New lower bounds for Ramsey numbers of graphs and hypergraphs

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Dedicated to the memory of Rodica Simion

Abstract

Let G be an r-uniform hypergraph. The multicolor Ramsey number $r_k(G)$ is the minimum n such that every k-coloring of the edges of the complete r-uniform hypergraph $K_n^{(r)}$ yields a monochromatic copy of G. Improving slightly upon results from [1], we prove that

$$tk^2 + 1 \leq r_k(K_{2,t+1}) \leq tk^2 + k + 2,$$

where the lower bound holds when t and k are both powers of a prime p. When $t = 1$, we improve the lower bound by 1, proving that $r_k(C_4) \geq k^2 + 2$ for any prime power k. This extends the result of [11] which proves the same bound when k is an odd prime power. These results are generalized to hypergraphs in the following sense.

Fix integers $r, s, t \geq 2$. Let $\mathcal{H}^{(r)}(s,t)$ be the complete r-partite r-graph with $r-2$ parts of size 1, one part of size s, and one part of size t (note that $\mathcal{H}^{(2)}(s,t) = K_{s,t}$). We prove

$$tk^2 - k + 1 \leq r_k(\mathcal{H}^{(r)}(2, t+1)) \leq tk^2 + k + r,$$

where the lower bound holds when t and k are both powers of a prime p; and

$$k^s - k^{s-1} \leq r_k(\mathcal{H}^{(r)}(s,t)) \leq O(k^s), \text{ for fixed } t, s \geq 2,\ t > (s-1)!;$$

$$r_k(\mathcal{H}^{(r)}(3,3)) = (1 + o(1))k^3,$$

where the lower bound holds when k is a prime power.

Some of our lower bounds are special cases of a family of more general hypergraph constructions obtained by algebraic methods. We describe these, thereby extending results of [12] about graphs.

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1 Introduction

The classical multicolor Ramsey number for the r-uniform hypergraph (or simply r-graph) G is the minimum n such that no matter how the edges of the complete r-graph $K_n^{(r)}$ are colored with k colors, there is a monochromatic copy of G. It is denoted $r_k(G)$. For $r \geq 2$, very little is known about the growth rate of these numbers when G is a complete r-graph. For example, the best known bounds even for the smallest nontrivial graph case are $c^k < r_k(K_3) < c'k!$, where c and c' are constants (Chung [3] and Erdős, Szekeres [5]).

Much more progress has been made in the graph case for $G = C_4$. In particular, the lower bound $r_k(C_4) \geq k^2 - k + 2$ when $k - 1$ is a prime power (see Chung, Graham [4], Irving [8]) has recently been improved by Lazebnik, Woldar [11] to $k^2 + 2$ for k an odd prime power. The upper bound $r_k(C_4) \leq k^2 + k + 1$ by Chung [3] and Irving [8] holds for all $k \geq 1$.

These results are generalized by Axenovich, Füredi, Mubayi in [1] to $G = K_{2,t+1}$, where an edge-coloring of K_n, $n = (q-1)^2/t$, q a prime power, is given using $k = (q-1)/t + O(\sqrt{q}\log q)$ colors, with no monochromatic copy of G. Solving for n in terms of k, this yields, for fixed t,

$$r_k(K_{2,t+1}) \geq tk^2 - O(k^{3/2}), \text{ as } k \to \infty. \tag{1}$$

In the other direction, classical results by Kővári, Sós, Turán [10] yield

$$r_k(K_{2,t+1}) \leq (t + o(1))k^2. \tag{2}$$

Our first theorem improves the lower bound in (1) slightly for some special values of t, k, and extends the results of [11] to not necessarily odd prime powers. We also precisely calculate the upper bound on $r_k(K_{2,t+1})$ implied from [10], thus sharpening (2).

Theorem 1. *Let p be a prime. Then*

$$tk^2 + 1 \leq r_k(K_{2,t+1}) \leq tk^2 + k + 2,$$

where the lower bound holds when t and k are both powers of p. If k is a prime power, then

$$r_k(C_4) \geq k^2 + 2.$$

Motivated by recent extensions of algebraic graph constructions to hypergraphs by Mubayi [13], we extend Theorem 1 to r-graphs.

Definition. Let $r, s, t \geq 2$ be integers. Then $\mathcal{H}^{(r)}(s,t)$ is the complete r-partite r-graph with $r-2$ parts of size 1, one part of size s, and one part of size t.

Theorem 2. *Let p be a prime. Then*

$$tk^2 - k + 1 \leq r_k(\mathcal{H}^{(r)}(2, t+1)) \leq tk^2 + k + r,$$

where the lower bound holds whenever t and k are both powers of p.

Using a completely different algebraic construction first introduced for graphs by Kollár, Rónyai, Szabó [9], we extend these results to a still larger class of complete r-partite r-graphs.

Theorem 3. (Section 4) *For fixed $t, s > 1$, and $t > (s-1)!$,*

$$(1-o(1))k^s \leq r_k(\mathcal{H}^{(r)}(s,t)) \leq O(k^s), \text{ and}$$

$$r_k(\mathcal{H}^{(r)}(3,3)) = (1+o(1))k^3.$$

Finally, in Section 5, we exhibit a very general family of algebraic constructions of r-uniform hypergraphs. Isomorphic copies of these hypergraphs often edge-decompose the complete or complete k-partite r-uniform hypergraphs that they span. The case $r = 2$, $t = 1$ of Theorem 1 can essentially be proved using special cases of these constructions. These generalize some results by Lazebnik and Woldar [12] to r-graphs.

2 Lower bounds in Theorems 1 and 2

A construction from [1] produces a coloring of the edges of K_n, $n = (q-1)^2/t$ (when q is a prime power congruent to 1 mod t), with $(q-1)/t + O(\sqrt{q}\log q)$ colors with no monochromatic copy of $K_{2,t+1}$. The construction below improves this (for certain values of t, q) by allowing n to be as large as q^2/t, and using only q/t colors. Moreover, the construction is slightly simpler than the one in [1] (but strongly motivated by the ideas of Füredi [7] and [1]). We describe it in the more general setting of hypergraphs.

Recall that the r-graph $\mathcal{H}^{(r)}(2, t+1)$ when $r = 2$ is just the bipartite graph $K_{2,t+1}$. The lower bounds in Theorems 1 and 2 are simultaneously proved in the following Theorem.

Theorem 4. *Let k, r, t be positive integers with k, t being powers of a prime p. Then*

$$r_k(\mathcal{H}^{(r)}(2,t+1)) \geq \begin{cases} tk^2 - k + 1 & r \geq 3 \\ tk^2 + 1 & r = 2 \\ k^2 + 2 & r = 2,\ t = 1. \end{cases}$$

Proof. Let $q = tk$ (if $t = 1$, then let $q = k$ be any prime power). Let

$$n = \begin{cases} q(q-1)/t & r \geq 3 \\ q^2/t & r = 2. \end{cases} \tag{3}$$

We define a coloring of the edges of $K_n^{(r)}$ by k colors with no monochromatic copy of $\mathcal{H}^{(r)}(2, t+1)$.

Let $\mathbf{F}$ be the q-element finite field and let H be a t-element (additive) subgroup of $\mathbf{F}$. Let $H_1, \ldots, H_{q/t}$ be the cosets of H. These cosets give the decomposition $\mathbf{F} = H_1 \cup \ \ldots \ \cup H_{q/t}$. Let $\mathbf{F}^*$ consist of the nonzero elements of $\mathbf{F}$. Set

$$V = \begin{cases} \mathbf{F}^* \times \mathbf{F} & r \geq 3 \\ \mathbf{F} \times \mathbf{F} & r = 2. \end{cases} \tag{4}$$

Define an equivalence relation on elements of V by letting $(a, b) \sim (x, y)$ if
(i) $a = x$, and
(ii) there is an $\alpha \in H$ such that $b = \alpha + y$.
The class represented by (a, b) is denoted by $\langle a, b \rangle$, and these classes form the vertex set of $K_n^{(r)}$. Observe that since a class $\langle a, b \rangle$ consists of exactly $|H| = t$ distinct (a, b), the value of n stated in (4) follows from (5).

Color the edge consisting of r distinct classes $\langle a_i, b_i \rangle$ $(1 \leq i \leq r)$ by j if

$$\prod_{i=1}^{r} a_i - \sum_{i=1}^{r} b_i \in H_j. \tag{5}$$

Note that this relation is well defined, since if $(a_i, b_i) \sim (a_i', b_i')$ for all i, and (6) holds, then for some $\alpha_i \in H$,

$$\prod_i a_i' - \sum_i b_i' = \prod_i a_i - \sum_i (b_i + \alpha_i) = \prod_i a_i - \sum_i b_i - \sum_i \alpha_i \in H_j.$$

Let G_α denote the graph consisting of the edges in color α. We claim that G_α contains no copy of $\mathcal{H}^{(r)}(2, t+1)$. First we argue that for $(p_1, s_1), (p_2, s_2) \in \mathbf{F} \times \mathbf{F}$, $(p_1, s_1) \not\sim (p_2, s_2)$, the system

$$\begin{aligned} p_1 x - s_1 - y &= \beta \qquad (6) \\ p_2 x - s_2 - y &= \gamma \end{aligned}$$

has at most one solution (x, y) for every $\beta, \gamma \in H_\alpha$. Suppose to the contrary we have

$$p_1 x - s_1 - y = \beta \tag{7}$$

$$p_2 x - s_2 - y = \gamma \tag{8}$$

$$p_1 x' - s_1 - y' = \beta \tag{9}$$

$$p_2 x' - s_2 - y' = \gamma \tag{10}$$

Adding (9) and (10) and subtracting from this (8) and then (11) yields $(p_2 - p_1)(x - x') = 0$, which implies that either $p_1 = p_2$ or $x = x'$. If $p_1 = p_2$, then (8) and (9) imply that $s_1 - s_2 = \gamma - \beta \in H$, yielding the contradiction $(p_1, s_1) \sim (p_2, s_2)$. On the other hand, if $x = x'$, then (9) and (11) imply that $y = y'$ which gives $(x, y) = (x', y')$.

There are t^2 possibilities for (β, γ), $\beta, \gamma \in H_\alpha$ in (7). The set of solutions form t-element equivalent classes, hence the system

$$\begin{aligned} p_1 x - s_1 - y &\in H_\alpha \\ p_2 x - s_2 - y &\in H_\alpha \end{aligned} \tag{11}$$

has at most t nonequivalent solutions (x, y).

Now suppose that $\langle a_i, b_i \rangle$, $1 \le i \le r-2$, $\langle u_j, v_j \rangle$, $1 \le j \le 2$, $\langle x_l, y_l \rangle$, $1 \le l \le t+1$, form the vertex set of a copy of $\mathcal{H} = \mathcal{H}^{(r)}(2, t+1)$, where the $\langle a_i, b_i \rangle$ form the $r-2$ parts of size 1, $\{\langle u_j, v_j \rangle\}$ forms the part of size 2, and $\{\langle x_l, y_l \rangle\}$ forms the part of size $t+1$. For $j = 1, 2$, set

$$p_j = \left(\prod_{i=1}^{r-2} a_i \right) u_j \quad \text{and} \quad s_j = \left(\sum_{i=1}^{r-2} b_i \right) + v_j.$$

Note that $(p_1, s_1) \not\sim (p_2, s_2)$, since otherwise by the definition of V in (5) we have $\langle u_1, v_1 \rangle = \langle u_2, v_2 \rangle$. The edges forming the copy of $\mathcal{H}$ yield, for $1 \le l \le t+1$,

$$\begin{aligned} p_1 x_l - s_1 - y_l &\in H_\alpha \\ p_2 x_l - s_2 - y_l &\in H_\alpha \end{aligned}$$

where $\langle x_m, y_m \rangle \ne \langle x_{m'}, y_{m'} \rangle$ when $m \ne m'$. But we have argued in (12) that such a system can have at most t nonequivalent solutions. This completes the proof of the lower bound except for the case $r = 2$, $t = 1$. To improve the construction by 1 in this case, we use the idea of [11].

Let v be a vertex of K_{k^2+1}, and let N denote the subgraph induced by its neighbors. Color N as shown above with k colors. For any edge vw with $w \in N$, color it with the first coordinate of the vector corresponding to w. To show that the resulting construction has no monochromatic C_4, it suffices to show that there is no C_4 with consecutive vertices $vwxy$.

Suppose that such a C_4 exists in color a and let $w = \langle a, w_2 \rangle$, $x = \langle x_1, x_2 \rangle$, and $y = \langle a, y_2 \rangle$. Then the edges wx and xy imply that $ax_1 - w_2 - x_2 = ax_1 - y_2 - x_2$. Consequently, $w_2 = y_2$, which yields the contradiction $w = y$. $\square$

3 Upper bounds in Theorems 1 and 2

We need bounds on the Turán numbers of r-graphs. The following Lemma sharpens the corresponding bound in [13].

Lemma 5. *Let $r \ge 2$, $t \ge 1$ be integers. Then*

$$\mathrm{ex}(n, \mathcal{H}^{(r)}(2, t+1)) \le \frac{\sqrt{t} n(n-1)\cdots(n-r+2)(n-r+2)^{1/2}}{r!} + \frac{n(n-1)\cdots(n-r+2)}{2r!}.$$

Proof. We proceed by induction on r. For $r = 2$, the result follows from [10], which yields $\mathrm{ex}(n, K_{2,t+1}) \leq \sqrt{t}n^{3/2}/2 + n/4$.

We now suppose that $r \geq 3$, and G is an r-graph with e edges containing no copy of $\mathcal{H}^{(r)}(2, t+1)$. Since $\sum_{v \in V(G)} \deg(v) = re$, there is a vertex x in G with $\deg(x) \geq re/n$. Let G' be the $(r-1)$-graph induced by the neighborhood of x; thus $V(G') = V(G) - \{x\}$, and $\{x_1, \ldots, x_{r-1}\} \in E(G')$ iff $\{x_1, \ldots, x_{r-1}, x\} \in E(G)$.

A copy of $\mathcal{H}^{(r-1)}(2, t+1)$ in G', together with x yields a copy of $\mathcal{H}^{(r)}(2, t+1)$ in G, hence we may assume that $|E(G')| \leq \mathrm{ex}(n-1, \mathcal{H}^{(r-1)}(2, t+1))$. By the induction hypothesis, we obtain

$$e \leq \frac{n \cdot \mathrm{ex}(n-1, \mathcal{H}^{(r-1)}(2, t+1))}{r} \leq \frac{\sqrt{t}n\ (n-1) \cdots (n-r+2)^{3/2}}{r!} + \frac{n\ (n-1) \cdots (n-r+2)}{2r!}\ .$$

This completes the proof. □

Proof of upper bounds in Theorems 1 and 2: Given a coloring of $E(K_n^{(r)})$ with k colors, the largest color class has at least $\binom{n}{r}/k$ colors. Thus it suffices to prove that for $n = tk^2 + k + r$,

$$\frac{\binom{n}{r}}{k} > \mathrm{ex}(n, \mathcal{H}^{(r)}(2, t+1)). \tag{12}$$

By Lemma 5, (13) holds if

$$\frac{n\ (n-1) \ldots (n-r+1)}{kr!} > \frac{\sqrt{t}n\ (n-1) \cdots (n-r+2)^{3/2}}{r!} + \frac{n\ (n-1) \cdots (n-r+2)}{2r!}.$$

This is equivalent to

$$(n-r+1) > k\sqrt{t}(n-r+2)^{1/2} + k/2.$$

Setting $n' = n - r + 2$, this is equivalent to $(n' - 1 - k/2)^2 > k^2tn'$. Because $n' = tk^2 + k + 2$,

$$\left(n' - 1 - \frac{k}{2}\right)^2 - k^2tn' = \frac{k^2}{4} + k + 1 > 0\ .$$

□

4 Norm Hypergraphs: Proof of Theorem 3

In [9] an algebraic construction was given which proved that $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$ for $t \geq s! + 1$. This construction was extended to $t \geq (s-1)! + 1$ $(s \geq 3)$ in Alon, Rónyai and Szabó [2]. In [13], the construction was extended to r-graphs. In [2] this construction was used to "tile" the edges of a complete graph, thereby obtaining lower bounds for Ramsey numbers. In this section we present the fairly straightforward generalization of these ideas to hypergraphs.

For an r-graph H, Let $f(H)$ be the minimum number of colors required to color the edges of $K_n^{(r)}$ with no monochromatic copy of H. We begin with the following observation due to Spencer (see [4]).

Proposition 6.

$$f(H) < \frac{n^r \log n}{\mathrm{ex}(n, H)} \ .$$

Let $s > 2$, $\mathbf{F}(q)$ be the finite field of q elements, and $\mathbf{F}(q)^* = \mathbf{F}(q) \setminus \{0\}$. Given $X \in \mathbf{F}(q^{s-1})$, let $N(X) = X^{1+q+\ldots+q^{s-2}}$ be the *norm* of $X \in \mathbf{F}(q^{s-1})$ over $\mathbf{F}(q)$. Note that for $X \in \mathbf{F}(q^{s-1})$, we have $(N(X))^q = N(X)$, so $N(X) \in \mathbf{F}(q)$ (indeed, $\mathbf{F}(q)$ consists of precisely the solutions to $x^q - x = 0$).

We need the following result that follows from a result in [9] and was proved in [2].

Lemma 7. *If $(D_1, d_1), \ldots, (D_s, d_s)$ are distinct elements of $\mathbf{F}(q^{s-1}) \times \mathbf{F}(q)^*$, then the system of s equations*

$$N(D_j + X) = d_j x, \qquad 1 \le j \le s$$

has at most $(s-1)!$ solutions $(X, x) \in \mathbf{F}(q^{s-1}) \times \mathbf{F}(q)^*$.

Proof of Theorem 3: Let $V(K_n^{(r)}) = \mathbf{F}(q^{s-1}) \times \mathbf{F}(q)^*$. We will produce a coloring of the edges of $K_n^{(r)}$ with $q + o(q)$ colors and no monochromatic copy of $\mathcal{H}^{(r)}(s,t)$. Then the bound for the Ramsey number will follow from the fact that there is a prime between N and $N + N^{1/3}$.

If $\sum_{i=1}^r S_i \neq 0$, then color the edge formed by vertices (S_i, s_i), $1 \le i \le r$, by

$$N(\sum_i S_i)/(\prod_i s_i) \in \mathbf{F}(q)^*.$$

This is a coloring of most of the edges of the complete r-graph on $q^s - q^{s-1}$ vertices with $q-1$ colors. We first show that there is no monochromatic copy of $\mathcal{H}^{(r)}(s,t)$ among the colored edges.

Suppose that (A_i, a_i), $1 \le i \le r-2$, (B_j, b_j), $1 \le j \le s$, (C_l, c_l), $1 \le l \le t$, form the vertex set of a copy of $\mathcal{H}^{(r)}(s,t)$ in color $\alpha \neq 0$, where the (A_i, a_i) form the parts of size 1, $\{(B_j, b_j)\}$ forms the part of size s, and $\{(C_l, c_l)\}$ forms the part of size t. For $1 \le j \le s$, set

$$P_j = \left(\sum_{i=1}^{r-2} A_i\right) + B_j \quad \text{and} \quad p_j = \left(\prod_{i=1}^{r-2} a_i\right) b_j \ \alpha.$$

Note that the ordered pair $(P_j, p_j) \neq (P_{j'}, p_{j'})$ for $j \neq j'$, since otherwise, $\alpha \neq 0$ implies that $(B_j, b_j) = (B_{j'}, b_{j'})$. The edges forming this copy of $\mathcal{H}^{(r)}(s,t)$ in color α yield, for $1 \le j \le s$, $1 \le l \le t$,

$$N(P_j + C_l) = p_j c_l$$

for all j, l. But Lemma 7 implies that such a system can have at most $t-1$ solutions (C_l, c_l). This completes the proof that there is no monochromatic copy of $\mathcal{H}^{(r)}(s,t)$ among the colored edges. Now we turn to the uncolored edges.

The r-graph G_0 of the uncolored edges consists of r-sets whose first coordinates sum to zero. For each such r-set (S_i, s_i), $1 \le i \le r$, with $\sum_i S_i = 0$, let $T_i = \{(S_i, x) : x \in \mathbf{F}(q)^*\}$. Then the complete r-partite r-graph with parts $T_1, \ldots, T_r$ is in G_0, and it is easy to see that G_0 consists of such edge-disjoint complete r-partite r-graphs with parts of size $q-1$. Call these $H_1, \ldots, H_\gamma$.

A result of Erdős [6] yields $\mathrm{ex}(r(q-1), \mathcal{H}^{(r)}(s,t)) = O(q^{r-1/s})$. Using this and Proposition 6 we conclude that for each α, $1 \le \alpha \le \gamma$, there is a coloring of H_α with $O(q^{1/s})$ colors and no monochromatic copy of $\mathcal{H}^{(r)}(s,t)$. Color each H_α with this same set C of $O(q^{1/s})$ new colors. We now check that there is no monochromatic copy H of $\mathcal{H}^{(r)}(s,t)$ in a color from C.

Suppose on the contrary that there is such an H. By construction, H is not entirely contained in any H_α. Let e and f be edges of H. Using the same notations for the vertices of H as before, we have that e and f must share the $r-2$ vertices $v_i = (A_i, a_i)$. Let $e = \{v_1, \ldots, v_m, (B_j, b_j), (C_l, c_l)\}$ and $f = \{v_1, \ldots, v_m, (B'_j, b'_j), (C'_l, c'_l)\}$. Since H is a complete r-partite graph, $\{v_1, \ldots, v_m, (B_j, b_j), (C'_l, c'_l)\}$ is also an edge of H. Therefore,

$$\sum_{i=1}^{r-2} A_i + B_j + C'_l = 0 = \sum_{i=1}^{r-2} A_i + B_j + C_l = \sum_{i=1}^{r-2} A_i + B'_j + C'_l.$$

This yields $C_l = C'_l$ and $B_j = B'_j$. Therefore the r-sets formed by the first coordinates of the vertices of e and of f are equal, and so e and f lie in the same H_α, a contradiction to our assumption. This completes the proof of the lower bound.

For the upper bounds, we must prove that every q-coloring of $K_n^{(r)}$ ($n = O(q^s)$ when $t \ge (s-1)! + 1 \ge s \ge 3$, $(s,t) \ne (3,3)$, and $n = (1+o(1))q^3$ when $s = t = 3$) yields a monochromatic copy of $\mathcal{H}^{(r)}(s,t)$. This follows from upper bounds for the Turán number of $\mathcal{H}^{(r)}(s,t)$. These were established by induction on r in [13], where it was shown that for fixed r, s, as $n \to \infty$,

$$\mathrm{ex}(n, \mathcal{H}^{(r)}(s,t)) = \begin{cases} \Theta(n^{r-1/s}) & s \ge 4 \\ (1/r! + o(1))n^{r-1/3} & s = 3 \end{cases}$$

Since the largest color class in a q-coloring of $K_n^{(r)}$ has at least $\binom{n}{r}/q > \mathrm{ex}(n, \mathcal{H}^{(r)}(s,t))$ edges, we immediately obtain the bounds claimed in Theorem 3. □

5 Edge-decomposition of complete k-partite r-graphs and complete r-graphs

In this section we generalize some constructions and results of [12] from 2-graphs to r-graphs, $r \ge 2$. Looking back, it is fair to say that most of these generalizations turned out to be rather straightforward and natural. Nevertheless it took us much longer to see this than we originally expected: some “clear” paths led eventually to nowhere, and several technical steps presented considerable challenge even after the “right” definitions had been found.

The constructions for 2-graphs have proven to be useful in Extremal Graph Theory (see the survey section in [12]). Therefore it is reasonable to expect that their generalizations for hypergraphs will also find applications. In particular, the construction used in our Theorem 4 can be interpreted as a natural extension of some constructions introduced in this section (see the remark at the end of the section).

Let $\mathbf{F} = \mathbf{F}(q)$ be the q-element finite field, $\mathbf{F}^d$ be the direct product of $d \geq 2$ copies of $\mathbf{F}$. For integers $i, r \geq 2$, let $f_i : \mathbf{F}^{(i-1)r} \to \mathbf{F}$ be a function. For $x^i = (x^i_1, \ldots, x^i_d) \in \mathbf{F}^d$, let $(x^1, \ldots, x^i)$ stand for $(x^1_1, \ldots, x^1_d, x^2_1, \ldots, x^2_d, \ldots, x^i_1, \ldots, x^i_d)$.

Suppose d, k, r are integers and $2 \leq r \leq k$, $d \geq 2$. First we define a k-partite r-graph $\mathcal{T} = \mathcal{T}(q, d, k, r, f_2, f_3, \ldots, f_d)$. Let the vertex set $V(\mathcal{T})$ be a disjoint union of sets, or color classes, $V^1, \ldots, V^k$, where each V^j is a copy of $\mathbf{F}^d$. By $a^j = (a^j_1, a^j_2, \ldots, a^j_d)$ we denote an arbitrary vertex from V^j. The edge set $E(\mathcal{T})$ is defined as follows: for every r-subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$ (the set of colors), we consider the family of all r-sets of vertices $\{a^{i_1}, \ldots, a^{i_r}\}$, where each $a^j \in V^j$, and such that the following system of $r - 1$ equalities hold:

$$\begin{aligned}
\sum_{j=1}^{r} a_2^{i_j} &= f_2(a_1^{i_1}, \ldots, a_1^{i_r}) \\
\sum_{j=1}^{r} a_3^{i_j} &= f_3(a_1^{i_1}, \ldots, a_1^{i_r}, a_2^{i_1}, \ldots, a_2^{i_r}) \\
\cdots \quad \cdots &\quad \cdots \quad \cdots \quad \cdots \\
\sum_{j=1}^{r} a_d^{i_j} &= f_d(a_1^{i_1}, \ldots, a_1^{i_r}, a_2^{i_1}, \ldots, a_2^{i_r}, \ldots, a_{d-1}^{i_1}, \ldots, a_{d-1}^{i_r})
\end{aligned} \tag{13}$$

The system (14) can also be used to define another class of r-graphs, $\mathcal{K} = \mathcal{K}(q, d, r, f_2, f_3, \ldots, f_d)$, but in order to do this, we have to restrict the definition to only those functions f_i which satisfy the following symmetry property: for every permutation π of $\{1, 2, \ldots, i-1\}$,

$$f_i(x^{\pi(1)}, \ldots, x^{\pi(i-1)}) = f_i(x^1, \ldots, x^{i-1}).$$

Then let the vertex set $V(\mathcal{K}) = \mathbf{F}^d$, and let the edge set $E(\mathcal{K})$ be the family of all r-subsets $\{a^{i_1}, \ldots, a^{i_r}\}$ of vertices which satisfy system (14). We impose the symmetry condition on the f_i to make the definition of an edge independent of the order in which its vertices are listed.

$\mathcal{K}$ can be also viewed as a q^d-partite r-graph, each partition having one vertex only. If $d = r$, then $\{i_1, \ldots, i_r\} = \{1, \ldots, d\}$.

Theorem 8. *Let q, d, r, k be integers, $2 \leq r \leq k$, $d \geq 2$, and q be a prime power. Then*

1. *$\mathcal{T}$ is a regular r-graph of order kq^d and size $\binom{k}{r} q^{dr-d+1}$. The degree of each vertex is $\binom{k-1}{r-1} q^{dr-2d+1}$.*

2. For odd q, $\mathcal{K}$ is an r-graph of order q^d and size $\frac{1}{q^{d-1}}\binom{q^d}{r}$.

Proof. Let $a^{i_1} = (a_1^{i_1}, a_2^{i_1}, \ldots, a_d^{i_1})$ be an arbitrary vertex of $\mathcal{T}$ of color i_1. To count its degree in $\mathcal{T}$, $\deg_{\mathcal{T}}(a^{i_1})$, we first choose the remaining $k-1$ color classes from which the neighbors of a^{i_1} are to be selected. There are $\binom{k-1}{r-1}$ ways to do this. Then we consider system (14). Assigning arbitrary values to $a_1^{i_2}, \ldots, a_1^{i_r}, a_2^{i_2}, \ldots, a_2^{i_{r-1}}$, one determines $a_2^{i_r}$ from the first equation. Then assigning arbitrary values to $a_3^{i_2}, \ldots, a_3^{i_{r-1}}$, one determines $a_3^{i_r}$ from the second equation. Continuing in this way one assigns arbitrary values to $a_j^{i_2}, \ldots, a_j^{i_{r-1}}$ and then determines $a_j^{i_r}$ from the jth equation. Therefore

$$\deg_{\mathcal{T}}(a^{i_1}) = \binom{k-1}{r-1} q^{(r-1)+(r-2)}(q^{r-2})^{d-2} = \binom{k-1}{r-1} q^{dr-2d+1},$$

and it is independent of a^{i_1}. Hence

$$|E(\mathcal{T})| = \frac{1}{r} \sum_{x \in V(\mathcal{T})} \deg_{\mathcal{T}}(x) = \frac{1}{r}(kq^d)\binom{k-1}{r-1} q^{dr-2d+1} = \binom{k}{r} q^{dr-d+1}.$$

This proves the first statement of the theorem.

Trying to determine the degree of a vertex of $\mathcal{K}$ in a similar way, we encounter a difficulty: an edge of $\mathcal{T}$ can contain two vertices of different colors which are equal as vectors, but this is not allowed for an edge of $\mathcal{K}$. Therefore we use another approach, namely induction on r.

Let a^{i_1} be a vertex of $\mathcal{K}$, and, for each $i = 2, \ldots, d$, let

$$f_i' = f_i'(x_2^1, \ldots, x_d^1, x_2^2, \ldots, x_d^2, \ldots, x_2^{i-1}, \ldots, x_d^{i-1}) =$$

$$f_i(a_1^{i_1}, x_2^1, \ldots, x_d^1, a_2^{i_1}, x_2^2, \ldots, x_d^2, \ldots, a_{i-1}^{i_1}, x_2^{i-1}, \ldots, x_d^{i-1}) - a_i^{i_1}.$$

The new functions f_i' are symmetric with respect to their variables, since the f_i are. If $r \geq 3$, we consider an $(r-1)$-graph $\mathcal{K}' = \mathcal{K}'(q, d, r-1, f_2', f_3', \ldots, f_d') = \mathcal{K}'(a^{i_1})$, with $V(\mathcal{K}') = V(\mathcal{K}) = \mathbf{F}^d$, and edges $\{a^{i_2}, \ldots, a^{i_r}\}$ defined by the following system:

$$\begin{aligned}
\sum_{j=2}^{r} a_2^{i_j} &= f_2'(a_1^{i_2}, \ldots, a_1^{i_r}) \\
\sum_{j=2}^{r} a_3^{i_j} &= f_3'(a_1^{i_2}, \ldots, a_1^{i_r}, a_2^{i_2}, \ldots, a_2^{i_r}) \\
\cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \\
\sum_{j=2}^{r} a_d^{i_j} &= f_d'(a_1^{i_2}, \ldots, a_1^{i_r}, a_2^{i_2}, \ldots, a_2^{i_r}, \ldots, a_{d-1}^{i_2}, \ldots, a_{d-1}^{i_r})
\end{aligned} \tag{14}$$

Removing vertex a^{i_1} from an edge $\{a^{i_1}, a^{i_2}, \ldots, a^{i_r}\}$ of $\mathcal{K}$ (incident with it), we obtain an edge $\{a^{i_2}, \ldots, a^{i_r}\}$ of $\mathcal{K}'$ not incident with a^{i_1}. This gives a bijection between the set of edges of $\mathcal{K}$

incident with a^{i_1} and the set of edges of $\mathcal{K}'$ which are not incident with a^{i_1}. Hence $\deg_{\mathcal{K}}(a^{i_1}) = |E(\mathcal{K}')| - \deg_{\mathcal{K}'}(a^{i_1})$. The independence of $|E(\mathcal{K}')|$ from both a^{i_1} and the functions which define $\mathcal{K}'$ is a part of the induction hypothesis. Therefore we can denote $|E(\mathcal{K}')|$ by e_{r-1}. Then for each $x \in V(\mathcal{K})$ and $r \geq 3$, we have

$$\deg_{\mathcal{K}}(x) = e_{r-1} - \deg_{\mathcal{K}'}(x). \tag{15}$$

Summing both sides of this equality over all $x \in V(\mathcal{K})$, we get

$$r|E(\mathcal{K})| = q^d e_{r-1} - (r-1)e_{r-1} = (q^d - r + 1)e_{r-1}. \tag{16}$$

This shows that $|E(\mathcal{K})|$ is independent of the functions f_i, and allows us to denote it by e_r. Thus

$$re_r = q^d e_{r-1} - (r-1)e_{r-1} = (q^d - r + 1)e_{r-1}. \tag{17}$$

Using this recurrence, one easily finds e_r if e_2 is known, and we concentrate on the base case.

For $2 = r \leq d$, the statement was proved in [12], but we present its proof here for completeness. Let $i_1 = 1$ and $i_2 = 2$. For a given $a^1 = (a^1_1, \ldots, a^1_d)$ we want to count the number of its neighbors in $\mathcal{K}$, i.e., the number of those $a^2 \neq a^1$ which are solutions of the system

$$\begin{aligned}
a^1_2 + a^2_2 &= f_2(a^1_1, a^2_1)\\
a^1_3 + a^2_3 &= f_3(a^1_1, a^2_1, a^1_2, a^2_2)\\
\cdots \quad \cdots &\quad \cdots \quad \cdots \quad \cdots\\
a^1_d + a^2_d &= f_d(a^1_1, a^2_1, a^1_2, a^2_2, \ldots, a^1_{d-1}, a^2_{d-1}).
\end{aligned} \tag{18}$$

Assigning an arbitrary value to a^2_1, defines a^2 uniquely, but in some cases it can be equal to a^1. The latter happens if and only if

$$a^1_i + a^2_i = 2a^1_i = f_i(a^1_1, a^1_1, a^1_2, a^1_2, \ldots, a^1_{i-1}, a^1_{i-1})$$

for all $i = 2, \ldots, d$. Since q is odd, each such a^1 is completely defined by a choice of a^1_1. Hence there are exactly q such a^1 and their degree in $\mathcal{K}$ is $q-1$. Therefore $\mathcal{K}$ contains $q^d - q$ vertices of degree q, and q vertices of degree $q-1$. Hence

$$|E(K)| = \frac{1}{2}\sum_{x \in V(\mathcal{K})} \deg_{\mathcal{K}}(x) = \frac{1}{2}((q^d - q)q + q(q-1)) = \frac{1}{2}(q^{d+1} - q) = \frac{1}{q^{d-1}}\binom{q^d}{2},$$

and is independent of the functions f_i. This establishes the base case.

Solving recurrence (18) with this value of e_2, we obtain

$$e_r = \frac{1}{r}(q^d - r + 1)e_{r-1} = \frac{1}{r}(q^d - r + 1) \cdot \frac{1}{r-1}(q^d - r + 2)e_{r-2} = \cdots = \frac{1}{q^{d-1}}\binom{q^d}{r}.$$

This proves the second statement of the theorem. □

The computation of e_2 in the above proof shows that the corresponding graph $\mathcal{K}$ is bi-regular. In general this is not true for $r \geq 3$. Nevertheless, it is true when $q = p$ is an odd prime, and the precise statement follows. In this case, the condition $(r, p) = 1$ implies $(r-1, p) = 1$. This allows us to prove the following theorem by induction on r. The proof is similar to the one above and we omit it.

Theorem 9. *Let p, d, r be integers, $2 \leq r < p$, $d \geq 2$, and p be a prime. Then $\mathcal{K}$ is a bi-regular r-graph of order p^d and size $\frac{1}{p^{d-1}}\binom{p^d}{r}$. It contains $p^d - p$ vertices of degree Δ and p vertices of degree $\Delta + (-1)^{r+1}$, where $\Delta = \frac{1}{p^{d-1}}\left(\binom{p^d-1}{r-1} + (-1)^r\right)$.* $\square$

Our further, and main results of this section, will be concerned with edge-decompositions of hypergraphs. Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs. An *edge-decomposition of $\mathcal{H}$ by $\mathcal{H}'$* is a collection $\mathcal{P}$ of subhypergraphs of $\mathcal{H}$, each isomorphic to $\mathcal{H}'$, such that $\{E(\mathcal{X}) \mid \mathcal{X} \in \mathcal{P}\}$ is a partition of $E(\mathcal{H})$.

We also say in this case that $\mathcal{H}'$ *decomposes* $\mathcal{H}$, and refer to the hypergraphs from $\mathcal{P}$ as *copies* of $\mathcal{H}'$.

Let $T^{(r)}_{kq^d}$, $2 \leq r \leq k$, $d \geq 1$, denote the complete k-partite r-graph with each partition class containing q^d vertices. This is a regular r-graph of order kq^d and size $\binom{k}{r}q^{dr}$, and the degree of each vertex is $\binom{k-1}{d-1}q^{dr-d}$.

As before, let $K^{(r)}_{q^d}$ denote the complete r-graph on q^d vertices.

A surprising feature of the following theorem is that it holds for *arbitrary* functions $f_2, \ldots, f_r$. The proof below is similar to the one for 2-graphs from [12].

Theorem 10. *Let q, d, r, k be integers, $2 \leq r \leq k$, $d \geq 2$, and q be a prime power. Then*

1. *$\mathcal{T} = \mathcal{T}(q, d, r, k, f_2, f_3, \ldots, f_d)$ decomposes $T^{(r)}_{kq^d}$.*

2. *$\mathcal{K} = \mathcal{K}(q, d, r, f_2, f_3, \ldots, f_d)$ decomposes $K^{(r)}_{q^d}$ provided that q is odd and $(r, q) = 1$.*

Proof. Clearly $\mathcal{T}$ spans $T^{(r)}_{kq^d}$, i.e., the two hypergraphs have the same vertex sets. We also assume that they share the same color classes. For each $\alpha = (\alpha_2, \ldots, \alpha_d) \in \mathbf{F}^{d-1}$, define the bijection $\phi_\alpha : V(T^{(r)}_{kq^d}) \to V(T^{(r)}_{kq^d})$ by

$$a^1 = (a^1_1, a^1_2, \ldots, a^1_d) \mapsto (a^1)^{\phi_\alpha} = (a^1_1, a^1_2 + \alpha_2, \ldots, a^1_d + \alpha_d) \in V^1 \quad \text{for all} \quad a^1 \in V^1,$$
$$\text{and} \quad x \mapsto x^{\phi_\alpha} = x \quad \text{for all} \quad x \in V^2 \cup V^3 \cup \ldots \cup V^k.$$

Define $\mathcal{T}^{\phi_\alpha}$ to be the subhypergraph of $T^{(r)}_{kq^d}$ having vertex set $V((\mathcal{T})^{\phi_\alpha}) = V(T^{(r)}_{kq^d})$ and edge set $E^{\phi(\alpha)} = E(\mathcal{T}^{\phi_\alpha}) = \{\{(x^1)^{\phi_\alpha}, \ldots, (x^r)^{\phi_\alpha}\} \mid \{x^1, x^2, \ldots, x^r\} \in E(\mathcal{T})\}$. It is immediate from the description of its edges that $\mathcal{T}^{\phi_\alpha}$ is isomorphic to $\mathcal{T}$, for each $\alpha \in \mathbf{F}^{d-1}$; indeed, ϕ_α is an explicit isomorphism. By Theorem 8, $|E(\mathcal{T})| = |E(\mathcal{T}^{\phi_\alpha})| = \binom{k}{r}q^{dr-d+1}$ for all $\alpha \in \mathbf{F}^{d-1}$. We verify that $\{E^{\phi(\alpha)} \mid \alpha \in \mathbf{F}^{d-1}\}$ is a partition of $E(T^{(r)}_{kq^d})$. Since there are q^{d-1} edges in $E^{\phi(\alpha)}$, and $|E(\mathcal{T})| = \frac{1}{q^{d-1}}|E(T^{(r)}_{kq^d})|$, it suffices to prove that the sets E^{ϕ_α} and E^{ϕ_β} are disjoint for each pair of distinct vectors $\alpha, \beta \in \mathbf{F}^{d-1}$.

Let $\{x^1, \ldots, x^r\} \in E^{\phi_\alpha} \cap E^{\phi_\beta}$. Then $\{x^1, \ldots, x^r\} = \{(a^1)^{\phi_\alpha}, \ldots, (a^r)^{\phi_\alpha}\} = \{(b^1)^{\phi_\beta}, \ldots, (b^r)^{\phi_\beta}\}$ for certain edges $\{a^1, \ldots, a^r\}$ and $\{b^1, \ldots, b^r\}$ of $E(\mathcal{T})$.

As each vertex from $V_2 \cup \cdots \cup V_d$ is fixed by both ϕ_α and ϕ_β, we immediately obtain $a^i = b^i$ for all $i = 2, \ldots, r$. Moreover, since

$$(a^1_1, a^1_2 + \alpha_2, \ldots, a^1_r + \alpha_d) = (a^1)^{\phi_\alpha} = (b^1)^{\phi_\beta} = (b^1_1, b^1_2 + \beta_2, \ldots, b^1_d + \beta_d),$$

we have $a^1_1 = b^1_1$. Therefore the edges $\{a^1, \ldots, a^r\}$ and $\{b^1, \ldots, b^r\}$ share $r-1$ vertices, and the remaining vertices a^1 and b^1 have equal first coordinates $a^1_1 = b^1_1$. The adjacency relation (14) implies that $a^1 = b^1$, and hence $\alpha = \beta$. This proves that the sets E^{ϕ_α} are indeed pairwise disjoint and that $\{E^{\phi_\alpha} \mid \alpha \in \mathbf{F}^{d-1}\}$ is a partition of $E(T^{(r)}_{kq^d})$, as claimed.

The proof of the second statement follows the same logic, but is a little different. Clearly $\mathcal{K}$ spans $K^{(r)}_{q^d}$. For each $\alpha = (\alpha_2, \ldots, \alpha_d) \in \mathbf{F}^{d-1}$, define the bijection $\psi_\alpha : V(K^{(r)}_{q^d}) \to V(K^{(r)}_{q^d})$ by

$$v = (v_1, v_2, \ldots, v_d) \mapsto (v)^{\psi_\alpha} = (v_1, v_2 + \alpha_2, \ldots, v_r + \alpha_d) \quad \text{for all} \quad v \in V(K^{(r)}_{q^d}).$$

Define $\mathcal{K}^{\psi_\alpha}$ to be the subhypergraph of $K^{(r)}_{q^d}$ having vertex set $V((\mathcal{K})^{\psi_\alpha}) = V(K^{(r)}_{q^d})$ and edge set $E^{\psi(\alpha)} = E(\mathcal{K}^{\psi_\alpha}) = \{((x^1)^{\psi_\alpha}, \ldots, (x^r)^{\psi_\alpha}) \mid (x^1, x^2, \ldots, x^r) \in E(\mathcal{K})\}$. It is immediate from the description of its edges that $\mathcal{K}^{\psi_\alpha}$ is isomorphic to $\mathcal{K}$, for each $\alpha \in \mathbf{F}^{d-1}$. We verify that $\{E^{\psi_\alpha} \mid \alpha \in \mathbf{F}^{d-1}\}$ is a partition of $E(K^{(r)}_{q^d})$. Since there are q^{d-1} edges in E^{ψ_α}, and, due to Theorem 8, $|E(\mathcal{K})| = \frac{1}{q^{d-1}}|E(K^{(r)}_{q^d})|$, the statement will follow from the fact that sets E^{ψ_α} and E^{ψ_β} are disjoint for each pair of distinct vectors $\alpha, \beta \in \mathbf{F}^{d-1}$. We prove this next.

Let $\{x^1, \ldots, x^r\} \in E^{\psi_\alpha} \cap E^{\psi_\beta}$. Then $\{x^1, \ldots, x^r\} = \{(a^1)^{\psi_\alpha}, \ldots, (a^r)^{\psi_\alpha}\} = \{(b^1)^{\psi_\beta}, \ldots, (b^r)^{\psi_\beta}\}$ for certain edges $a = \{a^1, \ldots, a^r\}$ and $b = \{b^1, \ldots, b^r\}$ of $E(\mathcal{K})$. Since the order of vertices in an edge does not matter, we may assume that $x^i = (a^i)^{\psi_\alpha} = (b^i)^{\psi_\beta}$ for all i. This gives

$$\begin{aligned} (a^1_1, a^1_2 + \alpha_2, \ldots, a^1_d + \alpha_d) &= (b^1_1, b^1_2 + \beta_2, \ldots, b^1_d + \beta_d) \\ (a^2_1, a^2_2 + \alpha_2, \ldots, a^2_d + \alpha_d) &= (b^2_1, b^2_2 + \beta_2, \ldots, b^2_d + \beta_d) \\ \cdots \quad \cdots &\quad \cdots \quad \cdots \quad \cdots \\ (a^r_1, a^r_2 + \alpha_2, \ldots, a^r_d + \alpha_d) &= (b^r_1, b^r_2 + \beta_2, \ldots, b^r_d + \beta_d) \end{aligned} \tag{19}$$

Our goal is to show that this implies $a = b$ and $\alpha = \beta$. We do it by induction on the number of the first t coordinates of the vertices (vectors) from a and b. More precisely, we prove that for all t, $1 \le t \le d$, $(a^i_1, a^i_2, \ldots, a^i_t) = (b^i_1, b^i_2, \ldots, b^i_t)$ for all i, $i = 1, 2, \ldots, r$.

The base step, $t = 1$, is obviously satisfied: (20) implies that $a_1^i = b_1^i$ for all $i = 1, \ldots, r$. Suppose the first t, $1 \leq t < d$, coordinates of a and b, and the first $t-1$ coordinates of α and β (here $2 \leq t < d$) are equal. Now, since both a and b are edges of $\mathcal{K}$, we have

$$\sum_{j=2}^{r} a_{t+1}^j = f_i(a_1^1, \ldots, a_1^r, \ldots, a_t^1, \ldots, a_t^r) \quad \text{and} \quad \sum_{j=2}^{r} b_{t+1}^j = f_i(b_1^1, \ldots, b_1^r, \ldots, b_t^1, \ldots, b_t^r).$$

The right hand sides of these two equalities are equal by the induction hypothesis. Therefore $\sum_{j=2}^r a_{t+1}^j = \sum_{j=2}^r b_{t+1}^j$. Adding the $(t+1)$-st coordinates of vectors in both sides of (20), we get $\sum_{j=2}^r a_{t+1}^j + r\alpha_{t+1} = \sum_{j=2}^r b_{t+1}^j + r\beta_{t+1}$. Hence $r\alpha_{t+1} = r\beta_{t+1}$, and, since $(r,q) = 1$, $\alpha_{t+1} = \beta_{t+1}$. Using (20) again, we have $a_{t+1}^i + \alpha_{t+1} = b_{t+1}^i + \beta_{t+1}$ for all i. Therefore $a_{t+1}^i = b_{t+1}^i$ for all i, and the induction is finished. Hence $\alpha = \beta$ and $a = b$.

This proves that the sets E^{ψ_α}, are indeed pairwise disjoint, and $\{E(\mathcal{K}^{\psi_\alpha}) \mid \alpha \in \mathbf{F}^{d-1}\}$ is a partition of $E(K_{q^d}^{(r)})$, as claimed. □

As an immediate corollary of this theorem one obtains constructive lower bounds for the Ramsey numbers.

Corollary 11. *Let $\mathcal{H}$ be any r-uniform hypergraph which is not a subhypergraph in $\mathcal{K} = \mathcal{K}(q, d, r, f_2, f_3, \ldots, f_d)$. Let $k = q^{d-1}$, q be odd and $(r,q) = 1$. Then*

$$r_k(\mathcal{H}) \geq q^d + 1 = k^{d/(d-1)} + 1.$$

Remark. The construction used in the proof of Theorem 4 can be viewed as a modification of the definition of $\mathcal{K}$. Let $d = r = 2$, $f_2 = f_2(a_1^{i_1}, a_1^{i_2}) = a_1^{i_1} a_1^{i_2}$. Let H be a subgroup of the additive group of the field $\mathbf{F}$, and let $x \bmod H$ denotes the coset in $(\mathbf{F} : H)$ containing x. Simplifying notations, we consider the hypergraph with the vertex set $\{(a,b) : a \in \mathbf{F},\ b \in (\mathbf{F} : H)\}$. Define an edge as an r-subset of vertices $\{(a_i, b_i) : i = 1, \ldots, r\}$ such that $b_1 + b_2 + \ldots + b_r = a_1 \cdot a_2 \cdot \ldots \cdot a_r \bmod H$. The coloring used in the proof of Theorem 4 is obtained by letting α_2 from the proof of Theorem 10 (part 2) vary over all elements of $(\mathbf{F} : H)$.

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