ON INDEPENDENT SETS IN HYPERGRAPHS

A. KOSTOCHKA, D. MUBAYI, AND J. VERSTRAËTE

Abstract. The *independence number* $\alpha(H)$ of a hypergraph H is the size of a largest set of vertices containing no edge of H. In this paper, we prove that if H_n is an n-vertex $(r+1)$-uniform hypergraph in which every r-element set is contained in at most d edges, where $0 < d < n/(\log n)^{3r^2}$, then

$$\alpha(H_n) \geq c_r \Big(\frac{n}{d} \log \frac{n}{d}\Big)^{1/r}$$

where $c_r > 0$ satisfies $c_r \sim r/e$ as $r \to \infty$. The value of c_r improves and generalizes several earlier results that all use a theorem of Ajtai, Komlós, Pintz, Spencer and Szemerédi [2]. Our relatively short proof extends a method due to Shearer [22] and Alon [1].

The above statement is close to best possible, in the sense that for each $r \geq 2$ and all values of $d \in \mathbb{N}$, there are infinitely many H_n such that

$$\alpha(H_n) \leq b_r \Big(\frac{n}{d} \log \frac{n}{d}\Big)^{1/r}$$

where $b_r > 0$ depends only on r. In addition, for many values of d we show $b_r \sim c_r$ as $r \to \infty$, so the result is almost sharp for large r. We give an application to hypergraph Ramsey numbers involving independent neighborhoods.

1. Introduction

In this paper, an *r-graph* is a set of r-element subsets of a finite set, where the sets are called edges and the elements of the finite set are called vertices. An independent set in an r-graph is a set of vertices containing no edge. The independence number $\alpha(H)$ of an r-graph H is the maximum size of an independent set in H.

A *partial Steiner $(n, r+1, r)$-system* is an n-vertex $(r+1)$-graph such that each r-element set of vertices is contained in at most one edge. The

---

A.K. was supported by NSF grant DMS-0965587 and by grant 09-01-00244-a of the Russian Foundation for Basic Research.

D.M. was supported in part by NSF grants DMS 0653946 and DMS 0969092.

J.V. was supported by NSF grant DMS-0800704.

This research originated when the authors participated in the AIM workshop *Hypergraph Turán Problem*, March 21-25 2011, Palo Alto, CA.

maximum r-degree of an $(r+1)$-graph H is the maximum number of edges that any r-set of vertices is contained in.

The independence number $\alpha(H)$ has been studied at length in Steiner systems, sometimes in the language of projective geometry, in terms of *maximum complete arcs*, and has applications to geometric problems, for instance the "orchard planting problem" (see [12, 13]) or Heilbronn's celebrated triangle problem [17]. Given a partial Steiner $(n, r+1, r)$-system H, Phelps and Rödl [19] were the first to show $\alpha(H) > c(n \log n)^{1/r}$ for some constant $c > 0$ depending only on r, answering a question of Erdős [9]. Rödl and Šinajová [21] proved that this result is tight, apart from the constant c.

One of the methods for finding large independent sets is the randomized greedy approach: one iteratively picks a small set of independent vertices, deletes the neighbors of this set and controls the statistics of the remaining hypergraph. The paper of Ajtai, Komlós, Pintz, Spencer and Szemerédi [2] gives a detailed analysis of such an algorithm for finding independent sets in r-graphs. This approach has been used successfully to attack the corresponding coloring problems for hypergraphs (see [5, 10, 11]).

1.1. **Main Theorem.** In this paper, we give a short proof of a general result for $(r+1)$-graphs with maximum r-degree d. This extends the aforementioned result of Phelps and Rödl, which is the case $d = 1$, without a randomized greedy approach. Shearer [22] gave an ingenious short proof that improved the lower bound of Ajtai, Komlós and Szemerédi [3] on the independence number of a K_r-free graph in terms of the number of vertices and average degree (this was later refined by Alon [1]). Shearer asked whether his method could be applied to the hypergraph setting, and we partially answer this question by proving our main result using his approach:

Theorem 1. *Fix $r \geq 2$. There exists $c_r > 0$ such that if H is an $(r+1)$-graph on n vertices with maximum r-degree $d < n/(\log n)^{3r^2}$, then*

$$\alpha(H) \geq c_r \Big(\frac{n}{d} \log \frac{n}{d}\Big)^{\frac{1}{r}}$$

where $c_r > 0$ and $c_r \sim r/e$ as $r \to \infty$.

Remarks.

(1) In the paper of Duke, Lefmann and Rödl (see [7], Theorem 3) a lower bound of the same order of magnitude as in Theorem 1 is given. However, our contribution in Theorem 1 is that the proof is much shorter, as it does not rely on the seminal result of Ajtai, Komlos, Pintz, Spencer and Szemeredi [2] which has a very lengthy and technical proof, or the more recent version in

[10] which has an even longer proof. Instead, we adapt the method of Shearer and combine it with the very elegant idea in [7] of randomly sampling vertices to give a short proof of Theorem 1.

(2) In the well-studied case of Steiner systems – namely $d = 1$, our proof gives substantially better constants than those in [7]. Due to the constants in [2] (see [2] page 334), the results of [7] give, for Steiner triple systems, $\alpha(H) \geq a\sqrt{n \log n}$ where $a = \frac{0.98}{10^{5/2}e} \approx 0.001$, as compared to our bound of approximately $0.417\sqrt{n \log n}$, which is itself not too far from our constructions provided in Section 3.

(3) In fact in Section 3 we shall see that in addition if $\log d = o(\log n)$ and $d/\log n \to \infty$, then there is an $(r+1)$-graph H on n vertices with maximum r-degree d and

$$\alpha(H) \leq b_r \left(\frac{n}{d} \log \frac{n}{d}\right)^{\frac{1}{r}}$$

where $b_r \sim c_r \sim r/e$. In this sense, as $r \to \infty$, Theorem 1 is best possible including the value of the constant c_r. An earlier upper bound of $4\sqrt{n \log n}$ on the independence number of n-vertex Steiner triple systems was given by Phelps and Rödl [19] and generalized to Steiner (n, r, k)-systems by Rödl and Šinajová [21].

(4) Finally, we also expect the method might extend to F-free hypergraphs as in the case of Shearer's proof for graphs. This is a wide open problem raised in [10] in the context of the chromatic number of F-free hypergraphs.

1.2. **Independent neighborhoods.** An r-graph H is said to have *independent neighborhoods* if for every set R of $r-1$ vertices, $\{e \backslash R : R \subset e \in H\}$ is an independent set. These hypergraphs have been studied from the point of view of extremal hypergraph theory [14, 15] and hypergraph coloring [5]. Denote by T_r the r-graph with vertex set $R \cup S$ with $|R| = r$ and $|S| = r-1$ and consisting of all edges containing S together with the edge R. Then an r-graph has independent neighborhoods if and only if it does not contain T_r as a subgraph. The *Ramsey number* $R(T_r, K_t^{(r)})$ is the minimum N such that in every red-blue coloring of the edges of the complete r-graph $K_N^{(r)}$ on N vertices, there is either a red T_r or a blue $K_t^{(r)}$. As a straightforward consequence of Theorem 1, we obtain the following result:

Theorem 2. *Let H be an r-graph on n vertices with independent neighborhoods. Then for some constant c, $\alpha(H) \geq c(n \log n)^{\frac{1}{r}}$. In particular,*

$$R(T_r, K_t^{(r)}) = O\Big(\frac{t^r}{\log t}\Big).$$

We believe that the Ramsey result is best possible up to the value of the implicit constant. In the case $r = 2$, for graphs, a graph has independent neighborhoods if and only if it is triangle-free. Theorem 2 therefore generalizes the well-known result of Ajtai, Komlós and Szemerédi [3] for triangle-free graphs to hypergraphs. It remains an open problem to show that Theorem 2 is best possible for all r. It is known to be best possible for graphs by a result of Kim [16] which establishes that $R(K_3, K_t^{(2)})$ has order of magnitude $t^2/(\log t)$.

1.3. **Organization.** This paper is organized as follows: we start with stating the Chernoff Bound in Section 2, which will be used repeatedly in the probabilistic methods to follow. In Section 3, we give the constructions which prove that Theorem 1 is tight up to the constant c_r. In Section 4 we will sketch the proof for the case $r = 2$ – the interested reader might want to read this section first to see the main ideas. In Sections 5 and 6, we establish some preliminaries for the proof of Theorem 1, which is in Section 7. In Section 8 we give an application to Ramsey numbers and hypergraphs with independent neighborhoods. We end with some concluding remarks.

1.4. **Notation.** A hypergraph H is a pair $(V(H), E(H))$ where $E(H) \subset 2^{V(H)}$; it is an r-graph if $E(H) \subset \binom{V(H)}{r}$. Sometimes we will abuse notation by associating H with its edge set $E(H)$. A *subgraph* or *subhypergraph* of a hypergraph $H = (V, E)$ is a hypergraph $H' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. For $X \subset V$, the subgraph of H *induced* by X is the subgraph $H[X]$ consisting of all edges of H that are contained in X. A *triangle* in an r-graph H is a subgraph of three edges $\{e, f, g\}$ such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and all the intersections are distinct. A hypergraph is *linear* if it has no pair of distinct edges sharing two or more vertices. A set $Z \subseteq V$ is an *independent set* of H if Z contains no edges of H. Two vertices of H are *adjacent* if they are contained in a common edge of H. Let $N(x)$ denote the set of vertices adjacent to $x \in V(H)$.

All logarithms in this paper are to the natural base, e. We write $f(n) \sim g(n)$ or $f(n) = (1+o(1))g(n)$ for functions $f, g : \mathbb{N} \to \mathbb{R}^+$ to denote $f(n)/g(n) \to 1$

as $n \to \infty$, and $f(n) = O(g(n))$ to denote that there is a constant c such that $f(n) \le cg(n)$ for all n. We also write $f(n) \lesssim g(n)$ if $\limsup f(n)/g(n) \le 1$ as $n \to \infty$. Similarly, $f(n) = o(g(n))$ means that $\lim f(n)/g(n) = 0$. Unless otherwise indicated, any asymptotic notation implicitly assumes $n \to \infty$.

2. Chernoff-type bounds

In this section we state the concentration inequalities that will be used in the paper. Throughout, $U \sim \text{binomial}(n,p)$ means U is a binomial random variable with success probability p in n trials, and if $(A_n)_{n\in\mathbb{N}}$ is a sequence of events in some probability space, then we say A_n occurs *with high probability* if $\lim_{n\to\infty} P(A_n) = 1$.

The following lemma is a generalization of the Chernoff Bound (see McDiarmid [18] Theorem 2.7 for Part 1, and Theorem 2.3 for Part 2).

Lemma 3. *Let U be a sum of independent random variables $U_1, U_2, \ldots, U_n$ such that $E(U) = \mu$ and $U_i \le E(U_i) + b$ for all i. Let V be the variance of U. Then for any $\lambda > 0$*

1. $P(U \ge \mu + \lambda) \le e^{-\frac{\lambda^2}{2V+b\lambda}}$.
2. *If $U \sim$ binomial(n,p), then* $P(|U-\mu| \ge \varepsilon\mu) \le 2e^{-\frac{\varepsilon^2\mu}{3}}$.

The inequality in Lemma 3 Part 2 will be referred to as the Chernoff Bound [6].

2.1. A technical lemma.

In the proof of Theorem 1, we require the following consequence of the Chernoff Bound:

Lemma 4. *Let $q \in (0,1]$. For $b, k \in \mathbb{N}$, define*

$$S := \sum_{j=0}^{k} \binom{k}{j} q^j (1-q)^{k-j} \min\{j, b\}. \tag{1}$$

Then as $k \to \infty$,

$$S \sim \min\{qk, b\}. \tag{2}$$

Proof. Let $Y \sim \text{binomial}(k, q)$. Note that $S \leq b$ since $\min\{j, b\} \leq b$, and $S \leq E(Y) = qk$ since $\min\{j, b\} \leq j$. By the definition of Y,

$$\begin{aligned} S = \sum_{j=0}^{k} P(Y = j) \min\{j, b\} &= E(\min\{Y, b\}) \\ &= \sum_{y<b} yP(Y = y) + bP(Y \geq b) \\ &= \sum_{y<b} P(y < Y < b) + bP(Y \geq b). \end{aligned}$$

Since we have already observed that $|S| \leq \min\{qk, b\}$, it suffices to show that $|S| \gtrsim \min\{qk, b\}$. Pick $\varepsilon > 0$ and let us show that $|S| \geq (1 - 3\varepsilon) \min\{qk, b\}$ for k large. First suppose that $b \geq (1 - \varepsilon)qk$. For $y \leq (1 - 2\varepsilon)qk$, Lemma 3 Part 2 gives $P(Y \leq y) < \varepsilon$ for k large so

$$P(y < Y < b) = P(Y < b) - P(Y \leq y) > P(Y < b) - \varepsilon.$$

Therefore

$$\begin{aligned} S &= \sum_{y<b} P(y < Y < b) + bP(Y \geq b) \\ &\geq \sum_{y \leq (1-2\varepsilon)qk} P(y < Y < b) + bP(Y \geq b) \\ &\geq (1 - 2\varepsilon)qk((P(Y < b) - \varepsilon) + bP(Y \geq b) \\ &> (1 - 2\varepsilon)qk(P(Y < b) + P(Y \geq b)) - \varepsilon qk \\ &= (1 - 2\varepsilon)qk - \varepsilon qk \\ &= (1 - 3\varepsilon)qk \\ &\geq (1 - 3\varepsilon) \min\{qk, b\}. \end{aligned}$$

Next suppose that $b < (1-\varepsilon)qk$. Then by Lemma 3 Part 2, $P(Y \geq b) > 1-\varepsilon$ for k sufficiently large, and again

$$S \geq bP(Y \geq b) > (1 - \varepsilon)b > (1 - 3\varepsilon) \min\{qk, b\}.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $S \sim \min\{qk, b\}$. □

3. Hypergraphs with low independence numbers

We show that Theorem 1 is tight for all $d \in \mathbb{N}$ up to the value of the constant c_r, using a "blowup" of a Steiner system. Furthermore, for many values of

d and large r, we shall see via a random hypergraph construction that the constant c_r is itself almost best possible.

3.1. **Blowup of a Steiner system.** Let S_n be any Steiner $(n, r+1, r)$-system with $V(S_n) = \{1, 2, \dots, n\}$, where $r \geq 2$. Define an $(r+1)$-graph $H = (V, E)$ with $N = dn$ vertices and with maximum r-degree d as follows: let V be a disjoint union of sets $V_1, V_2, \dots, V_n$ each of size d. For each edge $e = \{x_1, \dots, x_r, x_{r+1}\} \in S_n$ let B_e be the collection of all edges of the form $\{v_1, \dots, v_r, v_{r+1}\}$ where $v_i \in V_{x_i}$. Let E comprise all $(r+1)$-sets in each V_i together with all edges in each B_e. Note that every edge $e \in H$ has the property that either $e \subset V_i$ for some i or $|e \cap V_i| = 1$ for exactly $r+1$ values of i. We may refer to H loosely as a blowup of a Steiner system. We observe that $\alpha(H) = r\alpha(S_n)$ since every independent set X of H contains at most r vertices in each V_i, and $\{i : |X \cap V_i| \neq \emptyset\}$ is an independent set of S_n. It is known that there are Steiner $(n, r+1, r)$-systems S_n in which $\alpha(S_n) \leq a_r(n \log n)^{1/r}$ for some $a_r > 0$ depending only on r – see [19, 21]. Therefore blowing up these Steiner systems, we obtain $(r+1)$-graphs H with N vertices and maximum r-degree d such that

$$\begin{aligned}\alpha(H) &= r\alpha(S_n) \\ &\leq ra_r\Big(\frac{N}{d}\log\frac{N}{d}\Big)^{1/r} \\ &= b_r\Big(\frac{N}{d}\log\frac{N}{d}\Big)^{1/r}\end{aligned}$$

where $b_r > 0$ depends only on r. This shows Theorem 1 is tight up to the constant c_r.

3.2. **Random hypergraphs.** A natural candidate for an $(r+1)$-graph with low independence number is the random $(r+1)$-graph $H = H_{n,r+1,p}$. This probability space is defined by selecting randomly and independently with probability p edges of the complete $(r+1)$-uniform hypergraph on n vertices, and letting H be the $(r+1)$-graph of selected edges. We sketch a standard argument showing that a random hypergraph gives good examples of a hypergraph with low independence number. We take $p = d/(n-r)$, so that the expected r-degree of any r-element set in $V(H)$ is exactly d. By the Chernoff Bound, Lemma 3, part 2, if $\log n = o(d)$, then with high probability, every r-set in H has r-degree asymptotic to d. Next, using the bounds $(1-p)^y \leq e^{-py}$ for $p \in [0,1]$ and $y \geq 0$ and $(a-b+1)^b/b! \leq \binom{a}{b} \leq a^b$ for $a \geq b \geq 1$, the expected number of independent sets of size x in H is exactly

$$E := \binom{n}{x}(1-p)^{\binom{x}{r+1}} < \exp\Big(x\log n - \frac{d}{n}\cdot\frac{(x-r)^{r+1}}{(r+1)!}\Big).$$

Fix $\varepsilon > 0$ and let

$$x = (1+\varepsilon)(r+1)!^{1/r}\left(\frac{n}{d}\log n\right)^{1/r}.$$

Then, as $n \to \infty$, we see that $(x-r)^{r+1} \geq (1-\epsilon/2)0.5x^{r+1}$ and so

$$\frac{d}{n}\frac{(x-r)^{r+1}}{(r+1)!} > x\log n$$

and therefore $E < 1$. We conclude that with positive probability, $\alpha(H) < x$ and consequently,

$$\alpha(H) \lesssim (r+1)!^{1/r}\Big(\frac{n}{d}\log n\Big)^{1/r}$$

as required. If, in addition, $\log d = o(\log n)$, then $\log\frac{n}{d} \sim \log n$ and so

$$\alpha(H) \lesssim (r+1)!^{1/r}\Big(\frac{n}{d}\log\frac{n}{d}\Big)^{1/r}.$$

Note that $(r+1)!^{1/r} \sim r/e \sim c_r$ showing that Theorem 1 provides close to the right constant for large r.

4. Sketch Proof of Theorem 1

We outline the proof of Theorem 1 for linear triple systems – that is when $r = 2$ and $d = 1$ – since the general proof requires only slight modifications of the ideas in this case. For a contradiction, suppose there are n-vertex linear triple systems H such that $\alpha(H) = o(\sqrt{n\log n})$.

4.1. **Step 1 : Random sets.**

A *random set* is a set $X \subset V(H)$ whose vertices are chosen independently from H with probability

$$p = \frac{n^{-2/5}}{(\log\log\log n)^{3/5}}.$$

Then $E(|X|) = pn$ and $E(|T|) \leq p^6\binom{n}{3}$ where $T = T(X)$ is the set of triangles in $H[X]$. The second bound holds since a triangle is uniquely determined by the three vertices which are the pairwise intersections of its edges, since H is linear. The choice of p ensures $E(|T|) = o(pn)$. Let $b = \log n$, and for an independent set $Z \subset V(H)$ and $x \in X$, let

$$\omega_Z(x,b) = \min(\,b,\ |\{xyz \in E(H) : \{y,z\} \subset Z\}|\,).$$

Define

$$h_Z(X) = \sum_{x \in X \backslash Z} \omega_Z(x, b).$$

Since H is linear, each $\{y, z\} \subset Z$ accounts for at most one such triple $\{x, y, z\}$ and $x \in X$ with probability p, so

$$E(h_Z(X)) \leq p\binom{|Z|}{2} \leq p\alpha(H)^2 = o(pn \log n).$$

We use Lemma 3 – details are given in Section 6 – to show that X can be chosen so that

(1) $h_Z(X) = o(pn \log n)$ for all independent sets Z in H,

(2) $|X| \sim pn$ and

(3) $T(X) = 0$.

Henceforth, fix such a subset X and work in $H[X]$.

4.2. **Step 2 : Random weights.** Let Z be a *randomly and uniformly chosen independent set* in $H[X]$ and define for $x \in X$ the random variable

$$W_x = \begin{cases} p\sqrt{n} & \text{if } x \in Z \\ \omega_Z(x, b) & \text{if } x \in X \backslash Z \end{cases}$$

We bound the expected value of $W := \sum_{x \in X} W_x$ in two ways.

4.3. **Step 3 : Upper bound for random weights.** By definition we have $W = p\sqrt{n}|Z| + h_Z(X)$. The choice of X in Step 1 ensures that

$$W \leq p\sqrt{n}\alpha(H) + o(pn \log n) = o(pn \log n)$$

so $E(W) = o(pn \log n)$.

4.4. **Step 4 : Lower bound for random weights.** Fixing an $x \in X$, we condition on the value of $Z_x = Z \backslash (N(x) \cup \{x\})$. Fixing Z_x, let J be the set of vertices $v \in N(x)$ such that $Z_x \cup \{v\}$ is an independent set in $H[X]$. Since $H[X]$ is triangle-free and linear, no edge of $H[X]$ has two vertices in $N(x)$ except the edges on x. Therefore, for any independent set I in $H[J \cup \{x\}]$, $I \cup Z_x$ is an independent set. Furthermore, conditioning on x and Z_x, Z is a uniformly chosen independent set of the form $I \cup Z_x$. Let M be the set of pairs of vertices of J forming an edge with x and L be the set of vertices in J not in any pair in M, and let $\ell = |L|$. If $|M| = k$, then there are $4^k + 3^k$ independent sets in $H[\bigcup M \cup \{x\}]$ – those not containing x plus those containing x. There are 2^ℓ choices for $Z \cap L$ since any choice of vertices of L together with Z_x forms an independent set. If j pairs from M are contained in the independent set, then $\omega_Z(x, b) = \min\{j, \log n\}$. There are $\binom{k}{j}$ ways to pick those pairs from M, and then 3^{k-j} ways to pick vertices in the independent set from the remaining $k - j$ pairs in M without picking both vertices from any of those pairs. Therefore, by the definition of W_x,

$$E(W_x|Z_x) = \frac{2^\ell 3^k p\sqrt{n} + 2^\ell \sum_{j=0}^{k} \binom{k}{j} 3^{k-j} \min\{j, \log n\}}{2^\ell(3^k + 4^k)}.$$

Using Lemma 4, with $q = 1/4$, the sum is asymptotic to $\min\{k4^{k-1}, 4^k \log n\}$ if $k \to \infty$. By the choice of p, a calculation shows the minimum value of the right hand side is of order $\log n$ – see Section 5 for details. So for every $x \in X$, $E(W_x|Z_x) = \Omega(\log n)$. Therefore by the tower property,

$$E(W) = \sum_{x \in X} E(W_x) = \sum_{x \in X} E(E(W_x|Z_x)) = \Omega(pn \log n).$$

This contradicts the upper bound in Step 3, and completes the proof.

5. An inequality on independent sets

It will be shown that if H is an $(r+1)$-graph of maximum r-degree d, then H has a large induced linear triangle-free $(r+1)$-graph, and this $(r+1)$-graph will contain an independent set of the size stated in Theorem 1. In this section, we prove a general inequality for independent sets in linear triangle-free $(r+1)$-graphs. Let H be a linear triangle-free $(r+1)$-graph with m vertices and let X be a subset of $V = V(H)$. Let $\mathcal{Z}$ be the set of all independent sets of H. The key quantity we wish to control is defined as follows. For $Z \in \mathcal{Z}$, $b \in \mathbb{R}$, and $v \in V(H) \backslash Z$, define $\omega_Z(v, b)$ to be the minimum of b and the number of r-sets $e \subset Z$ such that $e \cup \{v\} \in H$. Then define for any set $X \subset V(H)$ and any independent set Z in $H[X]$,

$$h_Z(X) = \sum_{v \in X \backslash Z} \omega_Z(v, b).$$

Lemma 5. *Let H be a linear triangle-free $(r+1)$-graph with m vertices, and let Z be a uniformly randomly chosen independent set in H, and $b \in \mathbb{R}^+$. Then as $b \to \infty$,*

$$E(h_Z(V)) + e^b E(|Z|) \gtrsim \frac{bm}{-2^r \log(1 - 2^{-r})}. \tag{3}$$

Proof. Let $V = V(H)$ and $q = 1 - 2^{-r}$. For $v \in V$, define the random variable:

$$W_v = \begin{cases} e^b & \text{if } v \in Z \\ \omega_Z(v,b) & \text{if } v \in V \backslash Z \end{cases}$$

By the definition of W_v,

$$W := \sum_{v \in V(H)} W_v = \sum_{v \in Z} W_v + \sum_{v \in V \backslash Z} W_v = e^b|Z| + h_Z(V).$$

To complete the proof, we show $E(W_v) \gtrsim b/(-2^r \log q)$ for every $v \in V$.

Fixing $v \in V$ and $Z_v = Z \backslash (N(v) \cup \{v\})$, define

$$J = \{u \in N(v) : Z_v \cup \{u\} \in \mathcal{Z}\}.$$

Since H is linear and triangle-free, Z is obtained from Z_v by selecting an independent subset of $H[J \cup \{v\}]$. Let M be the set of r-sets in J forming an edge with v and let $L = J \backslash \bigcup M$. Since H is linear, M consists of disjoint r-sets. A set of vertices of $J \cup \{v\}$ containing v is independent in H if and only if it contains at most $r-1$ vertices from each of the sets in M together with any subset of L. Any independent set of H in $J \cup \{v\}$ not containing v consists of any subset of $\bigcup M \cup L$. So if $|M| = k$ and $|L| = \ell$, there are $2^\ell(2^{rk} + (2^r-1)^k)$ independent sets in $H[J \cup \{v\}]$. It follows from the definition of W_v that

$$\begin{aligned} E(W_v|Z_v) &= \frac{e^b 2^\ell (2^r-1)^k + 2^\ell \sum_{j=0}^k \binom{k}{j}(2^r-1)^{k-j}\min\{j,b\}}{2^\ell(2^{rk} + (2^r-1)^k)} \\ &= \frac{e^b q^k}{1+q^k} + \frac{\sum_{j=0}^k \binom{k}{j}(2^r-1)^{k-j}\min\{j,b\}}{2^{rk}+(2^r-1)^k}. \end{aligned} \tag{4}$$

We shall show $E(W_v|Z_v) \gtrsim b/(-2^r \log q)$. First suppose that $e^b q^k > 2b$. Then using the inequality $-\log(1-x) > x$ for $0 < x < 1$, we obtain

$$E(W_v|Z_v) \geq \frac{e^b q^k}{1+q^k} > \frac{e^b q^k}{2} > b > \frac{b}{-2^r \log q}.$$

Next suppose that $e^b q^k \leq 2b$. Then Lemma 4 gives

$$\sum_{j=0}^k \binom{k}{j}(2^r-1)^{k-j}\min\{j,b\} \sim 2^{rk}\min\{(1-q)k, b\}.$$

Consequently,

$$E(W_v|Z_v) \gtrsim \frac{2^{rk}\min\{(1-q)k,b\}}{2^{rk}+(2^r-1)^k}.$$

Since $e^b q^k \le 2b$, if $b \to \infty$, then also $k \to \infty$ and so

$$\frac{2^{rk}\min\{(1-q)k,b\}}{2^{rk}+(2^r-1)^k} \sim \min\{(1-q)k,b\}.$$

Since $e^b q^k \le 2b$, we have $k \ge (\log 2b - b)/\log q \sim -b/\log q$, and so

$$\min\{(1-q)k,b\} \gtrsim \min\left\{\frac{(1-q)b}{-\log q}, b\right\} = \min\left\{\frac{b}{-2^r\log q}, b\right\} \ge \frac{b}{-2^r\log q}.$$

Now (4) and the tower property of expectation implies,

$$E(W) = \sum_{v\in V} E(E(W_v|Z_v)) \gtrsim \frac{bm}{-2^r\log q}.$$

This completes the proof of Lemma 5. □

6. Random subsets of hypergraphs

To prove Theorem 1, we shall find an appropriate set $Y \subset V(H)$ such that $H[Y]$ is linear and triangle-free and then we apply Lemma 5. To do so, we need to find a set Y in which the quantity $h(Z,b)$ in Lemma 5 is not too large. The set Y will be found by random sampling. A *random set* refers to a set $X \subset V(H)$ whose vertices are chosen from V independently with probability p, where p is to be chosen later.

Lemma 6. *Let H be an n-vertex $(r+1)$-graph with maximum r-degree d and $\alpha(H) \le \alpha$. Suppose that for some $p \in [0,1]$ with $pn \to \infty$ and $b \in \mathbb{R}^+$,*

$$\alpha \log n = o\Big(\frac{pd^2\alpha^{2r}}{nb^2 + db\alpha^r}\Big) \quad \text{and} \quad d^3 n^{3r-3} p^{3r} = o(pn). \tag{5}$$

Then there exists a set $Y \subseteq V(H)$ with the following properties

- *$|Y| \sim pn$*
- *$H[Y]$ is linear and triangle-free and*
- *for every independent set Z in $H[Y]$,*

$$h_Z(Y) \lesssim pd\binom{\alpha}{r}. \tag{6}$$

Remark. The condition $d^3 n^{3r-3} p^{3r} = o(pn)$ is natural, as it implies that the expected number of triangles in a random subset of $V(H)$ with vertex probability p is $o(pn)$ – in particular, a random subset can be made triangle-free by deleting very few vertices.

Proof. Let X be a random subset of $V(H)$. For an independent set Z in $H[X]$, $h_Z(X)$ is a random variable determined by $H[X]$. The main part of the proof is to show that with high probability, $h_Z(X) \lesssim pd\binom{\alpha}{r}$ for every independent set Z in $H[X]$. Fixing an independent set $Z \subset H[X]$, it is convenient to let $\mathfrak{h} = h_Z(X)$. First we bound $E(\mathfrak{h})$ from above. Since H has maximum r-degree d,

$$E(\mathfrak{h}) \leq pd\binom{|Z|}{r} \leq pd\binom{\alpha}{r}.$$

Next we estimate $Var(\mathfrak{h})$. Let I_v be the indicator of the event $v \in X$ and $\omega_v = \omega_Z(v,b)$. Then

$$\mathfrak{h} = \sum_{v \in V \setminus Z} \omega_v I_v$$

is a sum of independent random variables $\omega_v I_v$ for $v \in V \setminus Z$. It follows that

$$\begin{aligned} Var(\mathfrak{h}) &= \sum_{v \in V \setminus Z} Var(\omega_v I_v) \\ &= \sum_{v \in V \setminus Z} \omega_v^2 p(1-p) \\ &\leq \sum_{v \in V} b^2 p(1-p) \quad \leq \quad pnb^2. \end{aligned}$$

By Lemma 3 Part 1, with $\varepsilon \in (0,1)$ and $\lambda = \varepsilon pd\binom{\alpha}{r}$,

$$\begin{aligned} -\log P(\mathfrak{h} > E(\mathfrak{h}) + \lambda) &\geq \frac{\lambda^2}{2pnb^2 + \lambda b} \\ &= \frac{(\varepsilon pd)^2 \binom{\alpha}{r}^2}{2pnb^2 + \varepsilon pd\binom{\alpha}{r} b} \\ &\geq \frac{(\varepsilon pd\alpha^r)^2}{3r!^2(pnb^2 + pdb\alpha^r)} \quad > \quad 2\alpha \log n \quad \text{by (5)} \end{aligned}$$

for large enough n. Note that $\alpha \to \infty$ since $d/n \to 0$. Since $|\mathcal{Z}| < n^{\alpha(H)}$, this shows by Markov's Inequality that with probability at least $1 - n^{-\alpha}$, $h_Z(X) \leq (1+\varepsilon)pd\binom{\alpha}{r}$ for every independent set Z in $H[X]$. Since $\varepsilon \in (0,1)$ is arbitrary, $h_Z(X) \lesssim pd\binom{\alpha}{r}$ for every independent set Z in $H[X]$.

Consider pairs of edges in X that intersect in at least two vertices. The number of pairs of edges in H that intersect in i vertices can be bounded from above as follows: First choose an i-set S of vertices that is the intersection of two edges – there are at most n^i ways of choosing S. Now consider the $(r+1-i)$-graph H_S consisting of edges of the form $E \setminus S$ where $E \in E(H)$. Since H has r-degree at most d, we conclude that H_S has $(r-i)$-degree at most d, so H_S has at most dn^{r-i} edges. Now we pick two edges in H_S that are disjoint. The number of ways of doing this is at most d^2n^{2r-2i}. Altogether, the number of pairs of edges in H sharing exactly i vertices is at most d^2n^{2r-i}, and the probability that one such pair lies in X is p^{2r+2-i}. We conclude, using $pn \to \infty$ and (5), that the expected number of pairs of edges in X intersecting in two or more vertices is at most

$$d^2(p^{2r}n^{2r-2} + p^{2r-1}n^{2r-3} + \cdots + p^{r+2}n^r) = O(d^2p^{2r}n^{2r-2}) = o(pn)$$

due to the assumption $d^3n^{3r-3}p^3 = o(pn)$ of (5).

Next we consider triangles in $H[X]$ which here are triples $\{e, f, g\}$ of edges of $H[X]$ such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. There are fewer than n^3 choices for $e \cap f, f \cap g, g \cap e$. Fixing $e \cap f$ and $e \cap g$, there are fewer than $n^{r-2}d$ choices for e since H has r-degree at most d. It follows that the expected number of triangles in $H[X]$ is less than $n^3(dn^{r-2})^3p^{3r} = o(pn)$, using (5). We conclude that the number T of triangles in $H[X]$ satisfies $E(T) = o(pn)$. Now if Y is obtained from X by deleting a vertex of X from each triangle in $H[X]$ and from each pair of edges of $H[X]$ intersecting in at least two vertices in $H[X]$, then $|Y| \sim pn$ with high probability and $H[Y]$ is triangle-free. Finally, we observe that the value of $h_Z(X)$ does not increase by deleting vertices from X, so (6) holds in $H[Y]$ with high probability. $\square$

7. Proof of Theorem 1

We are now ready to prove Theorem 1, using Lemmas 5 and 6. In the proof, all asymptotic notation refers to $n \to \infty$. Let H be an $(r+1)$-graph of maximum r-degree $d \leq n/(\log n)^{3r^2}$ on n vertices. We will show that $\alpha(H) \gtrsim \alpha$ where

$$\alpha := c_r\Big(\frac{n}{d}\log\frac{n}{d}\Big)^{1/r}$$

and

$$c_r^r = \frac{r!}{-r(3r-1)2^r\log(1-2^{-r})}. \tag{7}$$

Suppose for a contradiction that there exists $\epsilon > 0$ such that $\alpha(H) \lesssim (1-\varepsilon)\alpha$. Define $p \in [0,1]$ and $b \in \mathbb{R}^+$ by

$$pn = \Big(\frac{n}{d \log\log\log n}\Big)^{\frac{3}{3r-1}} \qquad \text{and} \qquad b = \frac{1}{r(3r-1)} \log \frac{n}{d}.$$

There are two steps to the proof: first we have to verify that the above choice of parameters allows us to apply Lemma 5 and Lemma 6, in particular (5).

We claim that the following hold, the first two of which allow us to apply the lemmas:

$$\alpha \log n = o\Big(\frac{pd^2\alpha^{2r}}{nb^2 + db\alpha^r}\Big) \tag{8}$$

$$d^3 n^{3r-3} p^{3r} = o(pn) \tag{9}$$

$$e^b \alpha = o(pd\alpha^r). \tag{10}$$

The inequality (9) follows immediately from the definition of pn, due to the $\log\log\log n$ term there. To prove (8), we note $nb^2 \leq db\alpha^r$, since $nb^2 = dbx^r$ implies

$$x^r = \frac{1}{r(3r-1)} \frac{n}{d} \log \frac{n}{d} \leq c_r^r \frac{n}{d} \log \frac{n}{d} = \alpha^r.$$

Therefore

$$\frac{pd^2\alpha^{2r}}{nb^2 + db\alpha^r} > \frac{pd^2\alpha^{2r}}{2db\alpha^r} = \frac{pd\alpha^r}{2b} = \frac{r(3r-1)c_r^r}{2} pn.$$

By the definition of pn and $d \leq n/(\log n)^{3r^2}$, a short calculation yields $\alpha \log n = o(pn)$, which proves (8). For (10), we have

$$\begin{aligned} e^b \alpha &= c_r \Big(\frac{n}{d}\Big)^{1/r(3r-1)} \cdot \Big(\frac{n}{d} \log \frac{n}{d}\Big)^{1/r} \\ &= c_r \Big(\frac{n}{d}\Big)^{3/(3r-1)} \Big(\log \frac{n}{d}\Big)^{1/r} \\ &= c_r (\log\log\log n)^{3/(3r-1)} pn \Big(\log \frac{n}{d}\Big)^{1/r} \quad = \quad o(pd\alpha^r) \end{aligned}$$

since $d \leq n/(\log n)^{3r^2}$ and $r \geq 2$. This verifies (10) and we now apply Lemma 6.

By Lemma 6, there is a linear triangle-free subgraph $H[Y]$ with $|Y| \sim pn$ and

$$h_Z(Y) \lesssim pd \binom{\alpha}{r}$$

for every independent set Z in $H[Y]$. In particular, using (10), selecting a random uniform independent set Z in $H[Y]$,

$$E(h_Z(Y)) + e^b E(|Z|) \lesssim pd\binom{\alpha}{r} + e^b \alpha \lesssim \frac{(1-\epsilon)^r c_r^r}{r!} pn\Big(\log \frac{n}{d}\Big). \tag{11}$$

We note that $b = \frac{1}{r(3r-1)} \log \frac{n}{d} \to \infty$ since $d \le n/(\log n)^{3r^2}$. Therefore by Lemma 5,

$$E(h_Z(Y)) + e^b E(|Z|) \gtrsim \frac{pnb}{-2^r \log(1-2^{-r})} = \frac{c_r^r}{r!} pn\Big(\log \frac{n}{d}\Big). \tag{12}$$

Now (12) contradicts (11), and this completes the proof.

8. Ramsey numbers and independent neighborhoods

Ajtai, Komlós and Szemerédi [3] proved that $R(K_3, K_t^{(2)}) = O(t^2/(\log t))$. Using Theorem 1, we can generalize part of this result to hypergraphs in the following manner.

Proof of Theorem 2. It is enough to show that if n is large enough, then there exists a constant $a > 0$ such that $\alpha(H) \ge a(n \log n)^{1/r}$ when H is an r-graph on n vertices with independent neighborhoods. We will show that $a = (c_r/2)^{(r-1)/r}$ works where $c_r > 0$ is the constant in Theorem 1. Let $t = a(n \log n)^{1/r}$. If H has maximum $(r-1)$-degree at least t, then the set of vertices adjacent to an $(r-1)$-set of degree t is an independent set, since H has independent neighborhoods, and we are done. Otherwise, by Theorem 1 and the definition of t, if n is large enough then

$$\begin{aligned} \alpha(H) &\ge c_r\Big(\frac{n}{t}\log\frac{n}{t}\Big)^{\frac{1}{r-1}} \\ &> \frac{c_r}{2}\Big(\frac{n\log n}{a}\Big)^{\frac{1}{r-1}} \\ &= a(n\log n)^{1/(r-1)} \end{aligned}$$

by definition of a. A short computation with the value of t shows this gives the required upper bound on Ramsey numbers. □

The above theorem is best possible for $r = 2$, as shown via a random construction of triangle-free graphs by Kim [16]. We believe Theorem 2 is best possible for $r > 2$ as well. It is straightforward to give an example with $\alpha(H) \le c' n^{1/r}(\log n)^{1/(r-1)}$ with $c' > 0$ using the random hypergraph $H_{n,p}$ with edge probability $p \approx n^{-(r-1)/r}$. One can then use the Local Lemma or the deletion method (see the proof of Theorem 4 in [5] for details using the latter approach).

9. Concluding remarks

• Duke, Lefmann and Rödl [7], based on a paper of Ajtai, Komlós, Pintz, Spencer and Szemerédi [2] showed that a linear $(r+1)$-graph on n vertices with average degree d has an independent set of size at least $c'n(\frac{\log d}{d})^{1/r}$. It would be interesting to find a way to extend the method of this paper to prove such a result.

• This paper was partly inspired by the following question of De Caen [8]. A 3-graph H is *c-sparse* if every set S of vertices spans at most $c|S|^2$ edges.

Conjecture 7. *For every c there is $f(n) \to \infty$ as $n \to \infty$ such that $\alpha(H) \geq f(n)\sqrt{n}$ for each c-sparse n-vertex 3-graph H.*

We pose the stronger conjecture that for some function $\omega(n) \to \infty$, $\alpha(H) > \frac{n\omega(n)}{\sqrt{d}}$ for a c-sparse n-vertex 3-graph H with average degree d. Both conjectures remain open.

• A related but more difficult problem than that considered in this paper is to obtain analogous results for chromatic number. Frieze and the second author [10] have conjectured that if H is an $(r+1)$-graph on n vertices with maximum degree d, and H does not contain a specific $(r+1)$-graph F, then H has chromatic number $O(d^{1/r}/(\log d)^{1/r})$ (it appears that this conjecture is more difficult when d is much less than n). In [10] and [11] the case when H is linear is dealt with using a randomized greedy approach. For $r = 1$ – i.e. for graphs – this is known to be true when F is a bipartite graph, or one vertex away from a bipartite graph [4]. It is open for graphs even in the case $F = K_4$, and in each case where the chromatic number conjecture of Frieze and the second author stated above is open, the corresponding question for independence number is also open.

10. Acknowledgments

We are very grateful to Fan Chung, Franklin Kenter and Choongbum Lee for helpful discussions about this problem and the referees for their many comments and corrections to an earlier draft of this paper.

References

[1] N. Alon, Independence numbers of locally sparse graphs and a Ramsey type problem, Random Structures and Algorithms 9 (1996), 271-278.
[2] M. Ajtai, J. Komlós, J. Pintz, J. Spencer and E. Szemerédi, Extremal uncrowded hypergraphs, J. Comb. Theory (Series A), 32 (1982), no. 3, 321–335.
[3] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, J. Comb. Theory (Series A), 29 (1980), 354-360.
[4] N. Alon, M. Krivelevich and B. Sudakov, Coloring graphs with sparse neighborhoods, Journal of Combinatorial Theory, Ser. B 77 (1999) 73–82.
[5] T. Bohman, A. Frieze and D. Mubayi, Hypergraphs with independent neighborhoods, Combinatorica 30 (2010), no. 3, 277–293.
[6] H. Chernoff, A Measure of Asymptotic Efficiency for Tests of a Hypothesis Based on the sum of Observations, Annals of Mathematical Statistics 23 (1952), no 4, 493-507.
[7] R. Duke, H. Lefmann and V. Rödl, On uncrowded hypergraphs, Random Structures and Algorithms, 6, (1995) 209-212.
[8] D. de Caen, The current status of Turán's problem on hypergraphs, In Extremal problems for finite sets (Visegrád, 1991), volume 3 of Bolyai Soc. Math. Stud., pages 187-197. János Bolyai Math. Soc., Budapest, 1994.
[9] P. Erdős, Some unsolved problems in graph theory and combinatorial analysis, Combinatorial Mathematics and its Applications (Proc. Conf. Oxford, 1969), Academic Press, London, New York, 1971 pp. 97–109.
[10] A. Frieze and D. Mubayi, On the chromatic number of simple triangle-free triple systems, Electron. J. Combin., 15(1) Research Paper 121, no. 27 (2008).
[11] A. Frieze and D. Mubayi, Coloring simple hypergraphs, submitted.
[12] Z. Füredi, Maximal independent subsets in Steiner systems and in planar sets, SIAM Journal on Discrete Mathematics 4 (1991), 196–199.
[13] Z. Füredi and I. Palásti, Arrangements of lines with a large number of triangles, Proceedings of the American Mathematical Society 92 no. 4 (1982) 561–566.
[14] Z. Füredi, D. Mubayi and O. Pikhurko, Quadruple Systems with Independent Neighborhoods, J. Combin. Theory (Series A), 115 (2008) 1552–1560.
[15] Z. Füredi, O. Pikhurko and M. Simonovits, On Triple Systems with Independent Neighborhoods, Combin. Probab. & Comput., 14 (2005) 795–813.
[16] J. H. Kim, The Ramsey Number $R(3,t)$ has order of magnitude $t^2/\log t$, Random Structures and Algorithms 7 (1995), 173–207.
[17] J. Komlós, J. Pintz and E. Szemerédi, A lower bound for Heilbronn's problem, J. London Math. Soc. 25 (1982), no. 2, 13–24.
[18] C. McDiarmid, Concentration, Probabilistic Methods for Algorithmic Discrete Mathematics (1998), 1–46.

[19] K. Phelps and V. Rödl, Steiner triple systems with minimum independence number, Ars Combin. 21 (1986), 167–172.

[20] V. Rödl, On a packing and covering problem, European J. Combin. 5 (1985), 69–78.

[21] V. Rödl and E. Šinajová, Note on independent sets in Steiner systems, Proceedings of the Fifth International Seminar on Random Graphs and Probabilistic Methods in Combinatorics and Computer Science (Poznan, 1991), Random Structures Algorithms 5 (1994), no. 1, 183–190.

[22] J. Shearer, On the independence number of sparse graphs, Random Structures and Algorithms 7 (1995), 269–271.

Department of Mathematics, University of Illinois, Urbana-Champaign.

E-mail address: kostochk@uiuc.edu

Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL 60607

E-mail address: mubayi@math.uic.edu

Department of Mathematics, University of California, San Diego (UCSD), La Jolla, CA 92093-0112, USA.

E-mail address: jacques@ucsd.edu