 3.1, the following shows that 0 is not a jump for r. For every $\delta > 0$, there exist $\varepsilon_0 > 0$ and $M_0 > 0$, such that for every $\ell \geq r-1$, there exist $n > \ell$ and an r-graph G_n satisfying

(1) $\mathcal{C}(G_n) \geq \varepsilon_0 n$,

(2) $\mathcal{C}(G'_m) < (\delta - \varepsilon_0)m$ for every $G'_m \subset G_n$ with $m > M_0$.

Remark. Note again that the order of quantifiers in the theorem is

$$\forall\,\delta \quad \exists\,\varepsilon_0, M_0 \qquad \forall\,\ell \quad \exists\, n > \ell, G_n \quad \forall\, G'_m, m > M_0$$

The following special r-graph is important to our proof.

Definition 3.3. *$B(\ell, t, r)$ is the r-graph (V, E) in which $V = V_0 \cup V_1 \cup \cdots \cup V_{t-1}$, $V_i \cap V_j = \emptyset$ for all $i \neq j$, $|V_i| = \ell$ for all i, and E comprises all $S \in \binom{V}{r}$ with $|S \cap V_i| \leq r-2$ for all i.*

Fix $\delta \in (0,1)$. Let

$$\varepsilon_0 = \frac{\delta}{3}, \quad M_0 = \frac{3(r-1)}{\delta}, \quad \text{and} \quad t = \left\lfloor \frac{1}{\varepsilon_0} \right\rfloor . \tag{2}$$

Therefore $\varepsilon_0 < 1/3$ and $t \geq 3$.

For every $\ell \geq r-1$, set $n = t\ell$. Starting from the r-graph $B(\ell, t, r)$, we add to the edge set the r-sets with $r-1$ vertices in V_i and one vertex in V_{i+1} for all i (here $V_t = V_0$). Denote the resulting r-graph by G_n. It is easy to see that $\mathcal{C}(G_n) \geq \varepsilon_0 n$. In fact, given an $(r-1)$-set $R \subset V(G_n)$, let $R_i = R \cap V_i$. If $|\{i : R_i \neq \emptyset\}| = 1$, i.e., $R \subset V_i$ for some i, then $N(R) = V_{i+1}$ and $d(R) = \ell \geq \varepsilon_0 n$. Otherwise, the definition of $B(l, t, r)$ implies that $d(R) \geq (t-2)\ell \geq \ell \geq \varepsilon_0 t\ell = \varepsilon_0 n$ when $|\{i : R_i \neq \emptyset\}| = 2$, and $d(R) \geq n - (r-1) > \varepsilon_0 n$ when $|\{i : R_i \neq \emptyset\}| \geq 3$.

To complete the proof, we show that

For every m-set S with $m > M_0$, $G' = G_n[S]$ satisfies $\mathcal{C}(G') < (\delta - \varepsilon_0)m$. $(\star)$

Suppose that $(\star)$ does not hold. Then $\mathcal{C}(G') = \mathcal{C}(G_n[S]) \geq (\delta - \varepsilon_0)m = 2\varepsilon_0 m$ for some m-set S. Let $S_i = S \cap V_i$ for all i. Since $m > M_0 \geq (r-1)/\varepsilon_0 \geq (r-1)t$, by the pigeonhole principle, there is an i_0 and an $(r-1)$-set $R_0 \subset S_{i_0}$. Because $d_{G'}(R_0) \geq 2\varepsilon_0 m$ and $N(R_0) \subset V_{i_0+1}$, we have $|S_{i_0+1}| \geq 2\varepsilon_0 m$. Since $2\varepsilon_0 m > r-1$, we may repeat this argument to i_0+1 and conclude $|S_i| \geq 2\varepsilon_0 m$ for all i. But this yields the contradiction $m \geq t\, 2\varepsilon_0 m \geq (1/\varepsilon_0 - 1)\, 2\varepsilon_0 m > m$ (using the fact $\varepsilon_0 < 1/3$).

$\square$

3.3. Proof of Theorem 1.6 for $0 < \alpha < 1$

Since the set of rational numbers is dense in the reals, it suffices to show that $\alpha = a/b$ is not a jump for every two positive integers $a < b$. As in the $\alpha = 0$ case, we will show that the negation of **S3** holds for every $\delta > 0$.

Given $\delta > 0$, let ε_0, M_0, t be as in (2). Set

$$\varepsilon = \frac{\varepsilon_0}{b} \qquad \text{and} \qquad M = \max\left\{\frac{rb^2}{(\delta-\varepsilon_0)a}\,,\ \frac{M_0 + rb}{\delta - \varepsilon_0}\,,\ M_0 b\right\}.$$

For every integer $\ell \geq r-1$, set $n = t\ell b$. Let $\mathcal{D}$ be the directed graph on $\{0, 1, \ldots, b-1\}$ with $E(\mathcal{D}) = \{(i,j) : j = i+1, \ldots, i+a\}$. The indices in this subsection are mod b unless stated differently. Let $G_{t\ell}$ be the r-graph constructed in Section 3.2. Let $H = (V, E)$ be the n-vertex r-graph obtained from $B(t\ell, b, r)$ by adding

- edges within each V_i so that $H[V_i] \cong G_{t\ell}$, and
- all edges with $r-1$ vertices in V_i and one vertex in V_j whenever $(i,j) \in E(\mathcal{D})$.

We claim that $\mathcal{C}(H) \geq (a/b + \varepsilon)n$. To see this, pick an $(r-1)$-set $R \subset V$. If $R \subset V_i$ for some i, then $R \cup \{v\} \in E$ for every $v \in \bigcup_{j=1}^{a} V_{i+j}$, and $d_{H[V_i]}(R) \geq \varepsilon_0 t\ell$ because $H[V_i] \cong G_{t\ell}$. Thus

$$d_{H_n}(R) \geq a|V_i| + \varepsilon_0 t\ell = (a + \varepsilon_0)t\ell = (a/b + \varepsilon)n.$$

Next suppose that $\max_i |R \cap V_i| < r-1$. We consider three cases: $a = b-1$, $a = b-2$ and $a \leq b-3$. If $a = b-1$, then the edges of $H[V_i]$ together with the edges of $B(t\ell, b, r)$ yield

$$d_{H_n}(R) \geq n - (r-1) \geq n - \ell = (b-1)t\ell + t\ell - \ell > (b-1)t\ell + \varepsilon_0 t\ell = (a/b + \varepsilon)n,$$

where the third inequality holds because $t > \varepsilon_0 t + 1$. Following a similar reasoning, when $a = b-2$, we have

$$d_{H_n}(R) \geq n - t\ell - (r-1) \geq n - t\ell - \ell > (b-2)t\ell + \varepsilon_0 t\ell = (a/b + \varepsilon)n,$$

and when $a \leq b-3$, the edges of $B(t\ell, b, r)$ yield

$$d_{H_n}(R) \geq n - 2t\ell - (r-1) \geq n - 2t\ell - \ell > (b-3)t\ell + \varepsilon_0 t\ell \geq (a/b + \varepsilon)n.$$

Let $S \in \binom{V}{m}$ with $m > M$ and $H' = H[S]$. Our goal is to show that $\mathcal{C}(H') < (a/b + \delta - \varepsilon)m$, i.e., there exists an $(r-1)$-set $R \subset S$ such that $d_{H'}(R) < (a/b + \delta - \varepsilon)m$.

Let $S_i = S \cap V_i$ for all i. We first claim that there exists i_0, such that $|S_{i_0}| \geq r-1$ and

$$\sum_{j=i_0+1}^{i_0+a} |S_j| < \frac{a}{b}m + rb. \tag{3}$$

In fact, if $\sum_{j=i+1}^{i+a} |S_j| > \frac{a}{b}m$ for all i, then by averaging, we obtain $|S| > \lceil (b\frac{a}{b}m)/a \rceil \geq m$, a contradiction. Next, assume that $\sum_{j=i+1}^{i+a} |S_j| \leq \frac{a}{b}m$ but $|S_i| \leq r-2$ for some i. Without loss of generality, let $i = 0$, so $\sum_{j=1}^{a} |S_j| \leq \frac{a}{b}m$ and $|S_0| \leq r-2$. Let i_0 be the largest integer less than b such that $|S_{i_0}| \geq r-1$ (such i_0 exists because $|S| = m > M > (r-2)b$). Then $\sum_{j=i_0+1}^{i_0+a} |S_j| \leq \sum_{j=i_0+1}^{b-1} |S_j| + |S_0| + \sum_{j=1}^{a} |S_j| \leq (r-2)b + \frac{a}{b}m$ and (3) follows.

Let R be an $(r-1)$-subset of S_{i_0}. We will show that $d_{H'}(R) < (a/b+\delta-\varepsilon_0)m < (a/b+\delta-\varepsilon)m$. If $|S_{i_0}| > M_0$, then $H[V_{i_0}] \cong G_{t\ell}$ and $(\star)$ implies that $d_{H'[S_{i_0}]}(R) < (\delta - \varepsilon_0)|S_{i_0}|$. Otherwise $d_{H'[S_{i_0}]}(R) \leq |S_{i_0}| \leq M_0$.

If $|S_{i_0}| \geq (1-\frac{a}{b})m$, then $b \geq a+1$ and $m \geq bM_0$ yield $|S_{i_0}| \geq \frac{m}{b} > M_0$. Therefore

$$d_{H'}(R) < (\delta - \varepsilon_0)|S_{i_0}| + (m - |S_{i_0}|) < (\delta - \varepsilon_0)m + \frac{a}{b}m.$$

Otherwise, $m - |S_{i_0}| > \frac{a}{b}m > \frac{rb}{\delta - \varepsilon_0}$, since $m > \frac{rb^2}{(\delta-\varepsilon_0)a}$. Consequently

$$(\delta - \varepsilon_0)|S_{i_0}| + rb < (\delta - \varepsilon_0)m. \tag{4}$$

By the structure of $\mathcal{D}$, we know that all the neighbors of R in $H'[S \setminus S_{i_0}]$ are in S_j, for $j = i_0+1, \ldots, i_0+a \pmod b$. Applying (3), we therefore obtain $d_{H'[S \setminus S_{i_0}]}(R) \leq \sum_{j=i_0+1}^{i_0+a} |S_j| < \frac{a}{b}m + rb$, and hence

$$\begin{aligned} d_{H'}(R) &= d_{H'[S_{i_0}]}(R) + d_{H'[S\setminus S_{i_0}]}(R) \\ &< \max\{M_0, (\delta-\varepsilon_0)|S_{i_0}|\} + \frac{a}{b}m + rb \\ &= \max\{M_0 + rb, (\delta-\varepsilon_0)|S_{i_0}| + rb\} + \frac{a}{b}m \end{aligned}$$

The hypothesis $m > \frac{M_0+rb}{\delta-\varepsilon_0}$ implies that $M_0 + rb < (\delta - \varepsilon_0)m$, and together with (4), we again derive that $d_{H'}(R) < (\delta - \varepsilon_0)m + \frac{a}{b}m$. □

4. Non-Principality

In this section, we prove Theorem 1.8 by an explicit construction. An r-graph G is called *2-colorable* (or with *chromatic number* two) if $V(G)$ can be partitioned into two disjoint sets A and B such that neither A nor B contains any edge. The main idea in our proof is to find $\gamma_0 < \frac{1}{2}$ and a 2-colorable r-graph F with $\gamma(F) \geq \frac{1}{2}$ such that every 2-colorable r-graph H_n with $c(H_n) \geq \gamma_0$ contains F as a subgraph.

Definition 4.1. *$K^r(t,t)$ is the r-graph with vertex set $V = A \cup B$, $A \cap B = \emptyset$, $|A| = |B| = t$, and edge set $\{S \in \binom{V}{r} : |S \cap A| = 1$ or $|S \cap B| = 1\}$.*

Proposition 4.2. *For $r \geq 3$, there exists a positive integer $\ell = \ell(r)$ such that $\gamma(K^r(\ell,\ell)) \geq \frac{1}{2}$.*

Proof. We need to show that for every $\varepsilon > 0$, there exists $N > 0$ such that for every $n > N$, there exists a $K^r(\ell, \ell)$-free r-graph H_n^r with $c(H_n^r) > \frac{1}{2} - \varepsilon$. We obtain H_n^r based on a random construction of Nagle and Rödl (see [7]). Let R be a random tournament on $[n]$, namely, an orientation of the complete graph on $\{1, \ldots, n\}$ such that $i \to j$ or $j \to i$, each with probability $1/2$ for every $i < j$. Nagle and Rödl define a random 3-graph G^3 on $[n]$ such that for all $i < j < k$, $\{i, j, k\} \in E(G^3)$ if and only if either $k \to i, i \to j$ or $j \to i, i \to k$. By using standard Chernoff bounds, we have $c(G^3) > \frac{1}{2} - \varepsilon$ with positive probability for any fixed $\varepsilon > 0$. On the other hand, G^3 contains no K_4^3 because for any $i < j < k < t$, two of ij, ik, it must have the same direction. Since $K_4^3 = K^3(2, 2)$, setting $\ell(3) = 2$, G^3 gives rise to the desired H_n^3.

For $r > 3$, we define a random r-graph G^r with vertex set $[n]$ and $E(G) = \{D \in \binom{[n]}{r} : D \supset T$ for some $T \in E(G^3)\}$. In other words, an r-subset $D \subset [n]$ is an edge if and only if D contains some $i < j < k$ such that either $k \to i, i \to j$ or $j \to i, i \to k$. As before we know that for any $\varepsilon > 0$, $c(G^r) > \frac{1}{2} - \varepsilon$ with positive probability. Let $\ell = 2R^3(4, r-1)$, where the Ramsey number $R^3(4, r-1)$ is the smallest m such that any 3-graph on m vertices either contains K_4^3 or $\bar{K}_{r-1}^3$ (the empty 3-graph on $r-1$ vertices). We claim that G^r contains no $K^r(\ell, \ell)$. That is, given two disjoint ℓ-subsets A and B of $[n]$, we show that some r-subset $S \subset A \cup B$ with $|S \cap A| = 1$ or $|S \cap B| = 1$ is not an edge of G^r. Without loss of generality, assume that $a_0 \in A$ is the smallest elements in $A \cup B$. Partition B into B_1 and B_2, where $B_1 = \{b \in B : a_0 \to b\}$ and $B_2 = \{b \in B : b \to a_0\}$. Without loss of generality, assume that $|B_1| \geq \ell/2$. Since $|B_1| \geq R^3(4, r-1)$ and G^3 is K_4^3-free, $G^3[B_1]$ contains a copy of $\bar{K}_{r-1}^3$ with the vertex set B_0. Together with the definitions of a_0 and B_1, this implies that $\{a_0\} \cup B_0$ contains no edge of G^3. Consequently $\{a_0\} \cup B_0$ is not an edge of G^r. □

Proposition 4.3. *Let $r \geq 3$, $\ell \geq r-1$ and $\rho = \frac{1}{2}\left(1 - \frac{1}{\binom{\ell}{r-1}} + \frac{1}{\binom{\ell}{r-1}2^{1/\ell}}\right) < \frac{1}{2}$. For any $\varepsilon > 0$ there exists N such that every 2-colorable r-graph G_n with $n > N$ and $\mathcal{C}(G_n) \geq (\rho + \varepsilon)n$ contains a copy of $K^r(\ell, \ell)$.*

Proof. Suppose to the contrary, that for arbitrarily large n, there exists a 2-colorable r-graph G_n such that

- $\mathcal{C}(G_n) = c \geq (\rho + \varepsilon)n$, and
- G_n contains no copy of $K^r(\ell, \ell)$.

Since G_n is 2-colorable, we may partition $V(G_n)$ into two sets A and B with $|A| = a \leq b = |B|$ such that no edges of G_n fall inside A or B. Thus for any $X \in \binom{A}{r-1}$, we have $N(X) \subseteq B$ and the same holds for $Y \in \binom{A}{r-1}$. This implies that $c \leq a \leq b \leq n - c$.

Let $X \in \binom{A}{\ell}$. We first estimate $|\bigcap_{X' \subseteq \binom{X}{r-1}} N(X')|$. Since every $X' \subset \binom{A}{r-1}$ has at most $b - c$ *non*-neighbors in B, the number of their common neighbors is at least $b - \binom{\ell}{r-1}(b - c)$. Similarly $|\bigcap_{Y' \subseteq \binom{Y}{r-1}} N(Y')| \geq a - \binom{\ell}{r-1}(a - c)$ for every $Y \in \binom{B}{\ell}$.

The key to our proof is to estimate Φ, the number of $X \cup Y$ with $X \in \binom{A}{\ell}$ and $Y \in \binom{B}{\ell}$ such that either

- $X' \cup \{y\} \in E(G_n)$ for all $X' \in \binom{X}{r-1}$ and $y \in Y$ or
- $Y' \cup \{x\} \in E(G_n)$ for all $Y' \in \binom{Y}{r-1}$ and $x \in X$.

Trivially $\Phi \leq \binom{a}{\ell}\binom{b}{\ell}$. On the other hand, because G_n does not contain $K^r(\ell, \ell)$, we have

$$\begin{aligned}
\Phi &\geq \sum_{X \in \binom{A}{\ell}} \binom{\left|\bigcap_{X' \subseteq \binom{X}{r-1}} N(X')\right|}{\ell} + \sum_{Y \in \binom{B}{\ell}} \binom{\left|\bigcap_{Y' \subseteq \binom{Y}{r-1}} N(Y')\right|}{\ell} \\
&\geq \binom{a}{\ell}\binom{b - \binom{\ell}{r-1}(b-c)}{\ell} + \binom{b}{\ell}\binom{a - \binom{\ell}{r-1}(a-c)}{\ell} \\
&= \binom{a}{\ell}\binom{ta - (tn - (t+1)c)}{\ell} + \binom{b}{\ell}\binom{tb - (tn - (t+1)c)}{\ell},
\end{aligned}$$

where $t = \binom{\ell}{r-1} - 1 \geq 0$ (equality holds if and only if $\ell = r - 1$).

For fixed t, c, n, define the function $f(x) = \binom{x}{\ell}\binom{tx - (tn - (t+1)c)}{\ell}$ for $x \in [c, n-c]$. After rewriting,

$$f(x) = \frac{\prod_{i=0}^{l-1}(x-i)\prod_{i=0}^{l-1}(tx - (tn - (t+1)c) - i)}{(l!)^2}.$$

We claim that the second derivative $f''(x) \geq 0$ for $x \in [c, n-c]$. By differentiation, this claim holds as long as each term in the products in the numerator is nonnegative and has nonnegative derivative. Since n is sufficiently large, $x - l + 1 \geq c - l + 1 > 0$ so each term in the first product is positive. To show the same for each term in the second product, it suffices to show that $tc \geq tn - (t+1)c + (\ell - 1)$. Since $c > \rho n$, it is enough to show that $\frac{1}{2}\left(1 - \frac{1}{t+1} + \frac{1}{2^{1/\ell}(t+1)}\right) > \frac{t}{2t+1}$. This holds since

$$\frac{1}{2}\left(1 - \frac{1}{t+1} + \frac{1}{2^{1/\ell}(t+1)}\right) > \frac{1}{2}\left(1 - \frac{1}{t+1} + \frac{1}{2(t+1)}\right) > \frac{1}{2}\left(1 - \frac{1}{2t+1}\right) = \frac{t}{2t+1}.$$

The derivatives of the terms are 1 and t, which are both nonnegative. We therefore conclude that $f''(x) \geq 0$ and consequently $f(x)$ is convex on $[c, n-c]$. Hence

$$\Phi \geq f(a) + f(b) \geq 2f\left(\frac{a+b}{2}\right) = 2f(n/2) = 2\binom{n/2}{\ell}\binom{(t+1)c - tn/2}{\ell}.$$

On the other hand, since $\ln\binom{x}{l}$ is a concave function, we have $\Phi \leq \binom{a}{l}\binom{b}{l} \leq \binom{n/2}{l}^2$. Putting the lower and upper bounds for Φ together yields

$$2\binom{n/2}{\ell}\binom{(t+1)c - tn/2}{\ell} \leq \binom{n/2}{\ell}^2,$$

But this implies that as $n \to \infty$,

$$c \le \frac{n}{2}\left(1 - \frac{1}{t+1} + \frac{1}{(t+1)2^{1/\ell}}\right) + o(n) = \frac{n}{2}\left(1 - \frac{1}{\binom{\ell}{r-1}} + \frac{1}{\binom{\ell}{r-1}2^{1/\ell}}\right) + o(n) = \rho n + o(n),$$

contradicting the assumption that $c \ge (\rho+\varepsilon)n$. □

Proof of Theorem 1.8. Let $\ell = \ell(r)$ be as in Proposition 4.2 and ρ as in Proposition 4.3. For any $0 < \varepsilon < \frac{1}{2} - \rho$ we will construct a finite family $\mathcal{F}$ of r-graphs such that

$$\frac{1}{4} \le \gamma(\mathcal{F}) \le \rho + \varepsilon < \frac{1}{2} \le \min_{F\in\mathcal{F}} \gamma(F). \tag{5}$$

Let $m = \max\{M(\varepsilon/2), N(\varepsilon/2)\} + 1$, where M is the threshold function in Lemma 2.1 and N is the threshold function in Proposition 4.3. Let $\mathcal{F}_0$ be the family of r-graphs on at most m vertices which are not 2-colorable. We observe that $\min_{F\in\mathcal{F}_0} \gamma(F) \ge \gamma(\mathcal{F}_0) \ge 1/2$. In fact, for any n, the following r-graph G_n is 2-colorable and satisfies $\mathcal{C}(G_n) = \lfloor n/2 \rfloor$: $V(G_n)$ contains two disjoint vertex sets A and B of sizes differing by at most 1, $E(G_n)$ contains all the edges intersecting both A and B.

We now show that (5) holds for $\mathcal{F} = \mathcal{F}_0 \cup \{K^r(\ell,\ell)\}$. Proposition 4.2 says that $\gamma(K^r(\ell,\ell)) \ge 1/2$. Together with $\min_{F\in\mathcal{F}_0} \gamma(F) \ge 1/2$, we conclude that $\min_{F\in\mathcal{F}} \gamma(F) \ge 1/2$. On the other hand, we claim that $\frac{1}{4} \le \gamma(\mathcal{F}) \le \rho + \varepsilon$ and thus (5) follows.

To see that $\gamma(\mathcal{F}) \ge \frac{1}{4}$, let H_n^r be the $K^r(\ell,\ell)$-free r-graph as in the proof of Proposition 4.2. We randomly partition $V(H_n^r)$ into two almost equal parts and remove all the edges within each part. The resulting r-graph $\tilde{H}_n^r$ is also $K^r(\ell,\ell)$-free and satisfies $\mathcal{C}(\tilde{H}_n^r) \ge \frac{n}{4} - o(n)$ with positive probability. Hence $\gamma(\mathcal{F}) \ge \frac{1}{4}$.

To see that $\gamma(\mathcal{F}) \le \rho + \varepsilon$, let G_n be an r-graph with $n > m$ and $\mathcal{C}(G_n) \ge (\rho+\varepsilon)n$. By Lemma 2.1, G_n has a subgraph G'_m with $\mathcal{C}(G'_m) \ge (\rho + \varepsilon/2)m$. If G'_m is not 2-colorable, then G'_m itself is a member of $\mathcal{F}$. Otherwise, since $m > N(\varepsilon/2)$ and $\mathcal{C}(G'_m) \ge (\rho+\varepsilon/2)m$, Proposition 4.3 guarantees that G'_m contains a copy of $K^r(\ell,\ell)$. Therefore G always contains a member of $\mathcal{F}$ as a subgraph. Consequently co-ex$(n,\mathcal{F}) \le (\rho+\varepsilon)n$ for all $n > m$ and thus $\gamma(\mathcal{F}) \le \rho + \varepsilon$. □

5. Concluding Remarks and open problems

- Theorem 1.6 and Proposition 3.1 together imply that the set $\{\gamma(\mathcal{F}) : \mathcal{F} \text{ is a finite family}\}$ is dense on $[0,1)$, i.e., for all $0 \le \alpha < \beta < 1$, there exists a finite family of r-graphs such that $\gamma(\mathcal{F}) \in (\alpha,\beta)$. It would be interesting to describe the set $\{\gamma(F) : F \text{ is an r-graph}\}$. For example, does Theorem 1.6 still hold when $\mathcal{F}$ in Definition 1.5 is replaced by a single r-graph F? This question is also related to the principality: if there exist $0 \le \alpha < \beta < 1$ such that $\gamma(F) \notin (\alpha,\beta)$ for every r-graph F, then *every* finite family $\mathcal{F}$ with $\gamma(\mathcal{F}) \in (\alpha,\beta)$ (such $\mathcal{F}$ exists by Theorem 1.6 and Proposition 3.1) is non-principal.

- We have mentioned the problem of verifying $\gamma(K_4^3) < \pi(K_4^3)$ in our introduction. Applying Proposition 4.3 with $r = 3$ and $\ell = 2$, we obtain that every 2-colorable 3-graph G_n with $c(G_n) > \frac{1}{2\sqrt{2}}$ (and large n) contains a copy of K_4^3. Is this constant $\frac{1}{2\sqrt{2}}$ sharp here? From (5) we know it can not be reduced to a number smaller than 1/4.

- Parallel to the situation for π, it would be interesting to construct two r-graphs F_1, F_2 such that $0 < \gamma(\{F_1, F_2\}) < \min\{\gamma(F_1), \gamma(F_2)\}$ (Sudakov pointed out that such a construction for even $r \geq 4$ can be obtained by following the ideas in [23]).

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