A new short proof of a theorem of Ahlswede and Khachatrian

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Abstract

Ahlswede and Khachatrian [5] proved the following theorem, which answered a question of Frankl and Füredi [3]. Let $2 \le t+1 \le k \le 2t+1$ and $n \ge (t+1)(k-t+1)$. Suppose that $\mathcal{F}$ is a family of k-subsets of an n-set, every two of which have at least t common elements. If $|\cap_{F\in\mathcal{F}} F| < t$, then $|\mathcal{F}| \le (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}$, and this is best possible.

We give a new, short proof of this result. The proof in [5] requires the entire machinery of the proof of the complete intersection theorem, while our proof uses only ordinary compression and an earlier result of Wilson [7].

1 Introduction

An intersecting family is a collection of sets, every two of which have a point in common. A family of sets is *trivial* if there is a fixed element that lies in all of its sets, otherwise it is *nontrivial*. The Erdős-Ko-Rado theorem [1] states that if $n \ge 2k$ and $\mathcal{F}$ is an intersecting family of k-sets of $[n]$, then $|\mathcal{F}| \le \binom{n-1}{k-1}$. If $n > 2k$ and equality holds, then $\mathcal{F}$ is trivial. If we consider nontrivial families, the corresponding result was proved by Hilton and Milner who showed that for $n \ge 2k$, $|\mathcal{F}| \le |\{F \subset [n] : k+1 \in F, F \cap [k] \ne \emptyset, |F| = k\} \cup [k]\}|$.

A family $\mathcal{F}$ is *t-intersecting* if for every $F, H \in \mathcal{F}$ we have $|F \cap H| \ge t$; if in addition $|\cap_{F\in\mathcal{F}} F| < t$, then $\mathcal{F}$ is *nontrivial t-intersecting*, otherwise it is a *trivial t-intersecting* family. The Erdős-Ko-Rado theorem was generalized in [1], where it was proved that for n sufficiently large, a t-intersecting family of k-sets of $[n]$ satisfies $|\mathcal{F}| \le \binom{n-t}{k-t}$. Frankl conjectured that the same bound holds if and only if $n \ge (t+1)(k-t+1)$, and this was proved by Wilson [7]. Later Ahlswede and Khachatrian [4] determined the maximum size for all n, by proving their Complete Intersection theorem.

Frankl [2] determined the maximum size of a nontrivial t-intersecting family of k-sets of $[n]$ for n sufficiently large, and later Frankl and Füredi [3] asked whether the same result holds for $n < ckt$

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for some constant c. There are two natural constructions, one of them is optimal for $k > 2t+1$ and the other for $t < k \leq 2t+1$. Almost 20 years after Frankl's theorem, the problem was solved by Ahlswede and Khachatrian [5] for the full range of n, however their proof contains the entire machinery of their earlier proof of the Complete Intersection theorem. For $n < (t+1)(k-t+1)$, the problem is already solved by the Complete Intersection theorem, since the largest t-intersecting families of k-sets in this range are nontrivial.

In this note, we provide a new, short proof of this theorem for the case $k \leq 2t+1$. It is somewhat based on ideas of Frankl and Füredi [3]. The only tools we use are ordinary compression and Wilson's result for the base case of our induction argument. Thus our proof is very similar to the original proof of the Erdős-Ko-Rado theorem.

Fix $2 \leq t+1 \leq k \leq 2t+1$. For $k \geq t$ let

$$f(n,k,t) = (t+2)\binom{n-t-2}{k-t-1} + \binom{n-t-2}{k-t-2}.$$

Note that $f(n,k,t)$ is the size of the family of all k-sets of $[n]$ that intersect a fixed $(t+2)$-set in at least $t+1$ points. Call such a family $\mathcal{B}(n,k,t)$. Throughout this note we set

$$n_0 = n(k,t) = (t+1)(k-t+1).$$

Theorem 1 *Let $1 \leq t < k \leq 2t+1$ and $n \geq n_0$. Suppose that $\mathcal{F}$ is a nontrivial t-intersecting family of k-sets of $[n]$. Then $|\mathcal{F}| \leq f(n,k,t)$. If $n > n_0$ and equality holds, then $\mathcal{F}$ is isomorphic to $\mathcal{B}(n,k,t)$, and possibly to $\{F \subset [n] : 4 \in F, F \cap [3] \neq \emptyset, |F| = 3\} \cup [3]\}$ when $(t,k) = (1,3)$.*

2 Proof

The following inequality will be needed.

Lemma 2 *Let $2 \leq t+1 < k \leq 2t+1$. Then for all $t \leq i < k$,*

$$\binom{n_0-t}{i-t} - \binom{n_0-k-1}{i-t} \leq f(n_0,i,t) = (t+2)\binom{n_0-t-2}{i-t-1} + \binom{n_0-t-2}{i-t-2}. \tag{1}$$

The inequality is strict when $i = k-1 > 2$.

Proof. Apply Pascal's identity three times to $\binom{n_0-t}{i-t}$, and then (1) is equivalent to

$$\binom{n_0-t-2}{i-t} \leq t\binom{n_0-t-2}{i-t-1} + \binom{n_0-k-1}{i-t}.$$

Define, for fixed k and t,

$$h(n,i) = t\binom{n-t-2}{i-t-1} + \binom{n-k-1}{i-t} - \binom{n-t-2}{i-t}.$$

We are to show that $h(n_0, i) \geq 0$ for all $t \leq i < k$. We will in fact show this for all $n \geq k+1$. Since $k > t+1$, we have $n_0 > k+1$. First, observe that $h(n,t) = 0$ for all $n \geq k+1$. We may henceforth assume that $i > t$. Next we show that $h(k+1, i) \geq 0$ for all $t < i < k$. Indeed, this is equivalent to

$$\binom{k-t-1}{i-t} \leq t\binom{k-t-1}{i-t-1},$$

which is equivalent to $i(t+1) \geq t^2 + k$. Since $i \geq t+1$, this follows as long as $k \leq 2t+1$, which holds. Finally, apply induction on n, and Pascal's identity to each term of $h(n,i)$ to obtain $h(n,i) = h(n-1,i) + h(n-1,i-1) \geq 0$. If $i = k-1 > 2$, then either $i > t+1$ or $k < 2t+1$, hence the inequality is strict for all $n > k+1$. In particular, it is strict for $n = n_0$. □

Proof of Theorem 1. When $t = 1$ and $k = 2$ then trivially $|\mathcal{F}| \leq 3$, and equality holds only for a triangle. When $t = 1$ and $k = 3$ then the theorem is the same as the Hilton-Milner theorem. So from now on we assume $t \geq 2$. Fixing t, we proceed by induction on k.

For $k = t+1$, the aim is to show that $|\mathcal{F}| \leq \binom{t+2}{t+1} = t+2 = f(n, t+1, t)$. Indeed, let $A, B \in \mathcal{F}$. Now if $\mathcal{F}$ is nontrivial t-intersecting, then it is easy to see that each $C \in \mathcal{F}$ contains $A\Delta B$ and $t-1$ vertices of $A \cap B$, hence $\mathcal{F}$ is isomorphic to $\mathcal{B}(n, t+1, t)$.

We may therefore assume that $k > t+1 > 2$. Let us order the underlying set of elements X linearly. For $x < y$, the family $S_{x,y}(\mathcal{F}) = \{S_{x,y}(F) : \ F \in \mathcal{F}\}$, where $S_{x,y}(F) = F - \{y\} \cup \{x\}$ if $x \notin F, y \in F, F - \{y\} \cup \{x\} \notin \mathcal{F}$ and F otherwise. It is well-known that $|S_{x,y}(\mathcal{F})| = |\mathcal{F}|$ and $S_{x,y}(\mathcal{F})$ is t-intersecting. Apply this compression procedure to $\mathcal{F}$ until we obtain either a family $\mathcal{H}$ such that $S_{x,y}(\mathcal{H})$ is trivial t-intersecting or a family $\mathcal{G}$ which is stable, i.e., $S_{x,y}(\mathcal{G}) = \mathcal{G}$ holds for all $x < y$.

In the first case - assuming that the operation S_{x_0,y_0} would result in the trivial t-intersecting family $S_{x_0,y_0}(\mathcal{H})$ - continue applying the operation $S_{x,y}$, for $x, y \notin \{x_0, y_0\}$. This procedure will terminate, and we call the resulting family $\mathcal{G}$. Define Y to be the smallest $n_0 - 2$ elements of $X - \{x_0, y_0\}$, together with $\{x_0, y_0\}$. In the second case, stop once we have a stable $\mathcal{G}$ and define Y to be the smallest n_0 elements.

Thus in both cases, we have defined a set Y of size n_0 and obtained a family $\mathcal{G}$ such that $S_{x,y}(\mathcal{G}) = \mathcal{G}$ for all $x, y \notin \{x_0, y_0\}$.

Claim 1. $\mathcal{G}$ is a nontrivial t-intersecting family.

Proof of Claim 1. This follows by our procedure if we are in the second case, so assume that we are in the first case. As $S_{x_0,y_0}(\mathcal{H})$ is a trivial t-intersecting family, every member of $\mathcal{H}$ contains either x_0 or y_0, and therefore every member of $\mathcal{G}$ contains either x_0 or y_0.

Suppose for contradiction that $\mathcal{G}$ is a trivial t-intersecting family. Then there is a t-set T contained in every member of $\mathcal{G}$. If $T \cap \{x_0, y_0\} = \emptyset$, then $T \cup \{x_0, y_0\}$ is a $(t+2)$-set, that contains at least $t+1$ elements of every member of $\mathcal{G}$. Since clearly $\mathcal{G}$ does not contain all such sets, $|\mathcal{G}| < f(n,k,t)$ and we are done. We cannot have $x_0 \in T$, since $S_{x_0,y_0}(\mathcal{H}) \neq \mathcal{H}$, and this implies that there exists $A \in \mathcal{G}$ with $x_0 \notin A$. So we may assume that $x_0 \notin T$ and $y_0 \in T$.

Since $S_{x_0,y_0}(\mathcal{H})$ is a trivial t-intersecting family, there is a t-set T' that is contained in all members of $S_{x_0,y_0}(\mathcal{H})$, and since $\mathcal{H}$ is a nontrivial t-intersecting family $x_0 \in T'$ and $y_0 \notin T'$. There exists a set $B \in \mathcal{H}$ that contains x_0 but not y_0, otherwise the t-set $T' - \{x_0\} \cup \{y_0\}$ is subset of every set of $\mathcal{H}$, making it trivial. This implies that there exists a $C \in \mathcal{G}$ that omits y_0, which contradicts the fact that $y_0 \in T$. $\square$

Claim 2 (see Lemma 2.2 of [3].) For every $A, B \in \mathcal{G}$, we have $|A \cap B \cap Y| \geq t$.

Proof of Claim 2. Assume that A, B is a counterexample with $|(A \cup B) \cap Y|$ maximum possible. Let us choose $x \in Y - (A \cup B) - \{x_0, y_0\}$ and $y \in (A \cap B) - Y$, which could be done as $n_0 - 2 > 2k - t - 1 \geq |(A \cup B) \cap Y|$ and $(A \cap B) - Y \neq \emptyset$. Since $S_{x,y}(\mathcal{G}) = \mathcal{G}$, we conclude that $B' = (B \cup \{x\} - \{y\}) \in \mathcal{G}$, and A, B' is a counterexample with $|(A \cup B') \cap Y| > |(A \cup B) \cap Y|$, a contradiction. $\square$

For $t < i \leq k$, let $\mathcal{A}_i := \{A \cap Y : A \in \mathcal{G} : \ |A \cap Y| = i\}$.

Claim 3. Let $t < i \leq k$. Then $|\mathcal{A}_i| \leq f(n_0, i, t)$.

Proof of Claim 3. By Claim 2, $\mathcal{A}_i$ is t-intersecting. For $i = k$ Wilson's theorem [7] yields $|\mathcal{A}_k| \leq \binom{n_0-t}{k-t} = f(n_0, k, t)$, where the equality follows by a short calculation. We may therefore assume that $t < i < k$.

If $\mathcal{A}_i$ is nontrivial t-intersecting, then since $n(k,t) > n(i,t)$, we may apply induction on k to obtain $|\mathcal{A}_i| \leq f(n_0, i, t)$. Hence we may assume that $\mathcal{A}_i$ is trivial t-intersecting. Then there is a t-set T contained in each member of $\mathcal{A}_i$. By Claim 1, $\mathcal{G}$ is non-trivial, so there is a k-set $S \in \mathcal{G}$ that does not contain T. Let $\mathcal{H}_{S,T}$ denote the family of all i-sets in Y that contain T and intersect $S \cap Y$ in at least t elements. By Claim 2, $|\mathcal{A}_i| \leq |\mathcal{H}_{S,T}|$. It is easy to see that $\mathcal{H}_{S,T}$ is strictly maximized when $|S \cap T| = t - 1$ and $S \subset Y$ (one can see this algebraically or by a simple combinatorial argument). Thus there are $k - t + 1$ points of S outside T, and we obtain

$$|\mathcal{A}_i| \leq |\mathcal{H}_{S,T}| \leq \binom{n_0 - t}{i - t} - \binom{n_0 - k - 1}{i - t} \leq f(n_0, i, t), \tag{2}$$

where the last inequality follows by Lemma 2. $\square$

Suppose there exists an $A \in \mathcal{G}$ with $|A \cap Y| = t$. Then by Claim 2, all other sets of $\mathcal{G}$ contain the t-set $A \cap Y$, and this contradicts Claim 1. Therefore every set of $\mathcal{G}$ intersects Y in at least $t + 1$ elements. For $i > t$, the number of sets in $\mathcal{G}$ that intersect $X - Y$ in $k - i$ elements is at most $\binom{n-n_0}{k-i}|\mathcal{A}_i|$. Consequently, by Claim 3,

$$|\mathcal{G}| \leq \sum_{i=t+1}^{k} f(n_0, i, t)\binom{n - n_0}{k - i} = (t+2)\sum_{i=t+1}^{k} \binom{n_0 - t - 2}{i - t - 1}\binom{n - n_0}{k - i} + \sum_{i=t+1}^{k} \binom{n_0 - t - 2}{i - t - 2}\binom{n - n_0}{k - i}$$

$$= (t+2)\binom{n - t - 2}{k - t - 1} + \binom{n - t - 2}{k - t - 2} = f(n, k, t).$$

Now suppose that $|\mathcal{F}| = |\mathcal{G}| = f(n, k, t)$ and $n > n_0$. Then we have equality in Claim 3, and in particular, $|\mathcal{A}_{k-1}| = f(n_0, k-1, t)$. Because $k > t+1 \geq 3$, we have $k-1 > 2$, hence by Lemma 2 we have strict inequality in (2). This implies that $\mathcal{A}_{k-1}$ is nontrivial t-intersecting. Since $n_0 > n(k-1, t)$, we conclude, by induction on k, that $\mathcal{A}_{k-1} \cong \mathcal{B}(n_0, k-1, t)$. From this and Claim 2, we easily obtain $\mathcal{A}_i \cong \mathcal{B}(n_0, i, t)$ when $\max\{t+1, k-n+n_0\} \leq i \leq k$ (clearly $\mathcal{A}_i = \emptyset$ if $i < k-n+n_0$), and then $\mathcal{G} \cong \mathcal{B}(n, k, t)$. Finally, if $\mathcal{H}$ is t-intersecting and $S_{x,y}(\mathcal{H}) \cong \mathcal{B}(n, k, t)$, then it is easy to show that $\mathcal{H} \cong \mathcal{B}(n, k, t)$. This implies that $\mathcal{F} \cong \mathcal{B}(n, k, t)$. $\square$

For $k > 2t+1$ and $n > n_0$, a nontrivial t-intersecting family of k-sets of maximum size is isomorphic to either $\mathcal{B}(n, k, t)$ or

$$\mathcal{C}(n, k, t) = \{F : \ [t] \subset F, \ [t+1, k+1] \cap F \neq \emptyset\} \cup \{F : \ [k+1] - \{i\}, \ i \in [k+1]\}.$$

This was proved by Ahlswede and Khachatrian [5]. For n sufficiently large, $\mathcal{C}(n, k, t)$ is the larger of these two families.

There are two reasons that our proof does not work for the case $k > 2t+1$. The first is that we could not prove a base case for the induction. While a short calculation can determine the smallest n_1 such that for all $n \geq n_1$, we have $|\mathcal{C}(n, k, t)| \geq |\mathcal{B}(n, k, t)|$, we do not know a way of proving that the maximum size of a nontrivial t-intersecting family of k-sets of $[n_1]$ is $|\mathcal{C}(n_1, k, t)|$ without using [5]. For $k \leq 2t+1$, we used Wilson's theorem to settle the case $n = n_0$.

The other obstacle would come from trying to prove an analogue of Claim 3. When we are trying to upper bound $|\mathcal{A}_i|$, we cannot use induction on i, since the formula differs depending on whether $i > 2t+1$.

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