A Ramsey-type result for geometric ℓ-hypergraphs

Dhruv Mubayi* Andrew Suk†

April 7, 2014

Abstract

Let $n \geq \ell \geq 2$ and $q \geq 2$. We consider the minimum N such that whenever we have N points in the plane in general position and the ℓ-subsets of these points are colored with q colors, there is a subset S of n points all of whose ℓ-subsets have the same color and furthermore S is in convex position. This combines two classical areas of intense study over the last 75 years: the Ramsey problem for hypergraphs and the Erdős-Szekeres theorem on convex configurations in the plane. For the special case $\ell = 2$, we establish a single exponential bound on the minimum N, such that every complete N-vertex geometric graph whose edges are colored with q colors, yields a monochromatic convex geometric graph on n vertices.

For fixed $\ell \geq 2$ and $q \geq 4$, our results determine the correct exponential tower growth rate for N as a function of n, similar to the usual hypergraph Ramsey problem, even though we require our monochromatic set to be in convex position. Our results also apply to the case of $\ell = 3$ and $q = 2$ by using a geometric variation of the stepping up lemma of Erdős and Hajnal. This is in contrast to the fact that the upper and lower bounds for the usual 3-uniform hypergraph Ramsey problem for two colors differ by one exponential in the tower.

1 Introduction

The classic 1935 paper of Erdős and Szekeres [13] entitled *A Combinatorial Problem in Geometry* was a starting point of a very rich discipline within combinatorics: Ramsey theory (see, e.g., [16]). The term *Ramsey theory* refers to a large body of deep results in mathematics which have a common theme: "Every large system contains a large well-organized subsystem." Motivated by the observation that any five points in the plane in general position[1] must contain four members in convex position, Esther Klein asked the following.

Problem 1.1. *For every integer $n \geq 2$, determine the minimum $f(n)$, such that any set of $f(n)$ points in the plane in general position, contains n members in convex position.*

Celebrated results of Erdős and Szekeres [13, 14] imply that

$$2^{n-1} + 1 \leq f(n) \leq \binom{2n-4}{n-2} \leq 2^{2n(1-o(1))}. \quad (1)$$

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Research partially supported by NSF grant DMS-0969092. Email: mubayi@math.uic.edu

†Massachusetts Institute of Technology, Cambridge, MA. Supported by an NSF Postdoctoral Fellowship and by Swiss National Science Foundation Grant 200021-125287/1. Email: asuk@math.mit.edu.

[1] A planar point set P is in *general position*, if no three members are collinear.

They conjectured that $f(n) = 2^{n-1} + 1$, and Erdős offered a \$500 reward for a proof. Despite much attention over the last 75 years, the constant factors in the exponents have not been improved.

In the same paper [13], Erdős and Szekeres gave another proof of a classic result due to Ramsey [24] on hypergraphs. An ℓ-uniform hypergraph H is a pair (V, E), where V is the vertex set and $E \subset \binom{V}{\ell}$ is the set of edges. We denote $K_n^\ell = (V, E)$ to be the complete ℓ-uniform hypergraph on an n-element set V, where $E = \binom{V}{\ell}$. When $\ell = 2$, we write $K_n^2 = K_n$. Motivated by obtaining good quantitative bounds on $f(n)$, Erdős and Szekeres looked at the following problem.

Problem 1.2. *For every integer $n \geq 2$, determine the minimum integer $r(K_n, K_n)$, such that any two-coloring on the edges of a complete graph G on $r(K_n, K_n)$ vertices, yields a monochromatic copy of K_n.*

Erdős and Szekeres [13] showed that $r(K_n, K_n) \leq 2^{2n}$. In [10], Erdős gave a construction showing that $r(K_n, K_n) > 2^{n/2}$. Despite much attention over the last 65 years, the constant factors in the exponents have not been improved.

Generalizing Problem 1.2 to q-colors and ℓ-uniform hypergraphs has also be studied extensively. Let $r(K_n^\ell; q)$ be the least integer N, such that any q-coloring on the edges of a complete N-vertex ℓ-uniform hypergraph H, yields a monochromatic copy of K_n^ℓ. We will also write

$$r(K_n^\ell; q) = r(\underbrace{K_n^\ell, K_n^\ell, ..., K_n^\ell}_{q \text{ times}}).$$

Erdős, Hajnal, and Rado [11, 12] showed that there are positive constants c and c' such that

$$2^{cn^2} < r(K_n^3, K_n^3) < 2^{2^{c'n}}. \tag{2}$$

They also conjectured that $r(K_n^3, K_n^3) > 2^{2^{cn}}$ for some constant $c > 0$, and Erdős offered a \$500 reward for a proof. For $\ell \geq 4$, there is also a difference of one exponential between the known upper and lower bounds for $r(K_n^\ell, K_n^\ell)$, namely,

$$\text{twr}_{\ell-1}(cn^2) \leq r(K_n^\ell, K_n^\ell) \leq \text{twr}_\ell(c'n), \tag{3}$$

where c and c' depend only on ℓ, and the tower function $\text{twr}_\ell(x)$ is defined by $\text{twr}_1(x) = x$ and $\text{twr}_{i+1} = 2^{\text{twr}_i(x)}$. As Erdős and Rado have shown [12], the upper bound in equation (3) easily generalizes to q colors, implying that $r(K_n^\ell; q) \leq \text{twr}_\ell(c'n)$, where $c' = c'(\ell, q)$. On the other hand, for $q \geq 4$ colors, Erdős and Hajnal (see [16]) showed that $r(K_n^\ell; q)$ does indeed grow as a ℓ-fold exponential tower in n, proving that $r(K_n^\ell; q) = \text{twr}_\ell(\Theta(n))$. For $q = 3$ colors, Conlon, Fox, and Sudakov [6] modified the construction of Erdős and Hajnal to show that $r(K_n^\ell, K_n^\ell, K_n^\ell,) > \text{twr}_\ell(c \log^2 n)$.

Interestingly, both Problems 1.1 and 1.2 can be asked simultaneously for geometric graphs, and a similar-type problem can be asked for geometric ℓ-hypergraphs. A *geometric ℓ-hypergraph H in the plane* is a pair (V, E), where V is a set of points in the plane in general position, and $E \subset \binom{V}{\ell}$ is a collection of ℓ-tuples from V. When $\ell = 2$ ($\ell = 3$), edges are represented by straight line segments (triangles) induced by the corresponding vertices. The sets V and E are called the *vertex set* and *edge set* of H, respectively. A geometric hypergraph H is *convex*, if its vertices are in convex position.

Geometric graphs ($\ell = 2$) have been studied extensively, due to their wide range of applications in combinatorial and computational geometry (see [23], [18, 19]). Complete convex geometric graphs are very well understood, and are some of the most *well-organized* geometric graphs (if not the most). Many long standing problems on complete geometric graphs, such as its crossing number [2], number of halving-edges [27], and size of crossing families [3], become trivial when its vertices are in convex position. There has also been a lot of research on geometric 3-hypergraphs in the plane, due to its connection to the *k-set problem* in $\mathbb{R}^3$ (see [22],[25],[8]). In this paper, we study the following problem which combines Problems 1.1 and 1.2.

Problem 1.3. *Determine the minimum integer $g(K_n^\ell; q)$, such that any q-coloring on the edges of a complete geometric ℓ-hypergraph H on $g(K_n^\ell; q)$ vertices, yields a monochromatic convex ℓ-hypergraph on n vertices.*

Another chromatic variant of the Erdős-Szekeres convex polygon problem was studied by Devillers et al. [7], where they considered colored points in the plane rather than colored edges.

We will also write

$$g(K_n^\ell; q) = g(\underbrace{K_n^\ell, ..., K_n^\ell}_{q \text{ times}}).$$

Clearly we have $g(K_n^\ell; q) \geq \max\{r(K_n^\ell; q), f(n)\}$. An easy observation shows that by combining equations (1) and (3), we also have

$$g(K_n^\ell; q) \leq f(r(K_n^\ell; q)) \leq \mathrm{twr}_{\ell+1}(cn),$$

where $c = c(\ell, q)$. Our main results are the following two exponential improvements on the upper bound of $g(K_n^\ell; q)$.

Theorem 1.4. *For geometric graphs, we have*

$$2^{q(n-1)} < g(K_n; q) \leq 2^{8qn^2 \log(qn)}.$$

The argument used in the proof of Theorem 1.4 above extends easily to hypergraphs, and for each fixed $\ell \geq 3$ it gives the bound $g(K_n^\ell; q) < \mathrm{twr}_\ell(O(n^2))$. David Conlon pointed out to us that one can improve this slightly as follows.

Theorem 1.5. *For geometric ℓ-hypergraphs, when $\ell \geq 3$ and fixed, we have*

$$g(K_n^\ell; q) \leq \mathrm{twr}_\ell(cn),$$

where $c = O(q \log q)$.

By combining Theorems 1.4, 1.5, and the fact that $g(K_n^\ell; q) \geq r(K_n^\ell; q)$, we have the following corollary.

Corollary 1.6. *For fixed ℓ and $q \geq 4$, we have $g(K_n^\ell; q) = \mathrm{twr}_\ell(\Theta(n))$.*

As mentioned above, there is an exponential difference between the known upper and lower bounds for $r(K_n^3, K_n^3)$. Hence, for two-colorings on geometric 3-hypergraphs in the plane, equation (2) implies

$$g(K_n^3, K_n^3) \geq r(K_n^3, K_n^3) \geq 2^{cn^2}.$$

Our next result establishes an exponential improvement in the lower bound of $g(K_n^3, K_n^3)$, showing that $g(K_n^3, K_n^3)$ does indeed grow as a 3-fold exponential tower in a power of n. One noteworthy aspect of this lower bound is that the construction is a geometric version of the famous stepping up lemma of Erdős and Hajnal [11] for sets. The lemma produces a q'-coloring χ' of $\binom{[2^n]}{\ell+1}$ from a q-coloring χ of $\binom{[n]}{\ell}$ for appropriate $q' > q$, where the largest monochromatic clique of $(\ell+1)$-sets under χ' is not too much larger than the largest monochromatic clique of ℓ-sets under χ (see [16] for more details about the stepping up lemma). While it is a major open problem to apply this method to $r(K_n^3, K_n^3)$ and improve the lower bound in equation (2), we are able to achieve this in the geometric setting as shown below.

Theorem 1.7. *For geometric 3-hypergraphs in the plane, we have*

$$g(K_n^3, K_n^3) \geq 2^{2^{cn}},$$

where c is an absolute constant. In particular, $g(K_n^3, K_n^3) = \mathrm{twr}_3(\Theta(n))$.

We systemically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. All logarithms are in base 2.

2 Proof of Theorems 1.4 and 1.5

Before proving Theorems 1.4 and 1.5, we will first define some notation. Let $V = \{p_1, ..., p_N\}$ be a set of N points in the plane in general position ordered from left to right according to x-coordinate, that is, for $p_i = (x_i, y_i) \in \mathbb{R}^2$, we have $x_i < x_{i+1}$ for all i. For $i_1 < \cdots < i_t$, we say that $X = (p_{i_1}, ..., p_{i_t})$ forms an *t-cup* (*t-cap*) if X is in convex position and its convex hull is bounded above (below) by a single edge. See Figure 1. When $t = 3$, we will just say X is a cup or a cap.

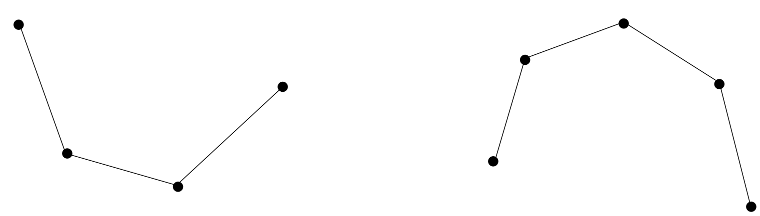


Figure 1: A 4-cup and a 5-cap.

Proof of Theorem 1.4. We first prove the upper bound. Let $G = (V, E)$ be a complete geometric graph on $N = 2^{8qn^2 \log(qn)}$ vertices, such that the vertices $V = \{v_1, ..., v_N\}$ are ordered from left to right according to x-coordinate. Let χ be a q-coloring on the edge set E. We will recursively construct a sequence of vertices $p_1, ..., p_t$ from V and a subset $S_t \subset V$, where $t = 0, 1, ..., qn^2$ (when $t = 0$ there are no vertices in the sequence), such that the following holds.

1. for any vertex p_i, all pairs (p_i, p) where $p \in \{p_j : j > i\} \cup S_t$ have the same color, which we denote by $\chi'(p_i)$,

2. for every pair of vertices p_i and p_j, where $i < j$, either (p_i, p_j, p) is a cap for all $p \in \{p_k : k > j\} \cup S_t$, or (p_i, p_j, p) is a cup for all $p \in \{p_k : k > j\} \cup S_t$,

3. the set of points S_t lies to the right of the point p_t, and

4. $|S_t| \geq \frac{N}{q^t t!} - t$.

We start with no vertices in the sequence, and set $S_0 = V$. After obtaining vertices $\{p_1, ..., p_t\}$ and S_t, we define p_{t+1} and S_{t+1} as follows. Let $p_{t+1} = (x_{t+1}, y_{t+1}) \in \mathbb{R}^2$ be the smallest indexed element in S_t (the left-most point), and let H be the right half-plane $x > x_{t+1}$. We define t lines $l_1, ...l_t$ such that l_i is the line going through points p_i, p_{t+1}. Notice that the arrangement $\cup_{i=1}^t l_i$ partitions the right half-plane H into $t+1$ cells. See Figure 2. Since V is in general position, by the pigeonhole principle, there exists a cell $\Delta \subset H$ that contains at least $(|S_t| - 1)/(t+1)$ points of S_t.

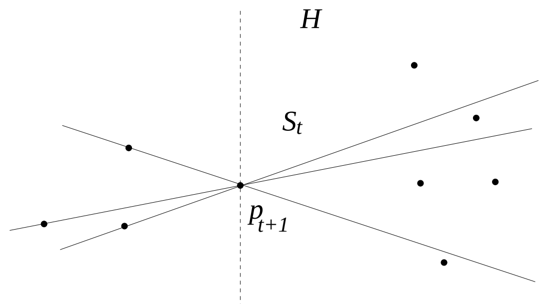


Figure 2: Lines partitioning the half-plane H.

Let us call two elements $v'_1, v'_2 \in \Delta \cap S_t$ *equivalent* if $\chi(p_{t+1}, v'_1) = \chi(p_{t+1}, v'_2)$. Hence, there are at most q equivalence classes. By setting S_{t+1} to be the largest of those classes, we have the recursive formula

$$|S_{t+1}| \geq \frac{|S_t| - 1}{(t+1)q}.$$

Substituting in the lower bound on $|S_t|$, we obtain the desired bound

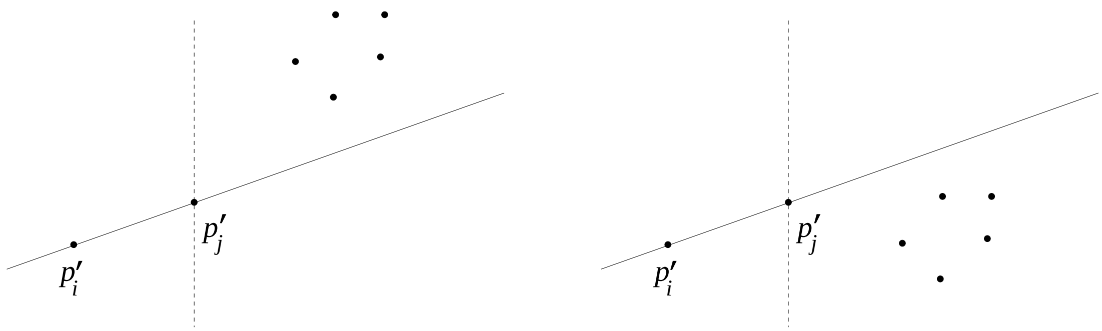
$$|S_{t+1}| \geq \frac{N}{(t+1)!q^{t+1}} - (t+1).$$

This shows that we can construct the sequence $p_1, \ldots, p_{t+1}$ and the set S_{t+1} with the desired properties. For $N = 2^{8qn^2 \log(qn)}$, we have

$$|S_{qn^2}| \geq \frac{2^{8qn^2 \log(qn)}}{(qn^2)!q^{qn^2}} - qn^2 \geq 1. \tag{4}$$

Hence, $P_1 = \{p_1, ..., p_{qn^2}\}$ is well defined. Since χ' is a q-coloring on P_1, by the pigeonhole principle, there exists a subset $P_2 \subset P_1$ such that $|P_2| = n^2$, and every vertex has the same color. By construction of P_2, every pair in P_2 has the same color. Hence these vertices induce a monochromatic geometric graph.

Now let $P_2 = \{p'_1, ..., p'_{n^2}\}$. We define partial orders $\prec_1, \prec_2$ on P_2, where $p'_i \prec_1 p'_j$ ($p'_i \prec_2 p'_j$) if and only if $i < j$ and the set of points $P_2 \setminus \{p'_1, ..., p'_j\}$ lie above (below) the line going through points p'_i and p'_j. See Figure 3. By construction of P_2, $\prec_1, \prec_2$ are indeed partial orders and every two elements in P_2 are comparable by either $\prec_1$ or $\prec_2$. By a corollary to Dilworth's Theorem [9] (see also Theorem 1.1 in [15]), there exists a chain $p^*_1, ..., p^*_n$ of length n with respect to one of the partial orders. Hence $(p^*_1, ..., p^*_n)$ forms either an n-cap or an n-cup. Therefore, these vertices induce a monochromatic convex geometric graph.



(a) Example of $p'_i \prec_1 p'_j$. (b) Example of $p'_i \prec_2 p'_j$.

Figure 3: Partial orders $\prec_1, \prec_2$.

For the lower bound, we proceed by induction on q. The base case $q = 1$ follows by taking the complete geometric graph on 2^{n-1} vertices, whose vertex set does not have n members in convex position. This is possible by the construction of Erdős and Szekeres [14]. Let G_0 denote this geometric graph. For $q > 1$, we inductively construct a complete geometric graph $G = (V, E)$ on $2^{(q-1)(n-1)}$ vertices, and a coloring $\chi : E \to \{1, 2, ..., q-1\}$ on the edges of G, such that G does not contain a monochromatic convex geometric graph on n vertices. Now we replace each vertex $v_i \in G$ with a small enough copy[2] of G_0, which we will denote as G_i, where all edges in G_i are colored with the color q, and all edges between G_i and G_j have color $\chi(v_i v_j)$. Then we have a complete geometric graph G' on

$$2^{(q-1)(n-1)} 2^{n-1} = 2^{q(n-1)}$$

vertices, such that G' does not contain a monochromatic convex graph on n vertices.

□

By following the proof above, one can show that $g(K^{\ell}_n; q) \leq \mathrm{twr}_{\ell}(O(n^2))$. However, the following short argument due to David Conlon gives a better bound. The proof uses an old idea of M. Tarsi (see [22] Chapter 3) that gave an upper bound on $f(n)$.

Lemma 2.1. *For geometric 3-hypergraphs, we have* $g(K^3_n; q) \leq r(K^3_n; 2q) \leq 2^{2^{cn}}$, *where* $c = O(q \log q)$.

Proof. Let $H = (V, E)$ be a complete geometric 3-hypergraph on $N = r(K^3_n; 2q)$ vertices, and let χ be a q coloring on the edges of H. By fixing an ordering on the vertices $V = \{v_1, ..., v_N\}$, we say that a triple $(v_{i_1}, v_{i_2}, v_{i_3})$, $i_1 < i_2 < i_3$, has a *clockwise (counterclockwise) orientation*, if $v_{i_1}, v_{i_2}, v_{i_3}$ appear in clockwise (counterclockwise) order along the boundary of $conv(v_{i_1} \cup v_{i_2} \cup v_{i_3})$. Hence by

[2]Obtained by an affine transformation.

Ramsey's theorem, there are n points from V for which every triple has the same color and the same orientation. As observed by Tarsi (see Theorem 3.8 in [26]), these vertices must be in convex position. $\square$

Lemma 2.2. *For $\ell \geq 4$ and $n \geq 4^\ell$, we have $g(K_n^\ell; q) \leq r(K_n^\ell; q+1) \leq \mathrm{twr}_\ell(cn)$, where $c = O(q \log q)$.*

Proof. Let $H = (V, E)$ be a complete geometric ℓ-hypergraph on $N = r(K_n^\ell; q+1)$ vertices, and let χ be a q coloring on the ℓ-tuples of V with colors $1, 2, ..., q$. Now if an ℓ-tuple from V is not in convex position, we change its color to the new color $q+1$. By Ramsey's theorem, there is a set $S \subset V$ of n points for which every ℓ-tuple has the same color. Since $n \geq 4^\ell$, by the Erdős-Szekeres Theorem, S contains ℓ members in convex position. Hence, every ℓ-tuple in S is in convex position, and has the same color which is not the new color $q+1$. Therefore S induces a monochromatic convex geometric ℓ-hypergraph.

$\square$

Theorem 1.5 now follows by combining Lemma 2.1 and 2.2.

3 A lower bound construction for geometric 3-hypergraphs

In this section, we will prove Theorem 1.7, which follows immediately from the following lemma.

Lemma 3.1. *For sufficiently large n, there exists a complete geometric 3-hypergraph $H = (V, E)$ in the plane with $2^{2^{\lfloor n/2 \rfloor}}$ vertices, and a two-coloring χ' on the edge set E, such that H does not contain a monochromatic convex 3-hypergraph on $2n$ vertices.*

Proof. Let G be the complete graph on $2^{n/2}$ vertices, where $V(G) = \{1, ..., 2^{n/2}\}$, and let χ be a red-blue coloring on the edges of G such that G does not contain a monochromatic complete subgraph on n vertices. Such a graph does indeed exist by a result of Erdős [10], who showed that $r(K_n, K_n) > 2^{n/2}$. We will use G and χ to construct a complete geometric 3-hypergraph H on $2^{2^{n/2}}$ vertices, and a coloring χ' on the edges of H, with the desired properties.

Set $M = 2^{n/2}$. We will recursively construct a point set P_t of 2^t points in the plane as follows. Let P_1 be a set of two points in the plane with distinct x-coordinates. After obtaining the point set P_t, we define P_{t+1} follows. We inductively construct two copies of P_t, $L = P_t$ and $R = P_t$, and place L to the left of R such that all lines determined by pairs of points in L go below R and all lines determined by pairs of points of R go above L. Then set $P_{t+1} = L \cup R$. See Figure 4.

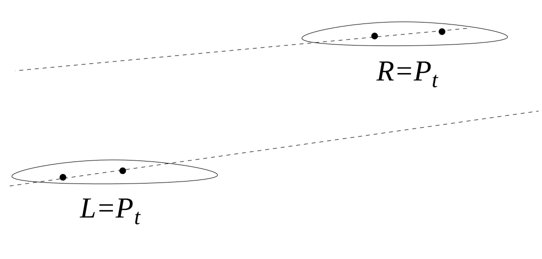


Figure 4: Constructing P_{t+1} from P_t.

Let $P_M = \{p_1, ..., p_{2^M}\}$ be the set of 2^M points in the plane, ordered by increasing x-coordinate, from our construction. Notice that P_M contains 2^{M-t} disjoint copies of P_t. For $i < j$, we define

$$\delta(p_i, p_j) = \max\{t : p_i, p_j \text{ lies inside a copy of } P_t = L \cup R, \text{ and } p_i \in L, p_j \in R\}.$$

Notice that

Property A: $\delta(p_i, p_j) \neq \delta(p_j, p_k)$ for every triple $i < j < k$,

Property B: for $i_1 < \cdots < i_n$, $\delta(p_{i_1}, p_{i_n}) = \max_{1 \leq j \leq n-1} \delta(p_{i_j}, p_{i_j+1})$.

Now we define a red-blue coloring χ' on the triples of P_M as follows. For $i < j < k$,

$$\chi'(p_i, p_j, p_k) = \chi(\delta(p_i, p_j), \delta(p_j, p_k)).$$

Now we claim that the geometric 3-hypergraph $H = (P_M, E)$ does not contain a monochromatic convex 3-hypergraph on $2n$ vertices. For sake of contradiction, let $S = \{q_1, ..., q_{2n}\}$ be a set of $2n$ points from P_M, ordered by increasing x-coordinate, that induces a red convex 3-hypergraph. Set $\delta_i = \delta(q_i, q_{i+1})$.

Case 1. Suppose that there exists a j such that $\delta_j, \delta_{j+1}, ..., \delta_{j+n-1}$ forms a monotone sequence. First assume that

$$\delta_j > \delta_{j+1} > \cdots > \delta_{j+n-1}.$$

Since G does not contain a red complete subgraph on n vertices, there exists a pair $j \leq i_1 < i_2 \leq j + n - 1$ such that $(\delta_{i_1}, \delta_{i_2})$ is blue. But then the triple $(q_{i_1}, q_{i_2}, q_{i_2+1})$ is blue, a contradiction. Indeed, by Property B,

$$\delta(q_{i_1}, q_{i_2}) = \delta(q_{i_1}, q_{i_1+1}) = \delta_{i_1}.$$

Therefore, since $\delta_{i_1} > \delta_{i_2}$ and $(\delta_{i_1}, \delta_{i_2})$ is blue, the triple $(q_{i_1}, q_{i_2}, q_{i_2+1})$ must also be blue. A similar argument holds if $\delta_j < \delta_{j+1} < \cdots < \delta_{j+n-1}$.

Case 2. Suppose we are not in Case 1. For $2 \leq i \leq 2n$, we say that i is a *local minimum* if $\delta_{i-1} > \delta_i < \delta_{i+1}$, a *local maximum* if $\delta_{i-1} < \delta_i > \delta_{i+1}$, and a *local extremum* if it is either a local minimum or a local maximum. This is well defined by Property A.

Observation 3.2. *For $2 \leq i \leq 2n$, i is never a local minimum.*

Proof. Suppose $\delta_{i-1} > \delta_i < \delta_{i+1}$ for some i, and suppose that $\delta_{i-1} \geq \delta_{i+1}$. We claim that $q_{i+1} \in \text{conv}(q_{i-1}, q_i, q_{i+2})$. Indeed, since $\delta_{i-1} \geq \delta_{i+1} > \delta_i$, this implies that $q_{i-1}, q_i, q_{i+1}, q_{i+2}$ lies inside a copy of $P_{\delta_{i-1}} = L \cup R$, where $q_{i-1} \in L$ and $q_i, q_{i+1}, q_{i+2} \in R$. Since $\delta_{i+1} > \delta_i$, this implies that q_i, q_{i+1}, q_{i+2} lie inside a copy $P_{\delta_{i+1}} = L' \cup R' \subset R$, where $q_i, q_{i+1} \in L'$ and $q_{i+2} \in R'$.

Notice that all lines determined by q_i, q_{i+1}, q_{i+2} go above the point q_{i-1}. Therefore q_{i+1} must lie above the line that goes through the points q_{i-1}, q_{i+2}, and furthermore, q_{i+1} must lie below the line that goes through the points q_{i-1}, q_i. Since $\delta_{i+1} > \delta_i$, the line through q_i, q_{i+1} must go below the point q_{i+2}, and therefore $q_{i+1} \in \text{conv}(q_{i-1}, q_i, q_{i+2})$. See Figure 5. If $\delta_{i-1} < \delta_{i+1}$, then a similar argument shows that $q_i \in \text{conv}(q_{i-1}, q_{i+1}, q_{i+2})$.

□

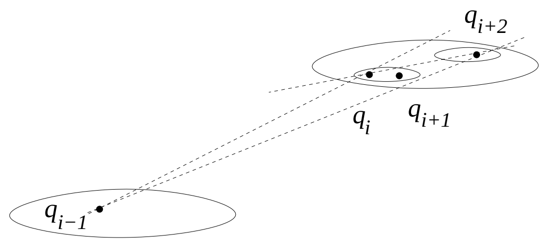


Figure 5: Point $q_{i+1} \in \text{conv}(q_{i-1}, q_i, q_{i+2})$.

Since $\delta_1, \ldots, \delta_{2n}$ does not have a monotone subsequence of length n, it must have at least two local extrema. Since between any two local maximums there must be a local minimum, we have a contradiction by Observation 3.2. This completes the proof.

□

4 Concluding remarks

For $q \geq 4$ colors and $\ell \geq 3$, we showed that $g(K_n^\ell; q) = \text{twr}_\ell(\Theta(n))$. Our bounds on $g(K_n^\ell; q)$ for $q \leq 3$ can be summarized in the following table.

	$q = 2$	$q = 3$
$\ell = 2$	$2^{\Omega(n)} < g(K_n, K_n) \leq 2^{O(n^2 \log n)}$	$2^{\Omega(n)} < g(K_n; 3) \leq 2^{O(n^2 \log n)}$
$\ell = 3$	$g(K_n^3, K_n^3) = 2^{2^{\Theta(n)}}$	$g(K_n^3; 3) = 2^{2^{\Theta(n)}}$
$\ell \geq 4$	$\text{twr}_{\ell-1}(\Omega(n^2)) \leq g(K_n^\ell, K_n^\ell) \leq \text{twr}_\ell(O(n))$	$\text{twr}_\ell(\Omega(\log^2 n)) \leq g(K_n^\ell; 3) \leq \text{twr}_\ell(O(n))$

Off-diagonal. The Ramsey number $r(K_s, K_n)$ is the minimum integer N such that every red-blue coloring on the edges of a complete N-vertex graph G, contains either a red clique of size s, or a blue clique of size n. The off-diagonal Ramsey numbers, i.e., $r(K_s, K_n)$ with s fixed and n tending to infinity, have been intensively studied. For example, it is known [1, 4, 5, 21] that $R_2(3, n) = \Theta(n^2/\log n)$ and, for fixed $s > 3$,

$$c_1(\log n)^{1/(s-2)} \left(\frac{n}{\log n}\right)^{(s+1)/2} \leq r(K_s, K_n) \leq c_2 \frac{n^{s-1}}{\log^{s-2} n}. \tag{5}$$

Another interesting variant of Problem 1.3 is the following off-diagonal version.

Problem 4.1. *Determine the minimum integer $g(K_s, K_n)$, such that any red-blue coloring on the edges of a complete geometric graph G on $g(K_s, K_n)$ vertices, yields either a red convex geometric graph on s vertices, or a blue convex geometric graph on n vertices.*

For fixed s, one can show that $g(K_s, K_n)$ grows single exponentially in n. In particular

$$2^{n-1} + 1 \leq g(K_s, K_n) \leq 4^{4^s n}.$$

The lower bound follows from the fact that $g(K_s, K_n) \geq f(n)$. The upper bound follows from the inequalities

$$g(K_s, K_n) \leq r(K_{4^s}, K_{4^n}) \leq (4^n)^{4^s}.$$

Indeed, by the Erdős-Szkeres theorem, if G contains a red-clique of size 4^s, then there must be a red convex geometric graph on s vertices. Likewise, If G contains a blue clique of size 4^n, then there must be a blue convex geometric graph on n vertices.

Higher dimensions. Generalizing Problem 1.1 to higher dimensions has also been studied. Let $f_d(n)$ be the smallest integer such that any set of at least $f_d(n)$ points in $\mathbb{R}^d$ in general position[3] contains n members in convex position. The following upper and lower bounds were obtained by Károlyi [17] and Károlyi and Valtr [20] respectively,

$$2^{cn^{1/(d-1)}} \leq f_d(n) \leq \binom{2n-2d-1}{n-d} + d = 2^{2n(1-o(1))}.$$

A *geometric ℓ-hypergraph H in d-space* is a pair (V, E), where V is a set of points in general position in $\mathbb{R}^d$, and $E \subset \binom{V}{\ell}$ is a collection of ℓ-tuples from V. When $\ell \leq d+1$, ℓ-tuples are represented by $(\ell-1)$-dimensional simplices induced by the corresponding vertices.

Problem 4.2. *Determine the minimum integer $g_d(K_n^\ell; q)$, such that any q-coloring on the edges of a complete geometric ℓ-hypergraph H in d-space on $g_d(K_n^\ell; q)$ vertices, yields a monochromatic convex ℓ-hypergraph on n vertices.*

When $d = 2$, we write $g_2(K_n^\ell; q) = g(K_n^\ell; q)$. Clearly $g_d(K_n^\ell; q) \geq \max\{f_d(n), R(K_n^\ell; q)\}$. One can also show that $g_d(K_n^\ell; q) \leq g(K_n^\ell; q)$. Indeed, for any complete geometric ℓ-hypergraph $H = (V, E)$ in d-space with a q-coloring χ on $E(H)$, one can obtain a complete geometric ℓ-hypergraph in the plane $H' = (V', E')$, by projecting H onto a 2-dimensional subspace $L \subset \mathbb{R}^d$ such that V' is in general position in L. Thus we have

$$g_d(K_n; q) \leq g(K_n; q) \leq 2^{cn^2 \log n},$$

where $c = O(q \log q)$, and for $\ell \geq 3$

$$g_d(K_n^\ell; q) \leq g(K_n^\ell; q) \leq \mathrm{twr}_\ell(c'n^2),$$

where $c' = c'(q, \ell)$.

Acknowledgment. We thank David Conlon for showing us an improved version of Theorem 1.5.

[3] A point set P in $\mathbb{R}^d$ is in general position, if no $d+1$ members lie on a common hyperplane.

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