Erdős-Ko-Rado for three sets

Dhruv Mubayi *

March 29, 2005

Abstract

Fix integers $k \geq 3$ and $n \geq 3k/2$. Let $\mathcal{F}$ be a family of k-sets of an n-element set so that whenever $A, B, C \in \mathcal{F}$ satisfy $|A \cup B \cup C| \leq 2k$, we have $A \cap B \cap C \neq \emptyset$. We prove that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality only when $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. This settles a conjecture of Frankl and Füredi [2], who proved the result for $n \geq k^2 + 3k$.

1 Introduction

We write $[n]$ for $\{1, \ldots, n\}$ and X^k for the family of all k-element subsets of a finite set X. A family of sets is a *star* if there is a fixed element contained in all members of the family. Our starting point is the fundamental result of Erdős-Ko-Rado (EKR) which states that for $n \geq 2k$, the maximum size of an intersecting family of k-sets of $[n]$ is $\binom{n-1}{k-1}$, and if $n > 2k$ then equality holds only for a star. Rephrasing, if $\mathcal{F} \subset [n]^k$ and for every $A, B \in \mathcal{F}$ (for which naturally $|A \cup B| \leq 2k$) we have $A \cap B \neq \emptyset$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Frankl [1] generalized this to more than two sets by proving the following result.

Theorem 1. (Frankl) *Let $\mathcal{F} \subset [n]^k$ and let $d \geq 2$ and $n \geq dk/(d-1)$. Suppose that every d sets of $\mathcal{F}$ have nonempty intersection. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

Katona asked whether the conclusion of Theorem 1 holds for an appropriately defined larger class of families $\mathcal{F}$. Specifically, he made the following definition.

Definition. Let $k \leq s \leq 3k$. Then $f(n, k, s)$ denotes the maximum size of a family $\mathcal{F} \subset [n]^k$ so that whenever $A, B, C \in \mathcal{F}$ satisfy $|A \cup B \cup C| \leq s$, we have $A \cap B \cap C \neq \emptyset$.

Frankl and Füredi [2] proved that for every $2k \leq s \leq 3k$, $f(n, k, s) = \binom{n-1}{k-1}$ as long as $n \geq k^2 + 3k$, and observed that $f(n, k, 2k-1) = \Omega(n^k)$ for fixed k. Note that the lower bound $f(n, k, s) \geq \binom{n-1}{k-1}$

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, 851 S. Morgan Street, Chicago, IL 60607-7045; research supported in part by the National Science Foundation under grant DMS-0400812, and an Alfred P. Sloan Research Fellowship

2000 Mathematics Subject Classification: 05C35, 05C65, 05D05

Keywords: *set system, nontrivial family, Erdős-Ko-Rado*

is valid for all $s \in \{2k, \dots, 3k\}$ by simply letting $\mathcal{F}$ be a maximum sized star. Moreover, by definition $f(n, k, s+1) \leq f(n, k, s)$, hence Frankl and Füredi's first result follows by proving the upper bound just for $s = 2k$. They conjectured that $f(n, k, 2k) = \binom{n-1}{k-1}$ for all $k \geq 3$ and $n \geq 3k/2$, with equality only for a star. The threshold $3k/2$ follows from the fact that for smaller n, three sets $A, B, C \in \mathcal{F}$ whose intersection is empty cannot exist (so in particular, we can have $|\mathcal{F}| = \binom{n}{k}$).

Frankl and Füredi [2] proved their conjecture for $k = 3$, and commented (without proof) that their approach also works for $k = 4, 5$ and more generally for $n > k^2/\log k$. Their proof is somewhat complicated since it uses the Hilton-Milner theorem on nontrivial intersecting families. In this note we prove their conjecture.

Theorem 2. *Let $k \geq 3$ and $n \geq 3k/2$. Then $f(n, k, 2k) = \binom{n-1}{k-1}$ with equality only for a star.*

Our proof is shorter than in [2], and perhaps simpler, since it uses only EKR. The main idea is to reduce the problem to a situation where we have a partition of the ground set into pairwise disjoint k-sets. The problem in this environment is then handled in the following (somewhat technical) lemma.

Lemma 3. *Fix $k, t \geq 2$ and $1 \leq l \leq k$. Let $S_1, \dots, S_t$ be pairwise disjoint k-sets and $X = \bigcup_i S_i$. Suppose that $\mathcal{F} \subset X^l$ with $S_i \in \mathcal{F}$ for all i if $l = k$ and $\mathcal{F} = \{S_1, S_2\}$ if $t = 2$ as well. Suppose that for every $A, B \in \mathcal{F}$ and $i \in [t]$, $A \cap B \cap S_i = \emptyset$ implies that $|(A \cup B) - S_i| > l$. Then $|\mathcal{F}| < \binom{tk-1}{l-1}$.*

Proof. We proceed by induction on t. For the base case, suppose that $t = 2$. If $l = k$, then $|\mathcal{F}| = 2 < \binom{2k-1}{k-1}$, so assume that $l < k$. If $A, B \in \mathcal{F}$ are disjoint l-sets, then either $|A \cap S_1| + |B \cap S_1| \leq l$ or $|A \cap S_2| + |B \cap S_2| \leq l$. Say the first inequality holds. Then $A, B, 2$ violate the hypothesis of the lemma, since $|(A \cup B) - S_2| = |A \cap S_1| + |B \cap S_1| \leq l$. Consequently, $\mathcal{F}$ is an intersecting family, and by EKR, we have $|\mathcal{F}| \leq \binom{2k-1}{l-1}$. If equality holds, then since $2l < 2k$, again by EKR we obtain $\bigcap_{F \in \mathcal{F}} F = \{x\}$, and $\mathcal{F}$ consists of all l-sets containing x. We may assume without loss of generality that $x \in S_1$. Now take two different sets $A, B \in \mathcal{F}$ with $A \subset S_1$ and $B \cap S_1 = \{x\}$. Then $A, B, 2$, violate the hypothesis of the lemma, hence equality cannot hold.

Next suppose that $t \geq 3$ and the result holds for $t - 1$. We first consider the case $l < k$. If there exist $A, B \in \mathcal{F}$ and $i \neq j$ for which $A \subset S_i$ and $B \subset S_j$, then A, B, i (and also A, B, j) violate the hypothesis. Hence we may assume that there is an i_0, such that no $A \in \mathcal{F}$ satisfies $A \subset S_{i_0}$. By relabelling if necessary, assume $i_0 = t$.

Now consider any $F \in \mathcal{F}$. Write F as $F_1 \cup F_2$, where $F_1 = F \cap S_t$ and $F_2 = F - F_1$. For a fixed F_1 of size $l - r$ $(1 \leq r \leq l)$, let $\mathcal{F}_r$ be the family of all r-sets $F_2 \subset \bigcup_{i=1}^{t-1} S_i$ such that $F_1 \cup F_2 \in \mathcal{F}$. If there exist $C, D \in \mathcal{F}_r$ and $i \in [t-1]$ for which $C \cap D \cap S_i = \emptyset$ and $|(C \cup D) - S_i| \leq r$, then $C_1 = C \cup F_1, D_1 = D \cup F_1, i$ violate the hypothesis of the lemma, since $C_1 \cap D_1 \cap S_i = \emptyset$ and

$$|(C_1 \cup D_1) - S_i| = |(C \cup D) - S_i| + |F_1| \leq r + (l - r) = l.$$

Hence by induction, we conclude that $|\mathcal{F}_r| < \binom{(t-1)k-1}{r-1}$. Recalling that $\mathcal{F} \cap (S_t)^l = \emptyset$, we obtain

$$|\mathcal{F}| < \sum_{r=1}^{l} \binom{|S_t|}{l-r} \binom{(t-1)k-1}{r-1} = \sum_{r=1}^{l} \binom{k}{l-r} \binom{(t-1)k-1}{r-1} = \binom{tk-1}{l-1}.$$

Next we consider the case $l = k$. In this case $S_t \in \mathcal{F}$, so a similar argument as above yields

$$|\mathcal{F}| \le \sum_{r=1}^{k} \binom{k}{k-r} \left(\binom{(t-1)k-1}{r-1} - 1 \right) + 1 = \binom{tk-1}{k-1} - \sum_{r=1}^{k} \binom{k}{r} + 1 < \binom{tk-1}{k-1},$$

where the last inequality holds since $k \ge 2$. $\Box$

To settle the cases of small n, we need a recent result of the author and Verstraëte [3] who proved that for $3k/2 \le n \le 2k$, every family $\mathcal{F} \subset [n]^k$ containing no three sets A, B, C for which $A \cap B \cap C = \emptyset$ satisfies $|\mathcal{F}| \le \binom{n-1}{k-1}$, and equality holds only if $\mathcal{F}$ is a star (the bound $|\mathcal{F}| \le \binom{n-1}{k-1}$ was proved much earlier by Frankl [1], but he didn't characterize the case of equality).

Proof of Theorem 2: The cases $3k/2 \le n \le 2k$ are settled by the result of [3], so we consider $n > 2k$. Suppose that $\mathcal{F} \subset [n]^k$ such that $A, B, C \in \mathcal{F}$ and $|A \cup B \cup C| \le 2k$ implies that $A \cap B \cap C \ne \emptyset$. We will show that $|\mathcal{F}| \le \binom{n-1}{k-1}$ with equality only if $\mathcal{F}$ is a star.

Let $S_1, \ldots, S_t$ be a maximum subfamily of pairwise disjoint k-sets from $\mathcal{F}$. If $t = 1$, then $\mathcal{F}$ is in fact intersecting, and the theorem follows from EKR, so assume that $t \ge 2$. If $n = tk$, then set $l = k$. The condition on $\mathcal{F}$ in the theorem implies the condition on $\mathcal{F}$ in Lemma 3 (with S_i in the statement of the lemma playing the role of $C \in \mathcal{F}$ above). Hence we may apply Lemma 3 directly and obtain $|\mathcal{F}| < \binom{n-1}{k-1}$.

We now suppose that $n > tk$ and let $Y = [n] - \bigcup_i S_i$. As in the proof of Lemma 3, each $F \in \mathcal{F}$ can be written as $F_1 \cup F_2$, where $F_1 = F \cap Y$, and $F_2 = F - F_1$. By construction of $S_1, \ldots, S_t$, no $A \in \mathcal{F}$ satisfies $A \subset Y$. Now fix $F_1 \subset Y^{k-l}$ $(1 \le l \le k)$, and consider the family $\mathcal{F}_l$ of all l-sets $F_2 \subset \bigcup_{i=1}^t S_i$ such that $F_1 \cup F_2 \in \mathcal{F}$. Suppose first that $l < k$. If there exist $A, B \in \mathcal{F}_l$ and $i \in [t]$ for which $A \cap B \cap S_i = \emptyset$ and $|(A \cup B) - S_i| \le l$, then consider the three sets $A_1 = A \cup F_1, B_1 = B \cup F_1, S_i$. Clearly $A_1 \cap B_1 \cap S_i = \emptyset$ and

$$|A_1 \cup B_1 \cup S_i| = |S_i| + |(A \cup B) - S_i| + |F_1| \le k + l + (k - l) = 2k.$$

If $l = k$ then observe that $S_i \in \mathcal{F}_k$ for all $i \in [t]$, and if in addition $t = 2$, then $\mathcal{F}_k = \{S_1, S_2\}$, since a third set in $\mathcal{F}_k$ is prohibited by the conditions on $\mathcal{F}$. Consequently, $\mathcal{F}_l$ satisfies the hypothesis of Lemma 3, and we obtain $|\mathcal{F}_l| < \binom{tk-1}{l-1}$ for all $1 \le l \le k$. Therefore, noting again that $\mathcal{F} \cap Y^k = \emptyset$,

$$|\mathcal{F}| < \sum_{l=1}^{k} \binom{|Y|}{k-l} \binom{tk-1}{l-1} = \sum_{l=1}^{k} \binom{n-tk}{k-l} \binom{tk-1}{l-1} = \binom{n-1}{k-1}. \quad \Box$$

We end by conjecturing an extension of this problem to more than three sets.

Conjecture 4. *Let $k \ge d \ge 3$ and $n \ge dk/(d-1)$. Suppose that $\mathcal{F} \subset [n]^k$ such that for every $A_1, \ldots, A_d \in \mathcal{F}$ satisfying $|\bigcup_i A_i| \le 2k$ we have $\bigcap_i A_i \ne \emptyset$. Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, with equality only if $\mathcal{F}$ is a star.*

References

[1] P. Frankl, On Sperner families satisfying an additional condition. J. Combinatorial Theory Ser. A 20 (1976), no. 1, 1–11

[2] P. Frankl, Z. Füredi, A new generalization of the Erdős-Ko-Rado theorem. Combinatorica 3 (1983), no. 3-4, 341–349

[3] D. Mubayi, J. Verstraëte, Proof of a conjecture of Erdős on triangles in set systems. to appear, Combinatorica.