An explicit construction for a generalized Ramsey problem

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Abstract

An explicit coloring of the edges of K_n is constructed such that every copy of K_4 has at least four colors on its edges. As $n \to \infty$, the number of colors used is $n^{1/2+o(1)}$. This improves upon the previous bound of $O(n^{2/3})$ due to Erdős and Gyárfás obtained by probabilistic methods. The exponent 1/2 is optimal, since it is known that at least $\Omega(n^{1/2})$ colors are required in such a coloring.

The coloring is related to constructions giving lower bounds for the multicolor Ramsey number $r_k(C_4)$. It is more complicated however, because of restrictions imposed on interactions between color classes.

1 Introduction

Given graphs G and H, an (H,q)-*coloring* of G is a coloring of the edges of G in which the edges of every subgraph of G that is isomorphic to H, together receive at least q colors. The minimum number of colors in an (H,q)-coloring of G has been denoted $r(G,H,q)$ [1]. When $G = K_n$ and $H = K_p$, we use the simpler notations (p,q)-coloring and $f(n,p,q)$ from [4].

Erdős [4] was the first to ask for the determination of $f(n,p,q)$ in the general case $2 \le q \le \binom{p}{2}$, however this problem when $q = 2$ reduces to determining the classical Ramsey number for multicolorings (and had been studied much earlier). For $k, p > 0$, the Ramsey number $r_k(p)$ is

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the minimum n such that no matter how the edges of K_n are colored with k colors, there is a monochromatic copy of K_p. It is easy to see that $f(n,p,2) = k$ is equivalent to $r_k(p) = n+1$ and $r_{k-1}(p) = n$, hence determining all $f(n,p,2)$ is equivalent to determining all $r_k(p)$. On the other hand, the numbers $r_k(p)$ seem extremely hard to determine. Even for the smallest nontrivial case $p = 3$, the best bounds are $c^k < r_k(3) < c'k!$ [2, 3] where c and c' are constants. This in turn translates to the bounds $d\frac{\log n}{\log\log n} < f(n,3,2) < d'\log n$ for some other constants d, d'.

The growth rate of $f(n,p,q)$ was more thoroughly investigated by Erdős and Gyárfás [5]. They considered the case when p is fixed and $n \to \infty$. Using the Local Lemma, they proved the upper bound $O(n^{c_{p,q}})$, where $c_{p,q} = \frac{p-2}{\binom{p}{2}-q+1}$. They also determined for each p the smallest q such that $f(n,p,q)$ is linear in n and the smallest q such that $f(n,p,q)$ is quadratic in n.

One of the problems posed in [5] is to determine the growth rates of $f(n,4,q)$.

- For $q = 2$, the only bounds we have are from $r_k(4)$. They are $d\frac{\log n}{\log\log n} < f(n,4,2) < d'\log n$.

- For $q = 3$, the probabilistic upper bound of $O(\sqrt{n}\ )$ from [5] has recently been improved to $e^{O(\sqrt{\log n}\ )}$ in [13] by an explicit construction. It is mentioned in [5] that it remains open whether $f(n,4,3)/\log n \to \infty$ however, it appears that even $f(n,4,3) > c\log n$ for some constant c is not known.

- For $q = 5$, it is proved in [5] that $5n/6 \le f(n,4,5) \le n+1$, where the upper bound holds for infinitely many n.

In this paper we improve the probabilistic construction from [5] that yields $f(n,4,4) < O(n^{2/3})$. Our construction is explicit.

Theorem 1.

$$f(n,4,4) < n^{1/2}\ e^{c\sqrt{\log n}},$$

where $c > 0$ is an absolute constant.

Since $e^{\sqrt{\log n}} = o(n^{\epsilon})$ for every $\epsilon > 0$, this implies the result in the abstract. Moreover, it is pointed out in [5] that $f(n,4,4) > cn^{1/2}$, so the exponent $1/2$ in Theorem 1 is optimal. We believe that the multiplicative factor $e^{\sqrt{\log n}}$ can be removed.

Conjecture 2.

$$f(n,4,4) = \Theta(n^{1/2}).$$

2 $(4,4)$-colorings and the Ramsey numbers $r_k(C_4)$

There is an intimate connection between $f(n,4,4)$ and Turán and Ramsey numbers for C_4. The Turán number $\text{ex}(n,G)$ of a graph G is the maximum number of edges in a subgraph of K_n that contains no copy of G. Classical results [7, 10] yield $\text{ex}(n,C_4) = (1/2+o(1))n^{3/2}$. This implies that $r_k(C_4) < (1+o(1))k^2$, indeed the more precise bound $r_k(C_4) \leq k^2+k+1$ [2, 9] holds for all $k \geq 1$.

On the other hand, obtaining a matching lower bound for $r_k(C_4)$ is more difficult than obtaining the corresponding lower bound for $\text{ex}(n,C_4)$: for the latter, we need only a single extremal graph with no copy of C_4, while for the former we need to essentially decompose the edges of K_n with copies of this extremal graph. This was accomplished independently by Chung and Graham [3] and Irving [9], where it is proved that $r_k(C_4) \geq k^2-k+2$ when $k-1$ is a prime power. Recently the lower bound has been improved by Lazebnik and Woldar [11] to k^2+2 when k is an odd prime power, and still more recently [12] the same bound has been proved when k is any prime power.

The connection with $f(n,4,4)$ becomes evident by observing that in a $(4,4)$-coloring of K_n, monochromatic C_4's are forbidden, since the four vertices forming such a copy of C_4 induce a K_4 with at most three colors. However, our problem is more difficult, since to obtain a $(4,4)$-coloring it does not suffice to merely forbid monochromatic C_4's. We must also consider how color classes interact with each other. Indeed, all known constructions yielding $r_k(C_4) \geq \Omega(k^2)$ are not $(4,4)$-colorings of complete graphs.

Nevertheless, in our approach the starting point is one such Ramsey decomposition. Then we further partition each color class suitably so as to destroy all remaining K_4's with three or fewer colors. The main ingredient for this step is a stronger version of the construction from [13] that provided a $(4,3)$-coloring of K_n with $e^{O(\sqrt{\log n})}$ colors.

In section 3 we describe a variation of the construction from [13], and prove that it forbids some special two-colored and three-colored copies of K_4. In section 4 we describe a slightly modified version of the construction from [11, 12], and again prove that certain three-colored configurations are absent. In section 5 we further modify (or partition) this construction to forbid another three-colored configuration. Finally, in section 6 we combine colorings to obtain a $(4,4)$-coloring. This will complete the proof of Theorem 1.

3 The Symmetric SR coloring

In this section, we describe a variation of the construction developed in [13]. Because the original coloring arose from the subsets of a specified set and used the notion of ranking these subsets, we called it the *Subset Ranking* (SR) coloring. We call the modified construction the *Symmetric Subset Ranking* (SSR) coloring.

For $m > 0$, let $[m] = \{1, \dots, m\}$, and let $[0] = \emptyset$. For $t \le m$, we write $\binom{[m]}{t}$ for the family of all subsets of $[m]$ with size t. The *symmetric difference* of the sets A and B is $A \triangle B = (A - B) \cup (B - A)$.

The SSR Coloring.
Let G be the complete graph with vertex set $\binom{[m]}{t}$. For each t-set $T \in \binom{[m]}{t}$, rank the $2^t - 1$ proper subsets of T according to some linear order. We may choose any linear order, and the linear orders for distinct elements of $\binom{[m]}{t}$ need not have any relationship to one another. Given distinct vertices A, $B \in \binom{[m]}{t}$, let R denote the member of $\{A, B\}$ that contains the minimum element of $A \triangle B$. Let $S \neq R$ denote the other member of $\{A, B\}$. In order to define the edge-coloring, we need to introduce four new parameters.
• $c_0(AB)$ is the minimum element of $A \triangle B$ (thus $c_0 \in R$),
• $c_1(AB)$ is the rank of $A \cap B$ in the linear order associated with the proper subsets of R,
• $c_2(AB)$ is any element of $S - R$,
• $c_3(AB)$ is the rank of $A \cap B$ in the linear order associated with the proper subsets of S.
Color the edge AB with the four-dimensional vector $c(AB) = (c_0(AB), c_1(AB), c_2(AB), c_3(AB))$. □

It is easy to see that the number of colors in the SSR coloring is at most $4^t m(m-1)$. Set $t = \lceil \sqrt{2 \log n} / \sqrt{\log 2} \rceil$ and choose m such that $\binom{m}{t} < n \le \binom{m+1}{t} = M$. Instead of coloring K_n we color the bigger K_M and restrict to K_n. Since $(m/t)^t < \binom{m}{t}$ for $t < m$, the number of colors used to color K_n is at most

$$4^t m(m+1) < (1 + o(1)) 4^t t^2 n^{2/t} < e^{2\sqrt{2 \log 4 \log n}(1+o(1))}.$$

In the remainder of the section we prove various properties of the SSR coloring.

Lemma 3. *Let A, B, C be vertices in an SSR colored K_n and suppose that $c(AB) = c(AC)$. Then $A \cap B = A \cap C$.*

Proof. First suppose that $x = c_0(AB) = c_0(AC) \in A$. Then $c_1(AB) = c_1(AC)$ implies that both $A \cap B$ and $A \cap C$ have the same rank in A. Since the rank of a subset in a set identifies the subset uniquely, the desired conclusion holds.

Now suppose that $x \in (B \cap C) - A$. Then $x' = c_2(AB) = c_2(AC) \in A$. But in this case $c_3(AB) = c_3(AC)$ implies that both $A \cap B$ and $A \cap C$ again have the same rank in A. □

Proposition 4. *Let A, B, C, D be vertices in an SSR colored K_n. Then none of the following four situations can occur (See Fig. 1):*

(i) $c(AB) = c(BC) = c(AC)$,

(ii) $c(AB) = c(BC) = c(CD)$ *and* $c(AC) = c(BD)$,

(iii) $c(AB) = c(BC) = c(CD)$ *and* $c(AD) = c(BD)$,

(iv) $c(AB) = c(AC)$, $c(BC) = c(BD)$, *and* $c(AD) = c(CD)$.

In particular, the SSR coloring is a $(4, 3)$*-coloring.*

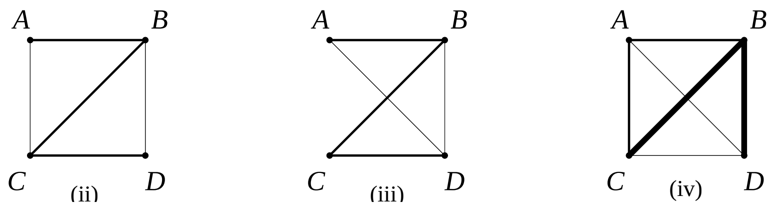

Fig. 1

Proof. Let $x = c_0(AB)$. By symmetry, we may assume that $x \in A$ in (i) and (ii).

(i) Since $c_0(AB) = c_0(BC)$, we have $x \in C$. But now $x \notin A \triangle C$, so $c(AC) \neq c(AB)$.

(ii) Let $z = c_0(BD) \in B - D$. By Lemma 3, $B \cap C = C \cap D$, which implies that $z \notin C$. Since $c_0(AC) = c_0(BD)$, we have $z \in A$. But this contradicts $A \cap B = B \cap C$, which must hold by Lemma 3. If $z \in D - B$, then let $z' = c_2(AB) \in B - D$. We now apply the preceding arguments with z replaced by z'.

(iii) Either $x \in (A \cap C) - (B \cup D)$, or $x \in (B \cap D) - (A \cup C)$. In either case, since $x = \min\{A \triangle B\}$,

$$A \cap [x - 1] = B \cap [x - 1] = C \cap [x - 1] = D \cap [x - 1].$$

This implies that $c_0(AD) = x \neq c_0(BD)$, and hence $c(AD) \neq c(BD)$.

(iv) Let $x' = c_2(AB)$. Since $c_0(AB) = c_0(AC)$ and $c_2(AB) = c_2(AC)$, there is a $y \in \{x, x'\}$ such that $y \in (B \cap C) - A$. Because $c(BC) = c(BD)$, Lemma 3 gives $B \cap C = B \cap D$, which implies that $y \in D$. Since $c(AD) = c(CD)$, Lemma 3 gives $A \cap D = C \cap D$ which yields the contradiction $y \in A$.

It is easy to observe that any two-coloring of the edges of K_4 yields one of the situations (i) or (ii). This proves that the SSR coloring is a $(4, 3)$-coloring of K_n □

4 The Algebraic coloring

In this section we define a variation of the construction from [11, 12]. Since it was originally motivated from Lie Algebras and is defined in terms of finite fields, we call it the *Algebraic* coloring. We always let $\mathbf{F} = \mathbf{F}_q$ denote the finite field with q elements, where q is an odd prime power.

The *Algebraic* Coloring.
Let G be the complete graph with vertex set $\mathbf{F} \times \mathbf{F}$. Given vertices $A = (a_1, a_2)$ and $B = (b_1, b_2)$, let $\delta(A, B) = 1$ if $a_1 = b_1$ and 0 if $a_1 \neq b_1$. Color the edge AB with the two dimensional vector $c(AB) = (a_1 b_1 - a_2 - b_2\ ,\ \delta(A, B))$. □

The Algebraic coloring gives at most $2q$ colors to the edges of K_{q^2}, since the first coordinate of the color vector is a field element. Using standard density results for primes (see, e.g. [8]), we obtain a coloring of $E(K_n)$ with at most $(2 + o(1))\sqrt{n}$ colors. The following Lemma from [11, 12] implies that every color class in the Algebraic coloring (even without the second coordinate) contains no C_4. We include a proof for completeness.

Lemma 5. *Let* $\mathbf{F}$ *be a finite field and* $a_1, a_2, b_1, b_2 \in \mathbf{F}$*, with* $(a_1, a_2) \neq (b_1, b_2)$*. Then the system of equations*

$$\begin{aligned} a_1\ x &= b_1 + x' \\ a_2\ x &= b_2 + x' \end{aligned}$$

has at most one solution $(x, x') \in \mathbf{F} \times \mathbf{F}$*.*

Proof. Suppose that we have two solutions (x, x') and (y, y'). Then

$$a_1\ x = b_1 + x' \tag{1}$$
$$a_2\ x = b_2 + x' \tag{2}$$
$$a_1\ y = b_1 + y' \tag{3}$$
$$a_2\ y = b_2 + y'. \tag{4}$$

Subtracting (1)−(2) from (3)−(4) yields $(x - y)(a_1 - a_2) = 0$. Consequently, $x = y$ or $a_1 = a_2$. In the first case, (1) and (3) yield $x' = y'$ giving $(x, x') = (y, y')$. In the second case, (1) and (2) yield $b_1 = b_2$ giving $(a_1, a_2) = (b_1, b_2)$. □

Proposition 6. *Let* $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2), D = (d_1, d_2)$ *be four vertices in an Algebraically colored* K_n*. Then neither of the following two situations can occur (See Fig. 2):*

(i) $c(AB) = c(AC) = c(AD)$ *and* $c(BD) = c(CD)$,

(ii) $c(AB) = c(AC)$, $c(BD) = c(CD)$, *and* $c(AD) = c(BC)$.

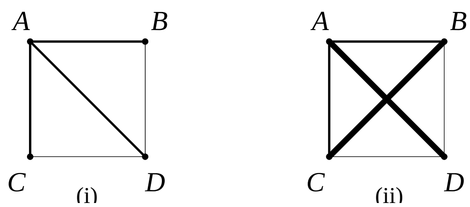

Fig. 2

Proof. Let $c(AB) = (\alpha, \delta(A,B))$ and $c(BD) = (\beta, \delta(B,D))$. Then from the first coordinates of the color vectors we have

$$a_1b_1 = a_2 + b_2 + \alpha \tag{5}$$

$$a_1c_1 = a_2 + c_2 + \alpha \tag{6}$$

$$b_1d_1 = b_2 + d_2 + \beta \tag{7}$$

$$c_1d_1 = c_2 + d_2 + \beta \tag{8}$$

Subtracting (6) from (5) and (8) from (7) yield

$$a_1(b_1 - c_1) = b_2 - c_2 \tag{9}$$

$$d_1(b_1 - c_1) = b_2 - c_2 \tag{10}$$

Now (9) and (10) yield $(a_1 - d_1)(b_1 - c_1) = 0$. If $b_1 = c_1$, then (5) and (6) imply that $b_2 = c_2$ and therefore $B = C$. This contradiction implies that $b_1 \neq c_1$. Therefore $a_1 = d_1$, and we conclude that $\delta(A,D) = 1$.

(i) Since $c(AB) = c(AC) = c(AD)$, we have $\delta(A,B) = \delta(A,C) = \delta(A,D) = 1$. This yields $b_1 = a_1 = c_1$ which we have already excluded.

(ii) Since $\delta(B,C) = \delta(A,D) = 1$, we again conclude that $b_1 = c_1$, a contradiction. □

5 Three matchings forming K_4

In this section we modify the Algebraic coloring to destroy all occurrences of a special three-colored configuration on four vertices. The configuration is made up of three monochromatic matchings of size two, and we henceforth call it a *striped* K_4.

For vertices X, Y in an Algebraically colored K_n, we let $c'(XY)$ denote the first coordinate of the color vector $c(XY)$; thus $c'(XY) \in \mathbf{F}$. Throughout this section we deal with an Algebraically colored K_n, and we let $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2), D = (d_1, d_2)$.

Proposition 7. *Let A, B, C, D form a striped K_4 in an Algebraically colored K_n. Then*

(i) *no two of a_1, b_1, c_1, d_1 are equal,*

(ii) $c_1 + d_1 = \dfrac{2(a_2 - b_2)}{a_1 - b_1}$,

(iii) $2(c_2 - d_2) = (c_1 - d_1)(a_1 + b_1)$.

Proof. We have

$$\begin{aligned} a_1 b_1 &= a_2 + b_2 + \alpha && (11) \\ c_1 d_1 &= c_2 + d_2 + \alpha && (12) \\ a_1 c_1 &= a_2 + c_2 + \beta && (13) \\ b_1 d_1 &= b_2 + d_2 + \beta && (14) \\ a_1 d_1 &= a_2 + d_2 + \gamma && (15) \\ b_1 c_1 &= b_2 + c_2 + \gamma && (16) \end{aligned}$$

Now (13) + (15) – (14) – (16) yields

$$(a_1 - b_1)(c_1 + d_1) = 2\,(a_2 - b_2). \tag{17}$$

If $a_1 = b_1$, then (17) implies that $A = B$, a contradiction. Hence by symmetry (using (11) and (12) if required) we may assume that no two of a_1, b_1, c_1, d_1 are equal, thereby proving (i). (17) also proves (ii).

(13) + (16) – (14) – (15) yields (iii). □

We now modify the Algebraic coloring by adding new colors to the striped K_4's. For each $q \in \mathbf{F}$, let G_q be the auxiliary graph whose vertices are the edges e with $c'(e) = q$. Vertices (A, B) and (C, D) in G_q are adjacent if $\{A, B, C, D\}$ forms a striped K_4. If a component C of G_q is bipartite, then partition the vertices of C into two (color) classes given by the bipartition of C. Let all nonbipartite components of G_q lie in a single class.

The partitioning of $V(G_q)$ yields a partition of all edges e in K_n with $c'(e) = q$ into two classes. Assign all edges in one of these classes a new color. This assignment, when performed on each color class of edges of K_n, results in a coloring with (at most) twice the number of colors as before. We call the resulting coloring the *Divided Algebraic coloring* (DAC).

It is easy (and crucial) to see that if two edges had different colors in the Algebraic coloring, then they also have different colors in the DAC. In particular, Proposition 6 remains true for the DAC. In the remainder of the section we prove that the DAC contains no striped K_4's.

Lemma 8. *Let u, v be vertices in a connected nonbipartite graph G. Then there is a u,v-walk of even length in G.*

Proof. Let C be an odd cycle in G. Let P be a shortest path from u to C, and let Q be a shortest path from u to v. If Q has even length, then Q is the required u,v-walk, so assume that Q has odd length. Let W be the walk obtained by first traversing P, then C, then P again in the opposite direction, and then Q. It is easy to see that W is a u,v-walk of even length. □

Lemma 9. *Let $(A,B),(C,D)$ be vertices in a nonbipartite component of G_q. Then*

$$\frac{a_2 - b_2}{a_1 - b_1} = \frac{c_2 - d_2}{c_1 - d_1}.$$

Proof. First suppose that (X,Y) is adjacent to both (A,B) and (A',B') in G_q, with $X = (x_1, x_2)$ and $Y = (y_1, y_2)$. Then Proposition 7 part (ii) implies that

$$\frac{2(a_2 - b_2)}{a_1 - b_1} = x_1 + y_1 = \frac{2(a'_2 - b'_2)}{a'_1 - b'_1}.$$

By Lemma 8, there is a walk in G_q from (A,B) to (C,D) of even length:

$$(A,B) = (A_0,B_0),(A_1,B_1),\ldots,(A_{2t-1},B_{2t-1}),(A_{2t},B_{2t}) = (C,D).$$

Applying the argument in the previous paragraph successively to (A_l,B_l) and (A_{l+2},B_{l+2}), we obtain the desired conclusion. □

Definition 10. *Given vertices A, B in K_n, let $A \sim B$ if $2(a_2 - b_2) = {a_1}^2 - {b_1}^2$.*

Lemma 11. *Suppose that $S = \{A,B,C,D\}$ forms a striped K_4 in the DAC. Let $X, Y \in S$ with $X \neq Y$. Then $X \sim Y$.*

Proof. By the symmetry of a striped K_4, it suffices to show that $A \sim B$. Because S forms a striped K_4, we have $q = c'(AB) = c'(CD)$. This implies that (A,B) and (C,D) are adjacent and are in a nonbipartite component of G_q.

From Lemma 9 and Proposition 7 part (iii) we get

$$(a_2 - b_2)(c_1 - d_1) = (a_1 - b_1)(c_2 - d_2) = (a_1 - b_1)(c_1 - d_1)(a_1 + b_1)/2.$$

Now Proposition 7 part (i) implies that $c_1 \neq d_1$, hence $A \sim B$. □

Proposition 12. *There are no striped K_4's in the DAC.*

Proof. Suppose on the contrary that A, B, C, D is a striped K_4. Then Lemma 11 implies that $A \sim B$ and $C \sim D$. Clearly the vertices (A, B) and (C, D) lie in the same component of G_q, and this component is nonbipartite. Therefore Lemma 9 applies and

$$\frac{a_1 + b_1}{2} = \frac{a_2 - b_2}{a_1 - b_1} = \frac{c_2 - d_2}{c_1 - d_1} = \frac{c_1 + d_1}{2}.$$

This gives $a_1 + b_1 = c_1 + d_1$. By a similar argument applied to edges AC and BD we get $a_1 + c_1 = b_1 + d_1$. These two equations yield $b_1 = c_1$, a contradiction to Proposition 7 part (i). $\square$

6 Partitioning the partitions

In this section we complete the proof of Theorem 1. We need a generalization of the "doubling" procedure used to construct the Divided Algebraic coloring in the previous section. It unifies the notions that have been implicitly used throughout this paper.

Definition 13. *Let c and c^* be two edge-colorings of G. Then the product $c \times c^*$ of c and c^* is the edge-coloring of G defined by*

$$(c \times c^*)(e) = (c(e), c^*(e))$$

for every edge e.

Note that if two edges have distinct colors in either c or c^*, then they have distinct colors in $c \times c^*$. Also, if c uses d colors, and c^* uses d^* colors, then $c \times c^*$ uses $d \times d^*$ colors.

Proof of Theorem 1: We will construct a $(4, 4)$-coloring of K_n with at most

$$Z = (4 + o(1))n^{1/2}\ e^{2\sqrt{2 \log 4 \log n}(1+o(1))}$$

colors. Let c be the SSR coloring and c^* be the Divided Algebraic coloring. We claim that the product $c \times c^*$ is a $(4, 4)$-coloring with at most Z colors. Since c uses at most $e^{2\sqrt{2 \log 4 \log n}(1+o(1))}$ colors, and c^* uses at most $(4 + o(1))\sqrt{n}$ colors, $c \times c^*$ uses at most Z colors. It remains to show that $c \times c^*$ is a $(4, 4)$-coloring.

Since c is a $(4, 3)$-coloring (Proposition 4), it suffices to consider copies of K_4 with exactly three colors on their edges. Moreover, by Proposition 4 there are also no monochromatic triangles, and by Lemma 5 there are no monochromatic C_4's in $c \times c^*$. This leaves six remaining possibilities (upto symmetries) for a three-colored K_4 with vertex set A, B, C, D (See Fig. 3):

(i) AB, BC, CD have the same color and AC, BD have the same color. This cannot occur by Proposition 4 part (ii).

(ii) AB, BC, CD have the same color and AD, BD have the same color. This cannot occur by Proposition 4 part (iii).

(iii) AB, AC have the same color, BC, BD have the same color, and AD, CD have the same color. This cannot occur by Proposition 4 part (iv).

(iv) AB, AC, AD have the same color and BD, CD have the same color. This cannot occur by Proposition 6 part (i).

(v) AB, AC have the same color, BD, CD have the same color, and AD, BC have the same color. This cannot occur by Proposition 6 part (ii).

(vi) AB, CD have the same color, AC, BD have the same color, and AD, BC have the same color. This cannot occur by Proposition 12 and the definition of c^*.

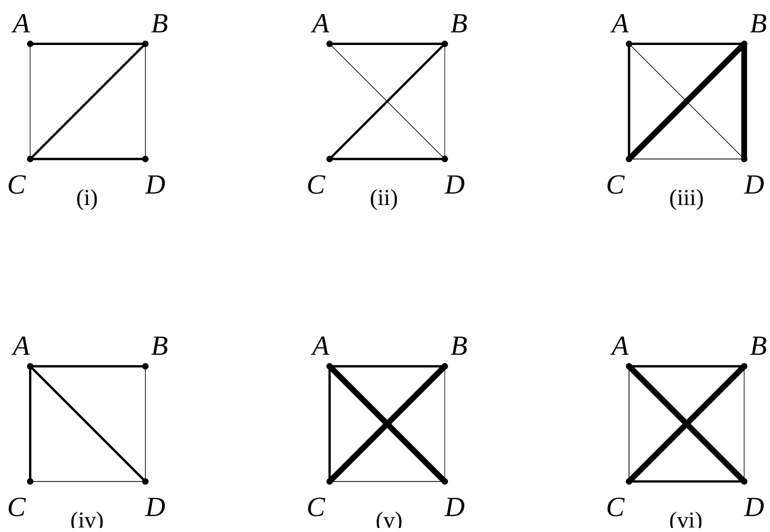

Fig. 3: The 3-colored K_4's

This completes the proof of the theorem. □

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