A Hypergraph Extension

of the

Bipartite Turán Problem

Dhruv Mubayi[1]
Jacques Verstraëte[2]

Abstract.

Let t, n be integers with $n \geq 3t$. For $t \geq 3$, we prove that in any family of at least $t^4\binom{n}{2}$ triples from an n-element set X, there exist $2t$ triples $A_1, B_1, A_2, B_2, \ldots, A_t, B_t$ and distinct elements $a, b \in X$ such that $A_i \cap A_j = \{a\}$ and $B_i \cap B_j = \{b\}$, for all $i \neq j$, and

$$A_i \cap B_j = \begin{cases} A_i - \{a\} = B_j - \{b\} & \text{for } i = j \\ \emptyset & \text{for } i \neq j. \end{cases}$$

When $t = 2$, we improve the upper bound $t^4\binom{n}{2}$ to $3\binom{n}{2} + 6n$. This improves upon the previous best known bound of $3.5\binom{n}{2}$ due to Füredi. Some results concerning more general configurations of triples are also presented.

1 Introduction

Let $\mathcal{F}$ be a family of r-graphs, some member of which is r-partite. A fundamental theorem due to Erdős states that there exists $\delta = \delta(\mathcal{F}) > 0$ such that the maximum number of edges in an r-graph on n vertices containing no member of $\mathcal{F}$ is $O(n^{r-\delta})$ as $n \to \infty$. The asymptotic order of this maximum, denoted $\mathrm{ex}(n, \mathcal{F})$, is generally very difficult to determine. For surveys, we refer the reader to Füredi [6] and to Frankl [8]. In this paper, we consider the above problem for the following specific classes of r-graphs.

[1]Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan Street, Chicago, IL 60607-7045 e-mail: mubayi@math.uic.edu

[2]Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON, Canada N2L 3G1 e-mail: jverstraete@math.uwaterloo.ca

This research was done while both authors were postdocs at Microsoft Research.

2000 AMS Subject Classification: 05B05, 05B07, 05C35, 05C65, 05D05.

Keywords: *triple system, r-graph, Steiner system, bipartite Turán problem.*

Definition. Let $X_1, X_2, \ldots, X_t$ be t pairwise disjoint sets of size $r-1$, and let Y be a set of s elements, disjoint from $\bigcup_{i\in[t]} X_i$. Then $K_{s,t}^{(r)}$ denotes the r-graph with vertex set $\bigcup_{i\in[t]} X_i \cup Y$ and edge set $\{X_i \cup \{y\} : i \in [t], y \in Y\}$.

Remark. Note that $K_{s,t}^{(r)}$ and $K_{t,s}^{(r)}$ are nonisomorphic when $r \geq 3$ and $s \neq t$. Our results apply to both cases, so for simplicity throughout this paper we let $t \geq s$.

Definition. Let $f_r(n)$ be the maximum number of edges in an n vertex r-graph containing no four edges A, B, C, D with $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$.

In the case $r = 3$, we note that $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)})$. Erdős [3] asked whether $f_r(n) = O(n^{r-1})$ when $r \geq 3$. Erdős and Frankl (unpublished) proved that $f_r(n) = O(n^{r-1/2})$. Füredi [4] later answered Erdős' question by the following Theorem:

Theorem 1.1 (Füredi) *For all integers n, r with $r \geq 3$ and $n \geq 2r$,*

$$\binom{n-1}{r-1} + \left\lfloor \frac{n-1}{r} \right\rfloor \leq f_r(n) < 3.5\binom{n}{r-1}.$$

The lower bound arises from the family of all r-element subsets of $[n]$ containing a fixed element of $[n]$ together with an arbitrary family of $\lfloor \frac{n-1}{r} \rfloor$ pairwise disjoint r-element subsets not containing that element. Füredi also observed that if we replace every 5-set in a Steiner $S_1(n, 5, 2)$ family by all its 3-element subsets, then the resulting triple system has $\binom{n}{2}$ triples and contains no copy of $K_{2,2}^{(3)}$ (for the existence of $S_1(n, 5, 2)$, see Ray-Chaudhuri and Wilson [12]). This slightly improves the lower bound above when $r = 3$ to $\binom{n}{2}$. Füredi conjectured that this construction gives a sharp lower bound when $n \equiv 1, 5$ modulo 20, and that the lower bound in Theorem 1.1 is sharp for $r \geq 4$ and n sufficiently large.

In this paper, we will concentrate on triple systems excluding a copy of $K_{2,t}^{(3)}$ and, more generally, excluding a copy of $K_{s,t}^{(3)}$. This is a common generalization of the problem of estimating both $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)})$ and $\mathrm{ex}(n, K_{s,t}^{(2)})$. The latter is a fundamental open problem in extremal graph theory. Our main result also improves Füredi's upper bound for $f_r(n)$ in Theorem 1.1.

Theorem 1.2 *Let $t \geq 2$ and $n \geq 3t$ be integers. Then*

$$\mathrm{ex}(n, K_{2,t}^{(3)}) < \begin{cases} 3\binom{n}{2} + 6n & \text{for } t = 2 \\ t^4\binom{n}{2} & \text{for } t > 2. \end{cases}$$

Moreover, for infinitely many n,

$$\mathrm{ex}(n, K_{2,t}^{(3)}) \geq \frac{2t-1}{3}\binom{n}{2}.$$

Remark: The expression $\mathrm{ex}(n, K_{2,t}^{(3)})/\binom{n}{2}$ has a limit $g(t)$ as $n \to \infty$. This follows by similar arguments to Proposition 6.1 in [4]. By Theorem 1.2, $\frac{2t-1}{3} \leq g(t) \leq t^4$. It would be interesting to determine the growth rate of $g(t)$.

Using Theorem 1.2 and Lemma 5.2 in [4] one can easily obtain the following improvement to Theorem 1.1 (see the remark at the end of Section 4).

Corollary 1.3 *Fix $r \geq 3$. Then $f_r(n) < 3\binom{n}{r-1} + O(n^{r-2})$.*

We will prove the lower bound of Theorem 1.2 in Section 2. In Section 3, we prove a fundamental lemma which enables us to establish the upper bound of Theorem 1.2 in Section 4.

Generalizing Theorem 1.2 to the larger class $K_{s,t}^{(3)}$ for $s > 2$ seems more difficult, and we are not able to determine the order of magnitude of $\mathrm{ex}(n, K_{s,t}^{(3)})$. We prove the following extension of the result of Erdős and Frankl that $f_3(n) = \mathrm{ex}(n, K_{2,2}^{(3)}) = O(n^{5/2})$.

Theorem 1.4 *Let $n/3 \geq t \geq s \geq 3$. Then*

$$\mathrm{ex}(n, K_{s,t}^{(3)}) < c_{s,t} n^{3-1/s}, \tag{1}$$

where $c_{s,t}$ depends only on s and t.

The proof of Theorem 1.4 (see Section 5) follows easily from the well-known bound $\mathrm{ex}(n, K_{s,t}) < c'_{s,t} n^{2-1/s}$ (see [10]) for graphs. However, we believe that the exponent $3-1/s$ in (1) is not the truth for any $s \geq 2$. Theorem 1.2 shows this for $s = 2$. As further evidence, we can also improve (1) when $s = 3$ (see Section 6). This proof does not yield improvements for larger values of s.

Theorem 1.5 *Let $t \geq 3$. Then*

$$\mathrm{ex}(n, K^{(3)}_{3,t}) < c_t n^{13/5},$$

where c_t depends only on t.

In the other direction, we prove the following Theorem in Section 2.

Theorem 1.6

$$\mathrm{ex}(n, K^{(3)}_{s,t}) > d_{s,t} \frac{[\mathrm{ex}(n, K_{s,t})]^2}{n}.$$

It is believed (see Füredi [6]) that $\mathrm{ex}(n, K^{(2)}_{s,t}) = \Omega_t(n^{2-1/s})$ whenever $2 \leq s \leq t$, but this has only been proved for $s \geq 2$ and $t > (s-1)!$ (see Kollár, Rónyai, Szabó [9] and Alon, Rónyai, Szabó [1]). This immediately implies the following Corollary to Theorem 1.6.

Corollary 1.7 *If $t > (s-1)! > 0$, then*

$$\mathrm{ex}(n, K^{(3)}_{s,t}) > d_t n^{3-2/s}.$$

We feel that $n^{3-2/s}$ is the correct order of magnitude of $\mathrm{ex}(n, K^{(3)}_{s,t})$.

Conjecture 1.8 *Let s, t be integers with $2 \leq s \leq t$. Then*

$$\mathrm{ex}(n, K^{(3)}_{s,t}) = \Theta_t(n^{3-2/s}).$$

Notations. The symbol $[n]$ denotes the set $\{1, 2, \ldots, n\}$, and $X^{(r)}$ denotes the family of all r-sets in set X. We write G for a simple finite undirected graph, unless indicated otherwise, and $G(A, B)$ to indicate that G is bipartite with parts A and B. The notation $\Gamma_G(v)$ is used for the set of vertices adjacent to a vertex v in a graph (or hypergraph) G, $e(G)$ is the number of edges in G, $\deg_G(v)$ is the number of edges incident with vertex v in G, and $n(G)$ is the number of vertices in G. For any vertex u of G, we write $G - u$ for the subgraph of G spanned by all edges of G disjoint from u. Similarly, if E is a set of edges of G, then $G - E$ denotes the subgraph of G spanned by all edges of G which are not in E. A hypergraph containing no subgraph isomorphic to a fixed hypergraph $\mathcal{F}$ is called $\mathcal{F}$-free.

2 Lower Bounds for $\mathrm{ex}(n, K_{s,t}^{(3)})$

In this section, we give lower bounds for the numbers $\mathrm{ex}(n, K_{s,t}^{(3)})$, by showing that they are related to the bipartite Turán numbers $\mathrm{ex}(n, K_{s,t}) = \mathrm{ex}(n, K_{s,t}^{(2)})$. We will establish this relationship using the following construction:

Construction. Let G be any $K_{s,t}^{(2)}$-free bipartite graph, with parts $A = \{a_1, a_2, \dots, a_n\}$ and B of size n. Let $A' = \{a'_1, a'_2, \dots, a'_n\}$. Define a 3-partite triple system $\mathcal{H}$ on $A \cup B \cup A'$ whose triples consist of those sets (a_i, b, a'_j) for which $a_i b a_j$ is a path in G.

We write $\mathrm{z}(n, K_{s,t}^{(r)})$ for the maximum number of edges in an r-partite $K_{s,t}^{(r)}$-free r-graph in which all parts have size n.

Proposition 2.1 *The triple system $\mathcal{H}$ contains no $K_{s,t}^{(3)}$ and*

$$\mathrm{z}(n, K_{s,t}^{(3)}) \geq e(\mathcal{H}) \geq \frac{1}{n}\mathrm{z}(n, K_{s,t})^2 - \mathrm{z}(n, K_{s,t}).$$

Proof. Choose G in the above construction to contain $\mathrm{z}(n, K_{s,t}^{(2)})$ edges. The number of triples in $\mathcal{H}$ is precisely twice the number of paths $a_i b a_j$ in G. Therefore, by the convexity of binomial coefficients,

$$\begin{aligned} e(\mathcal{H}) = 2\sum_{v \in B} \binom{\deg_G(v)}{2} &\geq 2n\binom{e(G)/n}{2} \\ &= 2n(e(G)^2/2n^2 - e(G)/2n) \\ &= e(G)^2/n - e(G). \end{aligned}$$

Thus $\mathcal{H}$ has the required number of edges. We now check that $\mathcal{H}$ is $K_{s,t}^{(3)}$-free.

Suppose, for a contradiction, that $\mathcal{F} \subset \mathcal{H}$ is isomorphic to $K_{s,t}^{(3)}$. We suppose the edges of $\mathcal{F}$ are $\{X_i \cup \{y\} : i \in [t], y \in Y\}$, where Y is an s-element set, $X_1, X_2, \dots, X_t$ are t pairwise disjoint 2-element sets, and Y is disjoint from $\bigcup_{i \in [t]} X_i$. It is not hard to see that there exists a 3-partition of $\mathcal{F}$ into parts Z_1, Z_2 and Z_3, each of which is contained entirely in a distinct part of $\mathcal{H}$. We suppose $|Z_1| = s$ and $|Z_2| = |Z_3| = t$. If $Z_1 \subset A$ ($Z_1 \subset B$), then we may assume $Z_2 \subset B$ ($Z_2 \subset A$) by symmetry. By the construction, $Z_1 \cup Z_2$ induces a copy of $K_{s,t}^{(2)}$ in G, a contradiction. Therefore $Z_1 = \{a'_1, a'_2, \dots, a'_s\} \subset A'$. By symmetry, we assume $Z_2 \subset B$. Let $Z = \{a_1, a_2, \dots, a_s\}$. Then $Z \cup Z_2$ induces a copy of $K_{s,t}^{(2)}$ in G, a contradiction. This completes the proof of Proposition 2.1. $\square$

The above proposition immediately gives the following result, by noting the $K_{s,t}^{(2)}$-free norm graph constructions of Kollár, Rónyai and Szabó [9] for $t > (s-1)!$ (see also Alon, Rónyai and Szabó [1]), and the constructions due to Füredi [7] of $K_{2,t}$-free graphs on n vertices with at least $(t-1)^{1/2}n^{3/2} - cn^{4/3}$ edges, where c is a positive constant:

Corollary 2.2 *Let s,t be integers with $s \geq 2$ and $t > (s-1)!$. Then $\mathrm{z}(n, K_{s,t}^{(3)}) = \Omega_t(n^{3-2/s})$. Moreover, for all $t \geq 2$, $\mathrm{z}(n, K_{2,t}^{(3)}) \geq (t-1)n^2 - dn^{11/6}$ for some constant $d > 0$.*

Proposition 2.1 also establishes the lower bound in Theorem 1.6, since $\mathrm{ex}(3n, K_{s,t}^{(3)}) \geq \mathrm{z}(n, K_{s,t}^{(3)})$ and $\mathrm{z}(n, K_{s,t}) \geq 2\mathrm{ex}(n, K_{s,t})$, where the last inequality can be found in [2].

Füredi's construction for $\mathrm{ex}(n, K_{2,2}^{(3)})$ can easily be generalized for $\mathrm{ex}(n, K_{2,t}^{(3)})$, thereby improving Proposition 2.1 in the case $s = 2$. Indeed, consider an $S_1(n, 2t+1, 2)$ Steiner system (i.e. every pair of elements is contained in precisely one $(2t+1)$-set) in which we replace each $(2t+1)$-set by all its 3-element subsets. The existence of such Steiner systems is established in Ray-Chaudhuri and Wilson [12] whenever $n = t$ modulo $t(t-1)$. The resulting triple system is $K_{2,t}^{(3)}$-free and the number of triples is

$$\binom{2t+1}{3}\binom{2t+1}{2}^{-1}\binom{n}{2} = \frac{2t-1}{3}\binom{n}{2}.$$

This verifies the lower bound in Theorem 1.2.

3 Main Lemma

In this section, we establish a generalization of a lemma due to Füredi (see Lemma 3.2 in [4]). This enables us to give an upper bound on the number of edges which may be deleted from a graph to obtain a $K_{s,t}$-free subgraph. This lemma will be of fundamental importance in the proofs of both Theorem 1.2 and Theorem 1.5. We begin with the following definition:

Definition. Let $t \geq s \geq 2$, and let $G = G(A,B)$ be a bipartite graph. Then $\mathcal{D}_{s,t}(G)$ denotes the s-graph on $A \cup B$ whose edge set consists of those $S \in A^{(s)} \cup B^{(s)}$ for which S lies in a $K_{s,t}$ in G.

Main Lemma. *Let $G = G(A, B)$ be a bipartite graph and let $t \geq s \geq 2$. Then we may delete at most $(s+t-3)e(\mathcal{D}_{s,t}(G))$ edges from G to obtain a $K_{s,t}$-free graph.*

Proof. We proceed by induction on $e(G)$. For convenience, we write $\mathcal{D}$ instead of $\mathcal{D}_{s,t}(G)$ and $\mathcal{D}_E$ instead of $\mathcal{D}_{s,t}(G - E)$. If $e(G) = 0$, then $e(\mathcal{D}) = 0$. We also suppose G has no isolated vertices. Suppose $e(G) > 0$. If some edge $f \in G$ is in no $K_{s,t}$ in G then, by induction, we may remove at most $(s+t-3)e(\mathcal{D}_f) = (s+t-3)e(\mathcal{D})$ edges from $G - f$ to delete all $K_{s,t}$ in $G - f$, and hence all $K_{s,t}$ in G. We therefore assume every edge of G is contained in a $K_{s,t}$ in G.

We now aim to define a non-empty set E of edges of G such that $|E| \leq (s+t-3)[e(\mathcal{D}) - e(\mathcal{D}_E)]$. Let us see that this will suffice to complete the proof. The induction hypothesis will apply to $G - E$: we delete a set E' of at most $(s+t-3)e(\mathcal{D}_E)$ edges from $G - E$ to obtain a $K_{s,t}$-free graph. The total number of edges deleted is $|E| + |E'| \leq (s+t-3)[e(\mathcal{D}) - e(\mathcal{D}_E)] + (s+t-3)e(\mathcal{D}_E) = (s+t-3)e(\mathcal{D})$. For a contradiction, we suppose that no such set E exists. That is, for any non-empty set E of edges of G,

$$|E| > (s+t-3)[e(\mathcal{D}) - e(\mathcal{D}_E)]. \qquad (*)$$

Claim 1. $\deg_{\mathcal{D}}(u) < \deg_G(u)$ *for every vertex* $u \in G$.

Proof. Suppose $\deg_G(u) \leq \deg_{\mathcal{D}}(u)$ for some $u \in G$. Let E be the set of edges of G incident with u. Then we certainly have $0 < |E| = \deg_G(u) \leq \deg_{\mathcal{D}}(u) \leq (s+t-3)\deg_{\mathcal{D}}(u)$, as $s, t \geq 2$. Since no $K_{s,t}$ in $G - E$ contains u, u is an isolated vertex of $\mathcal{D}_E$. This implies $\deg_{\mathcal{D}}(u) \leq e(\mathcal{D}) - e(\mathcal{D}_E)$. Consequently, $|E| \leq (s+t-3)\deg_{\mathcal{D}}(u) \leq (s+t-3)[e(\mathcal{D}) - e(\mathcal{D}_E)]$, contradicting $(*)$. So $\deg_{\mathcal{D}}(u) < \deg_G(u)$ for every vertex $u \in G$.

Fix a vertex $u \in A$, and let $B' = \Gamma_G(u)$. Let $A_1, \ldots, A_k$ denote the edges of $\mathcal{D}$ incident with u. Let $B_i = \bigcap_{x \in A_i} \Gamma_G(x)$ for $i \in [k]$, and define $\mathcal{A} = \{A_i : |B_i| \leq t-1\}$ and $\mathcal{B}$ to be the hypergraph spanned by the edge set $\{B_i : |B_i| \geq t\}$. Note that $B_i \subset B'$ for all i as $u \in A_i$, and $\mathcal{B}$ may have multiple edges.

Claim 2. $\mathcal{A} = \emptyset$.

Proof. Suppose $\mathcal{A} \neq \emptyset$. Let E_i be the set of edges from u to B_i, for each $i \in [k]$, and set $E = \bigcup_{A_i \in \mathcal{A}} E_i$. Since $B_i \neq \emptyset$ for each i, we have $E \neq \emptyset$. On the other hand

$$|E| \leq \sum_{A_i \in \mathcal{A}} |E_i| = \sum_{A_i \in \mathcal{A}} |B_i| \leq (t-1)|\mathcal{A}|,$$

where the last inequality follows from the definition of $\mathcal{A}$. Suppose $A_i \in \mathcal{A}$ lies in a $K_{s,t}$ in $G - E$. Then B_i contains the part of this $K_{s,t}$ in B'. However, all edges between u and B_i lie in E, so these edges are absent in $G - E$. This implies no $A_i \in \mathcal{A}$ is an edge in $\mathcal{D}_E$, and hence $|E| \leq (t-1)[e(\mathcal{D}) - e(\mathcal{D}_E)]$. This contradicts $(*)$.

Claim 3. *There exists $v \in B'$ such that $|\{B_i : v \in B_i\}| \leq |\Gamma_{\mathcal{B}}(v)|$.*

Proof. As each edge of G is in some $K_{s,t}$ in G and $\mathcal{A} = \emptyset$, $B' = \bigcup_{i\in[k]} B_i$. By Claim 1, $n(\mathcal{B}) = |B'| = \deg_G(u) > \deg_{\mathcal{D}}(u) = e(\mathcal{B})$. Applying Proposition A.1 to $\mathcal{B}$, there exists a vertex $v \in B'$ such that

$$|\{B_i : v \in B_i\}| = \deg_{\mathcal{B}}(v) < |\Gamma_{\mathcal{B}}(v)| + 1.$$

This completes the proof of Claim 3.

We now define

$$E = \{vx : x \neq u \text{ and } vx \text{ lies in a } K_{s,t} \text{ in } G \text{ with at least } s \text{ vertices in } B'\}.$$

Let $\mathcal{D}'$ be the subgraph of $\mathcal{D}$ induced by B', and let $\mathcal{D}'_E$ be the subgraph of $\mathcal{D}_E$ induced by B'. If some edge S of $\mathcal{D}'_E$ is incident with v, then there exists a $K_{s,t}$ in $G-E$ containing S. As this $K_{s,t}$ contains s vertices in B', all its edges (apart from uv) incident with v are in E. This contradiction shows v is an isolated vertex in $\mathcal{D}'_E$, and $\deg_{\mathcal{D}'}(v) \leq e(\mathcal{D}) - e(\mathcal{D}_E)$. It remains to verify that $|E| \leq (s+t-3)\deg_{\mathcal{D}'}(v)$ for E to contradict $(*)$. This fact is established in the next two claims. For each $G(A_i, B_i)$ with $v \in B_i$, select a single edge between v and A_i, distinct from uv. Let the collection of selected edges be E', and set $E'' = E - E'$.

Claim 4. $|E'| \leq (s-1)\deg_{\mathcal{D}'}(v)$.

Proof. Since $\mathcal{A} = \emptyset$, every set A_i has at least t common neighbors in B'. This implies that B_i induces a complete s-graph in $\mathcal{D}'$, for $i \in [k]$. Therefore

$$\deg_{\mathcal{D}'}(v) \geq \Big|\bigcup_{i:v\in B_i} (B_i - \{v\})^{(s-1)}\Big|.$$

By Lemma A.2, applied to the hypergraph $\mathcal{B}'$ with edge set $\bigcup_{i:v\in B_i}(B_i - \{v\})^{(s-1)}$, we find

$$|\Gamma_{\mathcal{B}}(v)| = \Big|\bigcup_{i:v\in B_i} (B_i - \{v\})\Big| \;\leq\; (s-1)\Big|\bigcup_{i:v\in B_i} (B_i - \{v\})^{(s-1)}\Big| \leq (s-1)\deg_{\mathcal{D}'}(v).$$

Consequently, by Claim 3 and the definition of E',

$$|E'| \leq |\{B_i : v \in B_i\}| \leq |\Gamma_{\mathcal{B}}(v)| \leq (s-1)\deg_{\mathcal{D}'}(v).$$

This completes the proof of Claim 4.

Claim 5. $|E''| \leq (t-2)\deg_{\mathcal{D}'}(v)$.

Proof. Suppose $|E''| > (t-2)\deg_{\mathcal{D}'}(v)$. Then there exists an edge S of $\mathcal{D}'$ incident with v and a set T of $t-1$ vertices of $A - \{u\}$ incident with all of S. However, u is also adjacent to all vertices of S so $G(S, T \cup \{u\})$ is a $K_{s,t}$ containing u. Consequently, $vx \in E'$ for some vertex $x \in T$. This contradicts $E' \cap E'' = \emptyset$, since we also have $vx \in E''$ for every $x \in T$. This completes the proof of Claim 5.

We have shown that $|E| = |E'| + |E''| \leq (s+t-3)\deg_{\mathcal{D}'}(v) \leq (s+t-3)[e(\mathcal{D}) - e(\mathcal{D}_E)]$, contradicting $(*)$. This completes the proof of the Main Lemma. □

4 Upper Bounds for $\mathrm{ex}(n, K_{2,t}^{(3)})$

In this section, we establish the upper bounds in Theorem 1.2. For integers q, s, t, we write $K_{q,s,t}^{(3)}$ for the complete 3-partite 3-graph with parts of sizes q, s, t. Our proof of Theorem 1.2 will use the counting technique from Mubayi [11] together with the Main Lemma in Section 3. This Main Lemma allows us to remove a small number of triples that destroy all copies of $K_{1,2,t}^{(3)}$ in a $K_{2,t}^{(3)}$-free triple system. An additional refinement of these ideas allows us to prove that $\mathrm{ex}(n, K_{2,2}^{(3)}) < 3\binom{n}{2} + 6n$.

It is sufficient to restrict our attention to 3-partite triple systems, in view of the following useful lemma of Erdős and Kleitman [5]:

Lemma 4.1 *Let $\mathcal{G}$ be a triple system on $3n$ vertices. Then $\mathcal{G}$ contains a 3-partite triple system, with all parts of size n, and with at least $\frac{2}{9}e(\mathcal{G})$ triples.*

Indeed, we prove the following theorem:

Theorem 4.2 *Let $t \geq 2$, $n \geq t$, and $g(t) = t - 1/2 + 2(t-1)^2\left[\binom{t-1}{2} + 1\right]$. Then*

$$\mathrm{z}(n, K_{2,t}^{(3)}) < \begin{cases} 3n^2 - \frac{3}{2}n & \text{for } t = 2 \\ g(t)n^2 & \text{for } t > 2. \end{cases}$$

Let us verify Theorem 1.2 from Theorem 4.2. By adding at most two isolated vertices to a $K_{s,t}^{(3)}$-free triple system $\mathcal{H}$ on n vertices, we obtain a triple system $\mathcal{G}$ such that $n(\mathcal{G})$ is divisible by three. Applying Lemma 4.1, we find a 3-partite triple system $\mathcal{G}'$ with at least $\frac{2}{9}e(\mathcal{G})$ edges. By Theorem 4.2 and Lemma 4.1,

$$\frac{2}{9}e(\mathcal{H}) = \frac{2}{9}e(\mathcal{G}) \leq e(\mathcal{G}') < g(t)\left(\frac{n(\mathcal{G})}{3}\right)^2 = \frac{1}{9}g(t)n(\mathcal{G})^2.$$

Consequently,

$$e(\mathcal{H}) = e(\mathcal{G}) \leq \frac{9}{2}\left[\frac{1}{9}g(t)n(\mathcal{G})^2\right] \leq \frac{1}{2}g(t)(n+2)^2 < t^4\binom{n}{2}.$$

The last inequality follows by some elementary calculations, using $t \geq 3$ and $n \geq 3t$. A similar argument applies to show that $\mathrm{ex}(n, K_{2,2}^{(3)}) < 3\binom{n}{2} + 6n$.

Before proving Theorem 4.2, we require the following definition:

Definition. Let $G_1, G_2, \dots, G_n$ be graphs on the same vertex set. Then $\sum_{i\in[n]} G_i$ denotes the multigraph in which a pair f of vertices is an edge whenever f is an edge of some G_i.

Proof of Theorem 4.2. Let $\mathcal{H}$ be a 3-partite $K_{2,t}^{(3)}$-free triple system, in which all three parts have size n. Suppose the parts of $\mathcal{H}$ are each copies of $[n]$, labeled A, B, C. For each $i \in [n]$, let $G_i = G_i(A, B)$ denote the bipartite graph with edge set

$$\Big\{(a, b) : a \in A, b \in B, (a, b, i) \in \mathcal{H}\Big\}.$$

Let $G = \sum_{i\in[n]} G_i$, $D(A) = \sum_{i\in[n]} D(G_i) \cap A$ and $D(B) = \sum_{i\in[n]} D(G_i) \cap B$, where $D(G_i) \cap A$ denotes the subgraph of $D(G_i) = \mathcal{D}_{2,t}(G_i)$ induced by A, and similarly for B.

Claim 1. *A pair of vertices $\{a, a'\}$ in $A^{(2)} \cup B^{(2)}$ forms a part of a copy of $K_{2,t}$ in G_i for at most $t-1$ integers $i \in [n]$.*

Proof. Suppose some pair $\{a, a'\} \in A^{(2)}$ forms a part of a copy of $K_{2,t}$ in G_i for at least t integers $i \in [n]$, say for $i \in [t]$. Then there exist t vertices $b_1, b_2, \dots, b_t \in B$ such that ab_ia' is a path of length two in G_i. However, the set of all edges of the form (a, b_i, i) and (a', b_i, i) forms a copy of $K_{2,t}^{(3)}$ in $\mathcal{H}$. This is a contradiction, so $\{a, a'\}$ forms a part of a copy of $K_{2,t}$ in G_i for at most $t-1$ integers $i \in [n]$.

Claim 2. *For $t \geq 3$, $D(A)$ and $D(B)$ have edge-multiplicity at most $(t-1)\binom{t-1}{2} + (t-1)$.*

Proof. Suppose, for a contradiction, that some edge $\{a, a'\}$ has edge multiplicity at least $r = (t-1)\binom{t-1}{2} + t$ in $D(A)$. Without loss of generality, we may assume $\{a, a'\}$ is an edge of $D(G_i) \cap A$ for each $i \in [r]$. By Claim 1, $\{a, a'\}$ is contained in a t-set forming a part of a $K_{2,t}$ in G_i for at least $r - t + 1$ integers $i \in [r]$, say for $i \in [r - t + 1]$. This gives a set of $r - t + 1$ edges in $D(B)$, corresponding to the part of a $K_{2,t}$ in G_i of size two, for $i \in [r - t + 1]$. These $r - t + 1$ edges together span a multigraph $M \subset D(B)$. The pairs of adjacent vertices in M form a part of a $K_{2,t}$ in different graphs G_i, for $i \in [r - t + 1]$. Now $e(M) \geq r - t + 1 = (t-1)\binom{t-1}{2} + 1$ and M has edge-multiplicity at most $t - 1$ by Claim 1. So we may apply Lemma A.4 (with $s = 2$ and $\mu = t - 1$) to M: there exist t vertices $b_1, b_2, \ldots, b_t \in B$, each incident with a different edge of M. Suppose this set of edges is $f_1, f_2, \ldots, f_t$ with $f_i \in D(G_i)$ for $i \in [t]$. Then ab_ia' is a path of length two in G_i for $i \in [t]$. But then, for each $i \in [t]$, all the edges (a, b_i, i) and (a', b_i, i) form a copy of $K_{2,t}^{(3)}$ in $\mathcal{H}$. This contradiction verifies Claim 2 for $D(A)$. Similar arguments apply to show that $D(B)$ has edge-multiplicity at most $(t-1)\binom{t-1}{2} + (t-1)$.

For each $i \in [n]$, let $G'_i = G'_i(A, C)$ denote the bipartite graph spanned by the edges (a, c), $a \in A, c \in C$, for which (a, i, c) is an edge of $\mathcal{H}$. Let $G' = \sum_{i \in [n]} G'_i$, $D'(A) = \sum_{i \in [n]} D(G'_i) \cap A$ and $D'(C) = \sum_{i \in [n]} D(G'_i) \cap C$. We note, by symmetry and applying the arguments of Claim 2, that $D'(A)$ and $D'(C)$ have edge-multiplicity at most $(t-1)\binom{t-1}{2} + (t-1)$ for $t \geq 3$.

Claim 3. *We may remove at most $4(t-1)^2[\binom{t-1}{2} + 1]\binom{n}{2}$ edges from $G \cup G'$ so that, for all $i \in [n]$, the resulting subgraph of $G \cup G'$ contains no $K_{2,t}$ in any G_i or G'_i. If $t = 2$, we may remove at most $3\binom{n}{2}$ edges from $G \cup G'$ for the same conclusions.*

Proof. Suppose $t \geq 3$. By Claim 2, no pair of vertices of A or B is an edge in more than $(t-1)\binom{t-1}{2} + (t-1)$ of the graphs $D(G_i)$. By the Main Lemma, we may delete at most $(t-1)e(D(G_i))$ edges from each G_i to obtain a $K_{2,t}$-free graph. The number of edges removed from G is therefore at most

$$(t-1)e(D(A)) + (t-1)e(D(B)) \leq 2\left[(t-1)^2\binom{t-1}{2} + (t-1)^2\right]\binom{n}{2}.$$

A similar argument applies for G', therefore the total number of edges removed is at most

$$4\left[(t-1)^2\binom{t-1}{2} + (t-1)^2\right]\binom{n}{2}.$$

Now suppose $t = 2$. We assert that $D(A) \cap D'(A) = \emptyset$ (this is the major new idea needed to improve the factor in Füredi's bound from 3.5 to 3). This suffices to prove Claim 3: as $D(A)$ has no multiple edges by Claim 1, the Main Lemma shows that the number of edges required to delete all $K_{2,2}$ in G_i and G'_i is at most

$$e(D(A)) + e(D'(A)) + e(D(B)) + e(D'(C)) \leq \binom{|A|}{2} + \binom{|B|}{2} + \binom{|C|}{2} = 3\binom{n}{2}.$$

Let us prove $D(A) \cap D'(A) = \emptyset$. Suppose, for a contradiction, that $\{a, a'\}$ is an edge of $D(G_i)$ and $D(G'_j)$, where $a, a' \in A$. Then there are edges $\{b, b'\} \in D(G_i)$ and $\{c, c'\} \in D(G'_j)$ such that $\{a, b, a', b'\}$ and $\{a, c, a', c'\}$ induce quadrilaterals in G_i and G'_j respectively. Now (a, b, i), (a, b', i), (a', b', i), (a', b, i) and (a, j, c), (a, j, c'), (a', j, c'), (a', j, c) are all edges of $\mathcal{H}$. These edges form a triple system containing a copy of $K^{(3)}_{2,2}$, a contradiction. This completes the proof of Claim 3.

We let H_G and $H_{G'}$ denote the subgraphs of G and G' obtained by removing all these edges from G and G'. For vertices x, y in a hypergraph, the *codegree* of x and y, written $\mathrm{codeg}(x, y)$ is the number of edges containing both x and y.

Claim 4. $e(H_G) < (t - 1/2)n^2$.

Proof. Suppose $e(H_G) \geq (t - 1/2)n^2$. Then the number of paths with two vertices in A and one vertex in B, contained in both H_G and some G_i, is exactly $\sum_{(b,c)\in B\times C} \binom{\mathrm{codeg}(b,c)}{2}$. As the average codegree is at least $t - 1/2$, the above expression is minimized when the codegree of half the pairs is $t - 1$, and the codegree of the other half of the pairs is t. Therefore

$$\sum_{(b,c)\in B\times C} \binom{\mathrm{codeg}(b,c)}{2} \geq \frac{1}{2}\binom{t-1}{2}n^2 + \frac{1}{2}\binom{t}{2}n^2 > (t-1)^2\binom{n}{2}.$$

This implies the existence of a set $P \subset B \times C$ of $(t-1)^2 + 1$ pairs and $a, a' \in A$ such that the triples (a, b, c) and (a', b, c) are edges of $\mathcal{H}$ whenever $(b, c) \in P$. Let G'' denote the bipartite graph on $B \cup C$ whose edges are the elements of P. By Lemma A.3, G'' contains a matching M with t edges or a star with t edges. In the former case, the set all of triples of the form (a, b, c) and of the form (a', b, c), with $\{b, c\} \in M$, form a copy of $K^{(3)}_{2,t}$ in $\mathcal{H}$, a contradiction. In the latter case, we obtain a $K_{2,t}$ in G_i or G'_j, according as the center of the star is $j \in B$, or $i \in C$. As $H_{G'}$ contains no $K_{2,t}$ in any G_i, by Claim 3, this is a contradiction. So $e(H_G) < (t-1/2)n^2$, and the proof of Claim 4 is complete.

We now complete the proof of Theorem 4.2. First suppose $t \geq 3$. Recall that $G = \sum G_i$ and H_G is the subgraph of G remaining on deleting edges from G using Claim 3. Let D denote the number of edges deleted in Claim 3. Thus, using Claims 3 and 4,

$$\begin{aligned} e(\mathcal{H}) &\leq e(H_G) + D \\ &< (t-1/2)n^2 + 4(t-1)^2\left[\binom{t-1}{2}+1\right]\binom{n}{2} \\ &< \left(t - 1/2 + 2(t-1)^2\left[\binom{t-1}{2}+1\right]\right)n^2. \end{aligned}$$

For $t = 2$, by Claims 3 and 4, we find $e(\mathcal{H}) < 3\binom{n}{2} + \frac{3}{2}n^2$. □

Remark: Corollary 1.3 can be proved in the same way as Theorem 4.2, using the generalization of Lemma 4.1 to r-partite subgraphs of r-uniform hypergraphs, due to Erdős and Kleitman [5], and using Lemma 5.2 in [4].

5 Upper Bound for $\mathrm{ex}(n, K_{s,t}^{(3)})$

In this section we prove Theorem 1.4.

Proof of Theorem 1.4: It suffices to prove that $\mathrm{z}(n, K_{s,t}^{(3)}) < c'_{s,t}n^{3-1/s}$. Let A, B, C be the three parts of size n of a 3-partite $K_{s,t}^{(3)}$-free triple-system $\mathcal{H}$. Suppose that $\mathcal{H}$ has more than $c'_{s,t}n^{3-1/s}$ triples, where $c'_{s,t}$ is defined as the smallest integer for which every bipartite graph with parts X and Y of size n with more than $c'_{s,t}n^{2-1/s}$ edges contains a $K_{s,t}$ with t vertices in X and s vertices in Y. Note that $c'_{s,t}$ is independent of n, since the number of edges between X and Y must satisfy

$$\sum_{v \in Y} \binom{d(v)}{s} \leq (t-1)\binom{|X|}{s}.$$

Partition the elements of $A \times B$ into n matchings $M_1, \ldots, M_n$, and let $\mathcal{H}_i$ be the subhypergraph of $\mathcal{H}$ induced by those edges that contain some pair of M_i. By the pigeonhole principle, $\mathcal{H}_i$ has more than $c'_{s,t}n^{2-1/s}$ edges for some i. Let G_i be the graph on vertex set $B \cup C$, with edge set $\{(b,c) : \exists a, (a,b,c) \in E(\mathcal{H}_i)\}$. Then by the choice of $c'_{s,t}$, we conclude that G_i contains a copy of $K_{s,t}$ with t vertices in B and s vertices in C, which extends via M_i to a $K_{s,t}^{(3)}$ in $\mathcal{H}$. This contradiction proves $\mathrm{z}(n, K_{s,t}^{(3)}) < c'_{s,t} n^{3-1/s}$. □

6 Upper Bound for $\mathrm{ex}(n, K_{3,t}^{(3)})$

We will use the techniques of Section 4 to prove Theorem 1.5. Because our bounds should be thought of as asymptotic results, we omit ceiling and floor symbols in this section. As in Section 4, we use Lemma 4.1 to obtain Theorem 1.5 from the following theorem:

Theorem 6.1 *For $n \geq t$,*

$$\mathrm{z}(n, K_{3,t}^{(3)}) < t^5 n^{13/5}.$$

Proof. Let $\mathcal{H}$ be a 3-partite triple system, each part of which is a copy of $[n]$, labeled A, B, C. As in the proof of Theorem 4.2, define the graphs $G_i = G_i(A,B)$, $G = G(A,B)$, $G'_i = G'_i(A,C)$ and $G' = G'(A,C)$. Partition A, B, and C into $m = n^{2/5}$ disjoint sets $A_1, A_2, \ldots, A_m$, $B_1, B_2, \ldots, B_m$, and $C_1, \ldots, C_m$ respectively, so that all sets have size $n/m = n^{3/5}$. We define $\mathcal{D}(A_i)$ to be the 3-graph (with multiple edges) on A_i such that $S \subset A_i$ is an edge of $\mathcal{D}(A_i)$ whenever S is an edge of $\mathcal{D}_{3,t}(G_j) = \mathcal{D}(G_j)$, for each $j \in [n]$. We define $\mathcal{D}(B_i)$ on vertex set B_i similarly. The first claim is similar to Claim 1 in the proof of Theorem 4.2:

Claim 1. *Any member of $A^{(3)} \cup B^{(3)}$ forms a part of a copy of a $K_{3,t}$ in G_i for at most $t-1$ integers $i \in [n]$.*

By using Lemma A.4 in the appendix, we prove the following Claim in a similar way to Claim 2 in Theorem 4.2. The case $t = 3$ also follows in this way.

Claim 2. *For $t \geq 3$ and $i \in [m]$, $\mathcal{D}(A_i)$ and $\mathcal{D}(B_i)$ have edge-multiplicity at most $\frac{1}{2}(t-1)^4$.*

Claim 3. *We may remove at most $2t^5 m^2 \binom{n/m}{3}$ edges from $G \cup G'$ so that, for all $i \in [n]$ and $j,k,l \in [m]$, the resulting graph contains no copy of $K_{s,t}$ in $G_i(A_j, B_k)$ or $G'_i(A_j, C_l)$.*

Proof. Fix $j,k \in [m]$ and let $H_i = G_i(A_j, B_k)$. By the Main Lemma, we may remove at most $te(\mathcal{D}_{3,t}(H_i))$ edges from H_i to obtain a $K_{3,t}$-free graph. As $\mathcal{D}(A_j)$ and $\mathcal{D}(B_k)$ have edge-multiplicity at most $\frac{1}{2}(t-1)^4$, the number of edges which may be removed to delete all copies of $K_{3,t}$ in the graphs H_i is at most

$$\sum_{i\in[n]} te(\mathcal{D}_{3,t}(H_i)) \le t \cdot \frac{1}{2}(t-1)^4 2\binom{n/m}{3} < t^5\binom{n/m}{3}.$$

Repeating this argument for all pairs $j,k \in [m]$, the number of edges removed from G is at most $t^5m^2\binom{n/m}{3}$. Applying a similar argument to G' gives the same result for G'. The total number of edges removed from $G \cup G'$ is therefore at most $2t^5m^2\binom{n/m}{3}$, completing the proof of Claim 3.

We let H_G and $H_{G'}$ denote the subgraphs of G and G' remaining after deleting the edges from G and G' in the application of Claim 3.

Claim 4. $e(H_G) < 3t^{2/3}n^{7/3}m^{2/3}$.

Proof. It is sufficient to prove that $e_{jk} = e(H_G(A_j, B_k)) \le 3t^{2/3}n^{7/3}m^{-4/3}$ for all $j,k \in [m]$. Suppose, for a contradiction, that $e_{jk} > 3t^{2/3}n^{7/3}m^{-4/3}$. The average codegree of pairs (b,c) in $B_k \times C$ is then at least

$$\frac{1}{|B_k||C|} \cdot \frac{3t^{2/3}n^{7/3}}{m^{4/3}} \ge \frac{3t^{2/3}n^{1/3}}{m^{1/3}}.$$

Consequently, the inequalities $a^b/b^b \le \binom{a}{b} < (3a)^b/b^b$, and convexity of binomial coefficients yield,

$$\sum_{(b,c)\in B_k\times C} \binom{\mathrm{codeg}(b,c)}{3} \ge \frac{n^2}{m} \cdot \binom{3t^{2/3}n^{1/3}m^{-1/3}}{3} > t^2\frac{n^3}{m^2} = t^2m\frac{n^3}{m^3} > t^2m\binom{n/m}{3}.$$

This implies the existence of a set $P \subset B_k \times C$ of t^2m pairs and vertices $a_1, a_2, a_3 \in A_j$ such that the triples (a_i, b, c) are edges of $\mathcal{H}$ whenever $(b,c) \in P$ and $i \in [3]$. Let G'' denote the bipartite graph on $B_k \cup C$ whose edges are the elements of P. By Lemma A.3, G'' contains a matching M with t edges or a star with tm edges. In the former case, the set of all triples (a_i, b, c) with $(b,c) \in M$ and $i \in [3]$ form a copy of $K_{3,t}^{(3)}$ in G. In the latter case, depending on where the center of the star lies, we obtain either

(i) a copy of $K_{3,mt}$ in $G_i(A_j, B_k)$ entirely contained H_G, or

(ii) a copy of $K_{3,mt}$ in $G'_i(A_j, C)$ entirely contained in $H_{G'}$.

In case (ii), the pigeonhole principle implies that for some $l \in [m]$, there is

(ii') a copy of $K_{3,t}$ in $G'_i(A_j, C_l)$ entirely contained in $H_{G'}$.

Both (i) and (ii') give contradictions, as all such subgraphs were removed from $G \cup G'$ in the application of Claim 3. Therefore $e_{jk} < 3t^{2/3}n^{7/3}m^{-4/3}$, as required. This completes the proof of Claim 4.

We now count the number of edges in $\mathcal{H}$. Recalling that $m = n^{2/5}$, this number is at most

$$e(H_G) + 2t^5m^2\binom{n/m}{3} < 3t^{2/3}n^{7/3}m^{2/3} + 2t^5m^2\frac{n^3}{6m^3} < 3t^{2/3}n^{39/15} + \frac{1}{3}t^5n^{3-2/5} < t^5n^{13/5}.$$

This completes the proof of Theorem 1.5. □

Remarks:

• Since we believe that the exponent 13/5 in Theorem 1.5 can be improved to 7/3, we have made no attempt to optimize the constants in the proof above.

• This approach gives the upper bound $\mathrm{ex}(n, K_{s,t}^{(3)}) < c_{s,t}n^{3-2/[(s-1)^2+1]}$ for all $s \geq 3$, but the bound $\mathrm{ex}(n, K_{s,t}^{(3)}) < c'_{s,t}n^{3-1/s}$ in Theorem 1.4 is better for $s > 3$.

Appendix

The following two results are due to Füredi (Lemma 3.1 in [4]):

Proposition A.1 *Let $\mathcal{B}$ be a hypergraph, possibly with multiple edges, in which* $\deg_{\mathcal{B}}(v) > |\Gamma_{\mathcal{B}}(v)|$ *for every vertex v. Then $e(\mathcal{B}) \geq n(\mathcal{B})$.*

Proof. We prove this by induction on $n(\mathcal{B})$. Let v be any vertex of $\mathcal{B}$, and let V be the vertex set of $\mathcal{B}$. Let $\mathcal{B}'$ be the hypergraph with vertex set $V' = V - \Gamma_{\mathcal{B}}(v) - \{v\}$ and edge set $\{B \cap V' : B \in E(\mathcal{B}), B \cap V' \neq \emptyset\}$. Then, by hypothesis and induction applied to $\mathcal{B}'$,

$$e(\mathcal{B}) \geq \deg_{\mathcal{B}}(v) + e(\mathcal{B}') > \deg_{\mathcal{B}}(v) + n(\mathcal{B}') = n(\mathcal{B}) - 1.$$

This completes the proof. □

Lemma A.2 *Let $s \geq 2$, and let $B_1, B_2, \ldots, B_k$ be sets of size at least s. Then*

$$|\textstyle\bigcup_{i\in[k]} B_i^{(s)}| \geq \frac{1}{s}|\bigcup_{i\in[k]} B_i|.$$

Proof. Form a hypergraph $\mathcal{B}$ with vertex set $\bigcup_{i\in[k]} B_i$ and edge set $\bigcup_{i\in[k]} B_i^{(s)}$. Then, as $|B_i| \geq s$ for $i \in [k]$, every vertex in $\mathcal{B}$ has degree at least one, so we have $n(\mathcal{B}) = |\bigcup_{i\in[k]} B_i| \leq \sum_{v\in\mathcal{B}} \deg_{\mathcal{B}}(v) = s \cdot e(\mathcal{B})$. □

Lemma A.3 *Let G be a simple bipartite graph with at least $(m-1)(s-1)+1$ edges. Then G contains a matching with m edges or a star with s edges.*

Proof. If every vertex of G has degree less than s, then we require at least m vertices to cover all the edges of G. Hence, by the König-Egerváry Theorem (see [13], page 112), G has a matching with at least m edges. □

Lemma A.4 *Let $t > s \geq 2$, $\mu \geq s$, and let $\mathcal{G}$ be an s-graph with $e(\mathcal{G}) \geq \mu\binom{t-1}{s} + 1$ and edge-multiplicity at most μ. Then $\mathcal{G}$ contains t vertices, each incident with a different edge of $\mathcal{G}$. If $s = t$, then the same conclusion holds as long as $e(\mathcal{G}) \geq s$.*

Proof. The case $s = t$ is trivial, so we focus on $t > s$. Fixing $\mu \geq s \geq 2$, we will prove the lemma by induction on $t > s$. We may assume $e(\mathcal{G}) = \mu\binom{t-1}{s} + 1$ and $\mathcal{G}$ contains no isolated vertices. Let G be the bipartite graph whose parts are the vertex set A of $\mathcal{G}$ and the edge set B of $\mathcal{G}$ and a vertex of $\mathcal{G}$ is joined to all the edges of $\mathcal{G}$ containing it. Thus every vertex in B has degree s. Therefore $|\Gamma_G(X)| \geq |X|$ for all $X \subset B$ with $|X| \leq s$.

Suppose that $t = s+1$. If $|\Gamma_G(X)| \geq s+1$ for some $X \subset B$ with $|X| = s+1$, then we can apply Hall's Theorem to the bipartite graph induced by $X \cup \Gamma_G(X)$. This gives $s+1 = t$ elements in B matched to t elements in A. However, such an X exists since $e(\mathcal{G}) \geq \mu + 1$, implies that $|\Gamma(B)| \geq s+1$, and this yields a set $X' \subset B$ with $|X'| = 2$ and $|\Gamma(X')| \geq s+1$. Now X' can be extended to a set X as required. We may therefore assume that $t \geq s+2$.

If some vertex v of $\mathcal{G}$ has degree at most $\mu\binom{t-2}{s-1}$, then we remove v from $\mathcal{G}$ to obtain a graph $\mathcal{G}'$ with at least $\mu\binom{t-2}{s} + 1$ edges. By induction, there exists a set $\mathcal{E}$ of $t-1$ edges in $\mathcal{G}'$ and a set S of $t-1$ vertices, each incident with a different edge in $\mathcal{E}$. As v is not an isolated vertex in $\mathcal{G}$, we may select any edge incident with v, distinct from any edge in $\mathcal{E}$. Then $S \cup \{v\}$ is the required set of t vertices of $\mathcal{G}$. We therefore suppose $\mathcal{G}$ contains no vertex of degree at most $\mu\binom{t-2}{s-1}$.

This implies that for any $X \subset A$, $s|\Gamma(X)| \geq \mu\binom{t-2}{s-1}|X| \geq s|X|$. By Hall's Theorem, we may find a matching of A into B. As $e(\mathcal{G}) > \mu\binom{t-1}{s}$ and $\mathcal{G}$ has edge-multiplicity at most

μ, $\mathcal{G}$ has at least t vertices. Therefore the vertex set of $\mathcal{G}$ satisfies the requirements of the lemma when $t \geq s+2$. □

This lemma is best possible as shown by the complete s-graph on $t-1$ vertices with every edge repeated μ times.

7 Acknowledgments

We are very thankful to Z. Füredi for helpful comments and to the referee for carefully reading and correcting errors in an earlier version of this manuscript. We would like to thank Microsoft research, where this work was done while both authors were postdocs.

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