On the Turán number of Triple-Systems

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Abstract

For a family of r-graphs $\mathcal{F}$, the Turán number $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n vertex r-graph that does not contain any member of $\mathcal{F}$. The Turán density

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

When $\mathcal{F}$ is an r-graph, $\pi(\mathcal{F}) \neq 0$, and $r > 2$, determining $\pi(\mathcal{F})$ is a notoriously hard problem, even for very simple r-graphs $\mathcal{F}$. For example, when $r = 3$, the value of $\pi(\mathcal{F})$ is known for very few (< 10) irreducible r-graphs.

Building upon a method developed recently by de Caen and Füredi [3], we determine the Turán densities of several 3-graphs that were not previously known. Using this method, we also give a new proof of a result of Frankl and Füredi [5] that $\pi(\mathcal{H}) = 2/9$, where $\mathcal{H}$ has edges $123, 124, 345$.

Let $\mathcal{F}(3, 2)$ be the 3-graph $123, 145, 245, 345$, let $\mathcal{K}_4^-$ be the 3-graph $123, 124, 234$, and let $\mathcal{C}_5$ be the 3-graph $123, 234, 345, 451, 512$. We prove

- $4/9 \ \leq\ \pi(\mathcal{F}(3, 2)) \ \leq\ 1/2$,
- $\pi(\{\mathcal{K}_4^-, \mathcal{C}_5\}) \ \leq\ 10/31 = 0.322581$,
- $0.464 \ <\ \pi(\mathcal{C}_5) \ \leq\ 2 - \sqrt{2} < 0.586$.

The middle result is related to a conjecture of Frankl and Füredi [6] that $\pi(\mathcal{K}_4^-) = 2/7$. The best known bounds are $2/7 \leq \pi(\mathcal{K}_4^-) \leq 1/3$.

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1 Introduction

Given a family of r-uniform hypergraphs (or r-graphs) $\mathcal{F}$, we say that an r-graph $\mathcal{G}$ is $\mathcal{F}$-free if $\mathcal{G}$ contains no subhypergraph isomorphic to any element in $\mathcal{F}$. The Turán number $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n vertex $\mathcal{F}$-free r-graph. In the case of complete r-graphs, it is easy to show that $t(r, s) = \lim_{n\to\infty} \mathrm{ex}(n, K_s^{(r)})/\binom{n}{r}$ exists, but determining it is perhaps the most fundamental open problem in extremal hypergraph theory. In fact, the numbers $t(r, s)$ are not known for any $s > r \geq 3$.

Research in this subject has therefore focused on determining Turán numbers for other hypergraphs. Throughout we focus on 3-graphs with *nondegenerate* Turán numbers, i.e., those whose Turán number is not $o(n^3)$. Indeed, for such a 3-graph $\mathcal{H}$, we let $\pi(\mathcal{H}) = \lim \mathrm{ex}(n, \mathcal{H})/\binom{n}{3}$ be its *Turán density.*

A recent breakthrough in this area was the determination of the Turán density of the Fano plane. Sós [11] conjectured over 20 years ago that this density is 3/4 and gave an example showing that this is a lower bound. de Caen and Füredi [3] proved this conjecture by a remarkably simple and short proof. The purpose of this paper is to expand on the ideas in their paper and use these techniques to compute the Turán density of several other 3-graphs. We also prove new best known bounds for several other Turán densities. For a positive integer k, we let $[k] = \{1, \ldots, k\}$, and for a set S, we write $\binom{S}{k}$ for the family of subsets of S of size k.

Definition 1.1. *Let $p, q > 0$. Then $\mathcal{F}(p, q)$ is the 3-graph with vertex set $P \cup Q$, where $P = [p]$ and $Q = [p + q] - [p]$, and edge set $\binom{P}{3} \cup \{xyz : x \in P, y, z \in Q\}$. Let $\mathcal{F}(5)$ be the 3-graph with edges* $123, 124, 345$.

As a generalization of the simplest case of Turán's theorem for graphs (this is referred to as Mantel's Theorem), Katona [9] proposed the problem of determining the maximum number of edges in an r-graph that contains no three edges A, B, C such that $A \triangle B \subseteq C$. The complete r-partite r-graph with parts as equal as possible shows that the Turán density of this family is at least $r!/r^r$. Bollobás [1] proved equality for the case $r = 3$, and Sidorenko [10] proved equality for the case $r = 4$.

When $r = 3$, there are two forbidden subgraphs in Katona's problem, $\mathcal{F}(5)$ and $\mathcal{F}(1, 3)$. Given Bollobás' result, it seems natural to refine his theorem by asking for the Turán density of these individual 3-graphs. Determining $\pi(\mathcal{F}(1, 3))$ is a well-studied open problem. de Caen proved an upper bound of 1/3 while Frankl and Füredi [6] gave a construction yielding a lower bound of 2/7. Frankl and Füredi [5] also proved

Theorem 1.2.
$$\mathrm{ex}(n, \mathcal{F}(5)) = \left\lceil \frac{n}{3} \right\rceil \left\lceil \frac{n+1}{3} \right\rceil \left\lceil \frac{n+2}{3} \right\rceil$$
for $n \geq 3000$*. In particular, this implies that*
$$\pi(\mathcal{F}(5)) = 2/9. \tag{1}$$

Our first application of the de Caen-Füredi method (Section 2) is to give another proof of (1). Although this is weaker than what is already known, we believe that it nicely illustrates the method. In section 3 we determine the Turán density of several other 3-graphs.

Definition 1.3. *Let* $\mathcal{F}'(3,3)$ *be the 3-graph made up of a copy of* $\mathcal{F}(3,3)$ *with vertices labeled as in Definition 1.1, two additional vertices, 7, 8, and four additional edges* $178, 278, 478, 578$*. Let* $\mathcal{F}''(3,3)$ *be obtained from* $\mathcal{F}'(3,3)$ *by adding two additional vertices, 9,a, and three additional edges,* $19a, 49a, 79a$*. Let* $\mathcal{F}^{-}(4,3)$ *be the 3-graph obtained from* $\mathcal{F}(4,3)$ *by deleting the edge* 156*. Let* $\mathcal{F}^{-'}(4,3)$ *be obtained from* $\mathcal{F}^{-}(4,3)$ *by adding two vertices* $8,9$*, and adding three edges* $289, 389, 589$*.*

By a relatively direct application of the de Caen-Füredi method we can prove

Theorem 1.4. (Section 3) *Let* $S = \{\mathcal{F}(3,3), \mathcal{F}'(3,3), \mathcal{F}''(3,3), \mathcal{F}^{-}(4,3), \mathcal{F}^{-'}(4,3)\}$*. Let* $\mathcal{A}, \mathcal{B} \in S$ *(possibly* $\mathcal{A} = \mathcal{B}$*), and* $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{B}$*. Then* $\pi(\mathcal{F}) = 3/4$*.*

The Turán density $\pi(\mathcal{F}(3,3)) = 3/4$ has been independently proven (using essentially the same proof) by J. Goldwasser.

For vertices x and y in a hypergraph $\mathcal{H}$, we write $x \sim y$ if
i) $xyu \notin E(\mathcal{H})$ for all u
ii) $xuv \in E(\mathcal{H}) \iff yuv \in E(\mathcal{H})$.
We say that $\mathcal{H}$ is *irreducible* if there is no pair x, y with $x \sim y$.

Let $x, y \in V(\mathcal{H})$ with $x \sim y$, and let $\mathcal{H}' = \mathcal{H} - x$. It is well-known that $\pi(\mathcal{H}) = \pi(\mathcal{H}')$, thus it makes sense to ask only for the Turán density of irreducible hypergraphs. The Turán density of very few (< 10, see the survey by Füredi [7]) irreducible 3-graphs is known. Theorem 1.4 determines the Turán density of several irreducible 3-graphs not in S, thus giving many Turán densities not previously known.

We try to use the same technique to determine $\pi(\mathcal{F}(3,2))$. Although we are not successful, we obtain reasonably good bounds.

Theorem 1.5. (Section 4) $4/9 \leq \pi(\mathcal{F}(3,2)) \leq 1/2$*.*

We believe that the upper bound in Theorem 1.5 can be improved.

Conjecture 1.6. $\pi(\mathcal{F}(3,2)) = 4/9$.

Definition 1.7. *The complete 3-graph on four vertices is $\mathcal{K}_4$. The 3-graph obtained from $\mathcal{K}_4$ by deleting a single edge is $\mathcal{K}_4^-$. The Pentagon $\mathcal{C}_5$ is the 3-graph with vertex set $[5]$ and edge set $\{123, 234, 345, 451, 512\}$.*

Perhaps the most famous conjecture in this area is that $\pi(\mathcal{K}_4) = 5/9$. It is known that $5/9 \leq \pi(\mathcal{K}_4) \leq (3+\sqrt{17}\,)/12 = .59359..$, where the lower bound is due to Turán [4] and the upper bound is due to Chung and Lu [2]. However, as mentioned earlier, even the Turán density $\pi(\mathcal{K}_4^-)$ is not known (note that $\mathcal{K}_4^- = \mathcal{F}(1,3)$). In some sense this is an even more fundamental problem, since $\mathcal{K}_4^-$ is the smallest (in every sense) 3-graph with positive Turán density.

The upper bound $\pi(\mathcal{K}_4^-) \leq 1/3$ was proved by de Caen, and Frankl and Füredi gave a fairly complicated recursive construction yielding $\pi(\mathcal{K}_4^-) \geq 2/7$. Although we are unable to improve upon these, we prove that density greater than $10/31$ ($< 0.323 < 1/3$) forces a copy of either $\mathcal{K}_4^-$ or a Pentagon, which is much harder to force by itself.

Theorem 1.8. (Section 5) $\quad \pi(\{\mathcal{K}_4^-, \mathcal{C}_5\}) \; \leq \; 10/31 = 0.322581.$

Given Theorem 1.8, it is natural to ask for the Turán density of the Pentagon. Also, the Pentagon is the smallest self-complementary 3-graph with positive Turán density (indeed, the only one on fewer than six vertices), and it therefore would be nice to determine $\pi(\mathcal{C}_5)$. Again, we are only able to give reasonable bounds. Our techniques are motivated by the de Caen-Füredi method, but are slightly different.

Theorem 1.9. (Section 6) $\quad 0.464 \; < \; \pi(\mathcal{C}_5) \; \leq \; 2 - \sqrt{2} < 0.586.$

Comparing Theorem 1.8, the results on $\pi(\mathcal{K}_4^-)$, and Theorem 1.9, a natural question that arises is whether the Turán density of a family of two 3-graphs is less than the minimum of the Turán densities of the individual 3-graphs. For graphs, this is not true, indeed, the famous Erdős-Stone-Simonovits theorem implies that the Turán density of a family of graphs is equal to the minimum of the Turán densities of the individual graphs in the family. We think that this phenomenon does not extend to hypergraphs.

Conjecture 1.10. [1] *There is a positive integer k and 3-graphs $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that*

$$\pi(\{\mathcal{F}_1, \ldots, \mathcal{F}_k\}) < \min_i \; \pi(\mathcal{F}_i).$$

We believe that Conjecture 1.10 should be true even with $k = 2$.

[1]Recently J. Balogh has announced a positive answer to this Conjecture

2 A new proof of an old result

Definition 2.1. *Let $\mathcal{H}$ be a 3-graph and $S \subseteq V(\mathcal{H})$. The link multigraph of S in $\mathcal{H}$ is the multigraph G with $V(G) = V(\mathcal{H}) - S$, and $E(G) = \{uv : xuv \in E(\mathcal{H})$ for some $x \in S\}$. The multiplicity of uv is the number of $x \in S$ with $xuv \in E(\mathcal{H})$.*

Proof of (1): For the lower bound, observe that the complete 3-partite 3-graph with parts of as equal size as possible is $\mathcal{F}(5)$-free. This 3-graph has density $2/9$.

For the upper bound, suppose that $\mathcal{G}$ is a 3-graph with n vertices and $(2/9)\binom{n}{3} + cn^2$ edges for some sufficiently large constant c ($c = 10$ suffices for our purposes, but we make no attempt to optimize its value). We must show that $\mathcal{G}$ contains a copy of $\mathcal{F}(5)$. Proceeding by induction, we may assume that the result holds for graphs with fewer than n vertices. We note that to apply the induction hypothesis, we need only find a vertex in $\mathcal{G}$ of degree at most $(2/9)\binom{n-1}{2} + c(2n-1)$.

By Bollobás' Theorem [1], $\mathcal{G}$ contains a copy of either $\mathcal{F}(5)$ or $\mathcal{F}(1,3)$. We may therefore assume that $\mathcal{G}$ contains a copy $\mathcal{K}_4^-$ of $\mathcal{F}(1,3)$. Consider the link multigraph G of $S = V(\mathcal{K}_4^-)$ in $\mathcal{G}$.

If G is not simple, then some pair xy appears twice, which means that we have triples axy, bxy in $\mathcal{G}$, where $a, b \in S$. Because every two vertices of $\mathcal{K}_4^-$ lie in an edge in $\mathcal{K}_4^-$, there is another triple abc with $a \in S$. Now the vertices a, b, c, x, y induce a subhypergraph containing a copy of $\mathcal{F}(5)$. We may therefore assume that G is simple.

As mentioned before, we may also assume that every vertex in S has degree in $\mathcal{G}$ greater than $(2/9)\binom{n-1}{2} + c(2n-1)$, otherwise we can apply induction. This yields at least $e = (8/9)\binom{n-1}{2} + 4c(2n-1) - 6(n-4) - 3$ edges in G, where the last two terms count edges with at least two vertices in S. Now we apply Turán's graph theorem to G: $e > (4/9)(n-4)^2$ edges force a subgraph $K_{10} \subseteq G$.

Suppose that there are $x, y, z \in V(G)$, and $a, b \in S$ with edges xya, yza, xzb. Then x, y, z, a, b induce a hypergraph containing $\mathcal{F}(5)$. We may therefore assume that for every $x, y, z \in V(G)$, there do not exist a, b with this property.

Pick a vertex v_1 in K_{10}. Edges incident to v_1 in K_{10} come from triples including one vertex in S. Hence we must have at least three edges from the same vertex of S, say $av_1v_2, av_1v_3, av_1v_4$. The observation in the previous paragraph implies that we also have triples $av_2v_3, av_2v_4, av_3v_4$. Now the vertices $a, v_1, \dots, v_4$ form a copy of $\mathcal{F}(1,4)$.

Consider the link multigraph G' formed by the vertex set of the copy of $\mathcal{F}(1,4)$ in $\mathcal{G}$. Since $\mathcal{F}(1,4)$ has the property that every two vertices lie in an edge, we may again assume that G' is simple. But this forces some vertex of the copy of $\mathcal{F}(1,4)$ to have degree (in $\mathcal{G}$) at most $(2/9)\binom{n-1}{2}+$

$c(2n-1)$ since G' cannot contain $(10/9)\binom{n-1}{2}+5c(2n-1)-10(n-5)-6 > \binom{n-5}{2}$ edges. Removing this vertex, we may apply induction. $\square$

3 More Turán densities

To prove Theorem 1.4, we need a special case of a recent result of Füredi and Kündgen [8]. Its proof appears in [3].

Lemma 3.1. *Let G be an n vertex multigraph with every three vertices spanning at most ten edges. Then G has at most $3\binom{n}{2}+n-2$ edges.*

Proof of Theorem 1.4: Recall that $\mathcal{A}, \mathcal{B} \in S$ and $\mathcal{A} \subseteq \mathcal{F} \subseteq \mathcal{B}$. We first argue that if $\chi(\mathcal{A}) > 2$, and $\pi(\mathcal{B}) \leq 3/4$, then $\pi(\mathcal{F}) = 3/4$.

Consider the n vertex 3-graph $\mathcal{H}$ obtained by splitting the vertices into two sets of size $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$; the edges are all triples that meet both these sets. $\mathcal{H}$ is clearly 2-colorable, and has $(3/4+o(1))\binom{n}{3}$ edges, hence it does not contain $\mathcal{A}$, therefore it does not contain $\mathcal{F}$. This shows that $\pi(\mathcal{F}) \geq 3/4$. For the other inequality we only need to observe that $\mathcal{F} \subseteq \mathcal{B}$.

It is easy to see that $\chi(\mathcal{F}(3,3)) > 2$. Since $\mathcal{F}(3,3)$ is a subgraph of every 3-graph considered in the Theorem, it suffices to prove that the Turán density of each 3-graph $\mathcal{F}$ in S is at most $3/4$.

Suppose that $\mathcal{G}$ is a 3-graph with more than $3/4\binom{n}{3}+cn^2$ edges for some appropriate constant c ($c = 10$ suffices). We will prove by induction on n that $\mathcal{G}$ contains a copy of $\mathcal{F}$. In order to do this, we need to only find a vertex of degree at most $(3/4)\binom{n-1}{2}+c(2n-1)$.

- $\mathcal{F} = \mathcal{F}^-(4,3)$: It is easy to see (by averaging) that $\mathcal{G}$ contains a copy K of $\mathcal{K}_4$. Let G be the link multigraph of $S = V(K)$ in $\mathcal{G}$. If some three vertices of G span at least eleven edges, then these three vertices together with K form a copy of $\mathcal{F}$. Hence we may assume that every three vertices in G span at most ten edges. Lemma 3.1 now implies that G has at most $3\binom{n-4}{2}+n-6$ edges. Hence some $v \in S$ has $d_{\mathcal{G}}(v) \leq (3/4)\binom{n-4}{2}+(n-6)/4+3(n-4)+4 < (3/4)\binom{n-1}{2}+c(2n-1)$.

- $\mathcal{F} = \mathcal{F}^{-'}(4,3)$: We have proved that $\mathcal{G}$ contains a copy K of $\mathcal{F}^-(4,3)$. Let G be the link multigraph of $S = V(K)$ in $\mathcal{G}$. If some pair of vertices of G span at least six edges, then this pair together with K forms a copy of $\mathcal{F}$. Hence we may assume that every pair of vertices in G spans at most five edges. Thus G has at most $5\binom{n-7}{2}$ edges. Hence some $v \in S$ has $d_{\mathcal{G}}(v) < (5/7)\binom{n-7}{2}+6(n-7)+11 < (3/4)\binom{n-1}{2}+c(2n-1)$.

- $\mathcal{F} = \mathcal{F}(3,3)$: It is enough to observe that $\mathcal{F} \subseteq \mathcal{F}^-(4,3)$.

• $\mathcal{F} = \mathcal{F}'(3,3)$: We have proved that $\mathcal{G}$ contains a copy K of $\mathcal{F}(3,3)$. Let G be the link multigraph of $S = V(K)$ in $\mathcal{G}$. If some pair of vertices of G span at least five edges, then these two vertices together with K form a copy of $\mathcal{F}$. Hence we may assume that every pair of vertices in G span at most four edges. Therefore G has at most $4\binom{n-6}{2}$ edges. Hence some $v \in S$ has $d_{\mathcal{G}}(v) < (4/6)\binom{n-6}{2} + 5(n-6) + 9 < (3/4)\binom{n-1}{2} + c(2n-1)$.

• $\mathcal{F} = \mathcal{F}''(3,3)$: We have proved that $\mathcal{G}$ contains a copy K of $\mathcal{F}'(3,3)$. Let G be the link multigraph of $S = V(K)$ in $\mathcal{G}$. If some pair of vertices of G span at least seven edges, then these two vertices together with K form a copy of $\mathcal{F}$. Hence we may assume that every pair of vertices in G span at most six edges. Therefore G has at most $6\binom{n-8}{2}$ edges. Hence some $v \in S$ has $d_{\mathcal{G}}(v) < (6/8)\binom{n-8}{2} + 7(n-8) + 13 < (3/4)\binom{n-1}{2} + c(2n-1)$. □

4 Proof of Theorem 1.5

We need some preliminaries to prove Theorem 1.5.

Definition 4.1. *Let $\mathcal{F}_2$ be the 3-graph obtained from $\mathcal{F}(1,3)$ by duplicating the vertex in P. For $t > 2$, let $\mathcal{F}_t$ be the family of all 3-graphs obtained as follows: let $\mathcal{F} \in \mathcal{F}_{t-1}$, add two new vertices, x, y and add any set of t edges of the form axy, where a lies in $\mathcal{F}$.*

The family $\mathcal{F}_t$ will play a crucial role in the proof of the upper bound of Theorem 1.5. We develop some properties of these families.

Proposition 4.2. *Let $\mathcal{F} \in \mathcal{F}_t$. Then*
1) $\mathcal{F}$ has $2t+1$ vertices
2) Every set of $t+2$ vertices in $\mathcal{F}$ spans at least one edge
3) $\pi(\mathcal{F}_2) \le 1/3$, and $\pi(\mathcal{F}_t) \le 1/2$ for $t \ge 3$.

Proof. 1) $\mathcal{F}_2$ has five vertices, and to form a 3-graph in $\mathcal{F}_t$, we add two vertices to a 3-graph in $\mathcal{F}_{t-1}$.

2) We proceed by induction on t. It is easy to verify it directly for $t = 2$. Suppose that the result holds for $t-1$. Let $\mathcal{F}' \in \mathcal{F}_t$ be obtained from $\mathcal{F} \in \mathcal{F}_{t-1}$ by adding vertices x, y and a set of t edges involving x, y. Let S be a set of $t+2$ vertices in $\mathcal{F}'$. If $|S \cap V(\mathcal{F})| \ge t+1$, then by induction, the $t+1$ vertices in $S \cap V(\mathcal{F})$ span an edge. Otherwise, $|S \cap V(\mathcal{F})| = t$. But now both $x, y \in S$, and since there are fewer than $2t$ vertices in $V(\mathcal{F})$, we have an edge of the form xya, where $a \in S$.

3) de Caen proved that $\pi(\mathcal{F}(1,3)) \le 1/3$, and it is well known that the same upper bound holds if we duplicate a vertex. Since $\mathcal{F}_2$ is formed by duplicating a vertex, the same upper bound holds

for $\pi(\mathcal{F}_2)$. Since $1/3 < 1/2$, we conclude that there is a constant $c > 2t$ such that $\mathrm{ex}(n, \mathcal{F}_2) \leq (1/2)\binom{n}{3} + cn^2$ for all n.

We will prove by induction on t that $\mathrm{ex}(n, \mathcal{F}_t) \leq (1/2)\binom{n}{3} + cn^2$ for all n. The base case is $t = 2$ above. Assume that the result holds for $t-1$, and let $\mathcal{G}$ be a 3-graph with n vertices and at least $(1/2)\binom{n}{3} + cn^2$ edges. We will prove that $\mathcal{G}$ contains a copy of some 3-graph in $\mathcal{F}_t$ by induction on n. The result is true for $n = 2t+1$ by the choice of c. Assume that the result holds for $n-1$.

By induction on t, $\mathcal{G}$ contains a copy $\mathcal{H}$ of a member $\mathcal{F} \in \mathcal{F}_{t-1}$. Let G be the link multigraph of $V(\mathcal{H})$ in $\mathcal{G}$. If G has a pair of vertices with multiplicity at least t, then this pair together with $\mathcal{H}$ yields a copy of some $\mathcal{F}' \in \mathcal{F}_t$. Hence we may assume that G has multiplicity at most $t-1$. But then there is a vertex v in $\mathcal{H}$ with

$$d_{\mathcal{G}}(v) \leq \frac{t-1}{2t-1}\binom{n-2t+1}{2} + (2t-2)(n-2t+1) + \binom{2t-1}{2} < \frac{t-1}{2t-1}\binom{n-1}{2} + 2tn. \qquad (2)$$

Removing v leaves a 3-graph $\mathcal{G}'$ on $n-1$ vertices with $|E(\mathcal{G}')|$ at least

$$|E(\mathcal{G})| - \frac{t-1}{2t-1}\binom{n-1}{2} - 2tn \geq \frac{1}{2}\binom{n}{3} + cn^2 - \frac{1}{2}\binom{n-1}{2} - 2tn > \frac{1}{2}\binom{n-1}{3} + c(n-1)^2.$$

By induction on n, $\mathcal{G}'$ contains a copy of some 3-graph in $\mathcal{F}_t$. $\square$

Proof of Theorem 1.5: For the lower bound, let $\mathcal{G}$ be the 3-graph with vertex set $X \cup Y$, where X and Y are disjoint, $|X| = \lfloor 2n/3 \rfloor$, and $|Y| = \lceil n/3 \rceil$. Let the edge-set of $\mathcal{G}$ be $\{xx'y : x, x' \in X, y \in Y\}$. It is easy to see that $\mathcal{G}$ has $(4/9 + o(1))\binom{n}{3}$ edges and contains no copy of $\mathcal{F}(3,2)$.

For the upper bound, pick $\epsilon > 0$, and let $t \geq 2$ be sufficiently large that $1/2+\epsilon > (t+1)/(2t+1)$. As in the proof of Proposition 4.2 part 3), let $c > 2t$ be such that $\mathrm{ex}(n, \mathcal{F}_t) \leq (1/2)\binom{n}{3} + cn^2$ for all $n > 4$. Let $\mathcal{G}$ be a 3-graph with n vertices and $(1/2 + \epsilon)\binom{n}{3} + cn^2$ edges. We will show that $\mathcal{G}$ contains a copy of $\mathcal{F}(3,2)$. Since this will be shown for each $\epsilon > 0$, the result follows.

We proceed by induction on n (the base case, $n = 5$, holds since $c > 2t \geq 4$). By the choice of c, $\mathcal{G}$ contains a copy K of some $\mathcal{F} \in \mathcal{F}_t$. Form the link multigraph G for K. If there is a pair x, y in G with multiplicity at least $t+2$, then, by Proposition 4.2 part 2), these two vertices together with some three of the $t+2$ vertices in K form a copy of $\mathcal{F}(3,2)$. Hence we may assume that G has multiplicity at most $t+1$. Similarly as in (2), this implies that some vertex v in K has $d_{\mathcal{G}}(v) < (t+1)/(2t+1)\binom{n-1}{2} + 2tn$. Removing v leaves a 3-graph $\mathcal{G}'$ on $n-1$ vertices with $|E(\mathcal{G}')|$ at least

$$|E(\mathcal{G})| - \frac{t+1}{2t+1}\binom{n-1}{2} - 2tn \geq \left(\frac{1}{2} + \epsilon\right)\binom{n}{3} + cn^2 - \frac{t+1}{2t+1}\binom{n-1}{2} - 2tn.$$

By the choice of t, this is at least

$$\left(\frac{1}{2}+\epsilon\right)\binom{n}{3}+cn^2-\left(\frac{1}{2}+\epsilon\right)\binom{n-1}{2}-2tn \geq \left(\frac{1}{2}+\epsilon\right)\binom{n-1}{3}+c(n-1)^2.$$

By the induction hypothesis, $\mathcal{G}'$ contains a copy of $\mathcal{F}(3,2)$. □

5 Either $\mathcal{K}_4^-$, or $\mathcal{C}_5$

We need two preliminary results for the proof of Theorem 1.8.

Lemma 5.1. *Suppose that $n, p_1, \ldots, p_k$ are all nonnegative with $\sum_i p_i = n$. Then*

$$\frac{\sum_i p_i^2}{4}+\sum_{i<j} p_i p_j \leq \frac{n^2}{4k}+\binom{k}{2}\frac{n^2}{k^2}\ .$$

Proof.

$$\frac{\sum_i p_i^2}{4}+\sum_{i<j} p_i p_j = \left(\sum_i \frac{p_i}{2}\right)^2+\sum_{i<j}\frac{p_i p_j}{2} = \frac{n^2}{4}+\sum_{i<j}\frac{p_i p_j}{2}.$$

It is easy to see by calculus that the final sum is maximized when the p_i's are all equal to n/k. This gives

$$\frac{\sum_i p_i^2}{4}+\sum_{i<j} p_i p_j \leq \frac{n^2}{4}+\sum_{i<j}\frac{n^2}{2k^2} = \frac{n^2}{4}+\binom{k}{2}\frac{n^2}{2k^2} = \frac{n^2}{4k}+\binom{k}{2}\frac{n^2}{k^2}.$$ □

Lemma 5.2. *Let G be a graph with vertex partition $A \cup B$, and $|A| \geq |B|-1$. Suppose that A is an independent set in G. If G is triangle free, then $|E(G)| \leq |A||B|$.*

Proof. Let $a = |A|$ and $b = |B|$. Suppose that $G[B]$ has s edges and G has $ab-r$ edges between A and B. We count triples u, v, w, where $u \in A$, $v, w \in B$, vw is an edge, and uv is not an edge.

Fix an edge xy in $G[B]$. For each $z \in A$, at least one of zx, zy is missing, since G is triangle-free. On the other hand, for each non-edge zx, $z \in A, x \in B$, there are at most $b-1$ vertices y in B with xy being an edge. Consequently, if t is the number of triples we are counting, then

$$sa \leq t \leq r(b-1).$$

Because $a \geq b-1$, we have $r \geq s$, and $|E(G)| = (ab-r)+s \leq ab$. □

Proof of Theorem 1.8: Let $\alpha = 10/31$, and suppose that $\mathcal{H}$ is a 3-graph with at least $\alpha\binom{n}{3}+cn^2$ edges for some sufficiently large constant c (for our purposes, $c = 10$ will do but we make no attempt to optimize c). We will prove by induction on n that $\mathcal{H}$ contains a copy of either $\mathcal{K}_4^-$, or $\mathcal{C}_5$. It therefore suffices to find a vertex in $\mathcal{H}$ of degree at most $\alpha\binom{n-1}{2} + c(2n-1)$.

Given vertices x, y, we let $N_{xy} = \{z : xyz \text{ is an edge}\}$, and let $d_{xy} = |N_{xy}|$. For an edge $e = xyz$, let

$$\mu(e) = d_{xy} + d_{yz} + d_{xz}.$$

If $\mu(e) > n$, then there is a vertex w in at least two of the sets N_{xy}, N_{xz}, N_{yz}, and $S = \{x, y, z, w\}$, contains a copy of $\mathcal{K}_4^-$. We may therefore assume that $\mu(e) \leq n$ for every edge e. Define $\epsilon > 0$ by

$$\max_{e \in E(\mathcal{H})} \mu(e) = (1-\epsilon)n. \tag{3}$$

Using $\sum_{u,v \in V(\mathcal{H})} d_{uv} = 3|E(\mathcal{H})|$, the upper bound from (3) on $\mu(e)$, and convexity of binomial coefficients, we obtain

$$|E(\mathcal{H})|(1-\epsilon)n \geq \sum_{e \in E(\mathcal{H})} \mu(e) \geq \sum_{u,v \in V(\mathcal{H})} 2\binom{d_{uv}}{2} \geq 2\binom{n}{2}\binom{\frac{3|E(\mathcal{H})|}{\binom{n}{2}}}{2}.$$

Together with the choice of c this implies that

$$\alpha\binom{n}{3} + cn^2 \leq |E(\mathcal{H})| \leq \frac{1}{3}\left[\frac{1-\epsilon}{3}n\binom{n}{2} + \binom{n}{2}\right] \leq \frac{1-\epsilon}{3}\binom{n}{3} + \frac{c}{2}n^2,$$

and therefore

$$\alpha < (1-\epsilon)/3. \tag{4}$$

Note that $\alpha = 10/31$ and (4) imply that $\epsilon < 1 - 3\alpha = 1/31 < 1/2$.

Let $e = uvw$ be an edge with $\mu(e) = (1-\epsilon)n$. Let G_u (resp. G_v, G_w) be the link graph of u (resp. v, w) in $\mathcal{H} - \{u, v, w\}$. Let $A = N_{uv} \cup N_{uw} \cup N_{vw} - \{u, v, w\}$ and let $B = V(\mathcal{H}) - A - \{u, v, w\}$. From now we also suppose that $\mathcal{H}$ contains neither $\mathcal{K}_4^-$ nor $\mathcal{C}_5$, and proceed toward a contradiction. We prove a series of three Claims.

Claim 1: If $\{x, y\} \subseteq A$, then the pair xy belongs to at most one of the link graphs G_u, G_v, G_w.
Proof: Suppose on the contrary that this pair belongs to two of the link graphs. By symmetry, we may assume that either $x, y \in N_{uv}$, or $x \in N_{uv}$ and $y \in N_{vw}$.

Clearly xy must belong to one of the link graphs G_u, G_v, assume by symmetry that it belongs to G_u. In the first case, the edges uvx, uvy, uxy form a copy of $\mathcal{K}_4^-$, a contradiction.

In the second case, if xy belongs to both the link graphs G_u and G_w, then vertices u, v, w, y, x from a copy of $\mathcal{C}_5$ taken in cyclic order. Otherwise by symmetry we may assume that xy belongs to both G_u and G_v (G_v and G_w is the other similar case). Now the edges uxy, vxy, uvx form a copy of $\mathcal{K}_4^-$. These contradictions prove the Claim. □

Let $G = G_u \cup G_v \cup G_w$.

Claim 2: Each set from $\{N_{uv}, N_{uw}, N_{vw}\}$ contains edges from at most one of the three link graphs forming G. Moreover, each of these link graphs is triangle-free.

Proof: Suppose that $x, y \in N_{uv}$. If the pair xy belongs to at least two of these three link graphs, then it belongs to at least one of G_u or G_v. Consequently, vertices u, v, x, y contain a copy of $\mathcal{K}_4^-$, a contradiction. If the link graph of any vertex p contains a triangle xyz, then edges pxy, pxz, pyz form a copy of $\mathcal{K}_4^-$, a contradiction. $\square$

Claim 3: Each of G_u, G_v, G_w has the following property: the number of edges with at most one endpoint in A is at most $\epsilon(1-\epsilon)n^2$.

Proof: Without loss of generality we consider only G_u. Let G be the graph with $V(G) = A \cup B$ and $E(G) = \{e \in G_u : e \subseteq B \text{ or } |e \cap A| = |e \cap B| = 1\}$.

By definition, A is an independent set in G. Since $\epsilon < 1/31 < 1/2$ by (4), we have $|A| \geq |B| - 1$. By Claim 2, G is triangle-free. Therefore G satisfies the hypothesis of Lemma 5.2, and the Lemma implies that $|E(G)| \leq |A||B| < (1-\epsilon)n\epsilon n = \epsilon(1-\epsilon)n^2$. Since $|E(G)|$ is precisely the number of edges in G_u with one endpoint in A, the Claim follows. $\square$

Lemma 5.1 with $k = 3$, and (3) and the definition of uvw imply that

$$\frac{(d_{uv})^2}{4} + \frac{(d_{uw})^2}{4} + \frac{(d_{vw})^2}{4} + d_{uv}d_{uw} + d_{uv}d_{vw} + d_{uw}d_{vw} \leq \frac{5}{12}(1-\epsilon)^2 n^2 < \frac{5}{6}(1-\epsilon)^2 \binom{n-1}{2} + 2n. \quad (5)$$

For a vertex z, let $\delta(z)$ be the number of edges in the link graph formed by z with at most one endpoint in A. Let $e(A)$ be the number of edges in $E(G_u) \cup E(G_v) \cup E(G_w)$ with both endpoints in A. Equation (5) and Claims 1–3 imply that

$$\begin{aligned} d(u) + d(v) + d(w) &\leq |E(G_u)| + |E(G_v)| + |E(G_w)| + \mu(uvw) \\ &\leq e(A) + \delta(u) + \delta(v) + \delta(w) + n \\ &\leq \frac{(d_{uv})^2}{4} + \frac{(d_{uw})^2}{4} + \frac{(d_{vw})^2}{4} + d_{uv}d_{uw} + d_{uv}d_{vw} + d_{uw}d_{vw} + 3\epsilon(1-\epsilon)n^2 + n \\ &\leq \left(\frac{5}{6}(1-\epsilon)^2 + 6\epsilon(1-\epsilon)\right)\binom{n-1}{2} + 3c(2n-1) \end{aligned}$$

Thus one of u, v, w has degree at most

$$\left(\frac{5}{18}(1-\epsilon)^2 + 2\epsilon(1-\epsilon)\right)\binom{n-1}{2} + c(2n-1).$$

If this is at most $\alpha\binom{n-1}{2} + c(2n-1)$, then we may apply induction, so we may assume that

$$\alpha < \frac{5}{18}(1-\epsilon)^2 + 2\epsilon(1-\epsilon). \quad (6)$$

Equations (4) and (6) yield

$$\frac{10}{31} = \alpha < \min\left\{\frac{1-\epsilon}{3}\ ,\ \frac{5}{18}(1-\epsilon)^2 + 2\epsilon(1-\epsilon)\right\}.$$

This is impossible since

$$\max_{0\le\epsilon<1/2} \min\left\{\frac{1-\epsilon}{3}\ ,\ \frac{5}{18}(1-\epsilon)^2 + 2\epsilon(1-\epsilon)\right\} = \alpha,$$

with the maximum of the minimum of these two functions of ϵ occurring at $\epsilon = 1/31$. This contradiction completes the proof of the Theorem. □

6 Just the Pentagon

The following Lemma is a special case of a result of Füredi and Kündgen [8].

Lemma 6.1. *Let G be an n vertex multigraph with maximum multiplicity two and every three vertices spanning at most four edges. Then G has at most $\binom{n}{2} + n$ edges.*

Proof. We use induction on n. The cases $n \le 3$ are easily checked. If G has no parallel edges, then $|E(G)| \le \binom{n}{2}$, so we may assume that there are two edges between vertices x and y. Every other vertex has at most two edges connecting it to $\{x, y\}$. Let $G' = G - \{x, y\}$. Then

$$|E(G)| \le |E(G')| + 2(n-2) + 2 \le \left(\binom{n-2}{2} + (n-2)\right) + 2n - 2 \le \binom{n}{2} + n,$$

where the second inequality follows from the induction hypothesis. □

We need an asymmetric variation of the above result.

Lemma 6.2. *Let G be a multigraph with vertex partition $A \cup B$ and maximum multiplicity two. For $S \subseteq V(G)$, let $e(S)$ be the number of edges (counting multiplicities) induced by S. Suppose that, for all S of size three,*
(i) *if $|S \cap A| \ge 2$, then $e(S) \le 4$, and*
(ii) *if $|S \cap A| = 1$, then $e(S) \le 5$.*
Then

$$|E(G)| \le \binom{a}{2} + 2\binom{b}{2} + ab + a.$$

Proof. We proceed by induction on $a+b$. The cases $a+b \le 4$ are easily checked. If every A, B-pair has multiplicity at most one, then by Lemma 6.1

$$|E(G)| \le e(A) + ab + e(B) \le \binom{a}{2} + a + ab + 2\binom{b}{2}.$$

We may therefore assume that the pair xy has multiplicity two, where $x \in A$ and $y \in B$. Conditions (i) and (ii) imply that every vertex in A has at most two edges to $\{x, y\}$, and every vertex in B has at most three edges to $\{x, y\}$. Let $G' = G - \{x, y\}$. This observation, together with the induction hypothesis applied to G', implies that

$$\begin{aligned} |E(G)| &\le |E(G')| + 2(a-1) + 3(b-1) + 2 \\ &\le \binom{a-1}{2} + 2\binom{b-1}{2} + (a-1)(b-1) + a - 1 + 2(a-1) + 3(b-1) + 2 \\ &= \binom{a}{2} + 2\binom{b}{2} + ab + a. \qquad \square \end{aligned}$$

Construction 1: Let $0 < b < 1$. Let $\mathcal{H}(m, b)$ be the 3-graph on m vertices partitioned into two sets B, B' of size bm and $(1-b)m$, respectively. Let $E(\mathcal{H}(m, b))$ consist of all triples uvw, where $u, v \in B$ and $w \in B'$. It is easy to check that $\mathcal{H}(m, b)$ is $\mathcal{C}_5$-free for all m, b.

It is still possible to add edges to $\mathcal{H}(m, b)$ maintaining the property that it is $\mathcal{C}_5$-free. Indeed, we can add any 3-graph $\mathcal{H}$ that is itself $\mathcal{C}_5$-free within B'. Consider a set S of five vertices in the resulting 3-graph.

If $|S \cap B'| < 3$, then edges among S are all edges in $\mathcal{H}(m, b)$, and S therefore contains no copy of $\mathcal{C}_5$. If $|S \cap B'| = 3$, then every two vertices of $S \cap B'$ lie in at most one edge in S, and $\mathcal{C}_5$ has the property that for every three vertices, at least one pair of them lies in two edges. Thus S does not contain a copy of $\mathcal{C}_5$. If $|S \cap B'| = 4$, then the vertex in $S \cap B$ has no edge in S, and clearly S therefore contains no copy of $\mathcal{C}_5$. If $|S \cap B'| = 5$, then by the construction within B', S again contains no copy of $\mathcal{C}_5$.

We repeat this construction recursively. This results in a 3-graph with $(a + o(1))\binom{n}{3}$ edges, where a is given by

$$\binom{bn}{2}(1-b)n + a\binom{(1-b)n}{3} = (a + o(1))\binom{n}{3}.$$

Solving gives $a = 3b^2(1-b)/(1-(1-b)^3) + o(1)$. The choice of b that maximizes a is $b = 0.633975$ and for this choice of b, we get $a = 0.464102$. $\square$

Proof of Theorem 1.9: The lower bound follows from Construction 1. We now prove the upper bound. Let $\alpha = 2 - \sqrt{2}$, and suppose that $\mathcal{H}$ is a 3-graph with at least $\alpha\binom{n}{3} + cn^2$ edges for some sufficiently large constant c (for our purposes, $c = 10$ will do). We will prove by induction on n that $\mathcal{H}$ contains a copy of the Pentagon $\mathcal{C}_5$. It therefore suffices to find a vertex in $\mathcal{H}$ of degree at most $\alpha\binom{n-1}{2} + c(2n-1)$.

Given vertices $x, y \in V(\mathcal{H})$, recall that $N_{xy} = \{z : xyz \text{ is an edge}\}$, and that $d_{xy} = |N_{xy}|$. Since $\sum_{x,y\in V(\mathcal{H})} d_{xy} = 3|E(H)|$, there is a pair x, y with $d_{xy} \geq \alpha n$. Let A be a subset of N_{xy} of size αn, and let $B = V(\mathcal{H}) - A$. Let G_x and G_y be the link graphs in $\mathcal{H} - \{x, y\}$ of x and y, respectively. Consider the multigraph $G = G_x \cup G_y$.

Claim: If $\mathcal{H}$ is $\mathcal{C}_5$-free, then G, A, and B satisfy the hypotheses (i) and (ii) in Lemma 6.2.

Proof of Claim: Let $S = \{u, v, w\}$. We first verify (i). Suppose that $|S \cap A| \geq 2$ and $e(S) \geq 5$. This means that at most one pair from S in not in $E(G_x) \cap E(G_y)$, and all pairs from S are in $E(G_x) \cup E(G_y)$. By symmetry, we may assume that the pair uv is (possibly) not in $E(G_x)$, and all other pairs from S are in both $E(G_x)$ and $E(G_y)$. Either $u, v \in A$, or $u, w \in A$ and $v \in B$. In either case, vertices x, u, y, v, w form a copy of $\mathcal{C}_5$ with edges xuy, uyv, yvw, vwx, wxu, a contradiction.

To verify (ii), suppose that $|S \cap A| = 1$ and $e(S) = 6$, i.e., all pairs in S appear with multiplicity two in G. Assume that $u \in A$ and $v, w \in B$. Again, vertices x, u, y, v, w form a copy of $\mathcal{C}_5$ as shown above. $\square$

By the Claim, we may assume that G satisfies the conclusion of Lemma 6.2. Consequently,

$$\begin{aligned}
|E(G)| &< \binom{|A|}{2} + 2\binom{|B|}{2} + |A||B| + n \\
&\leq \binom{\alpha n}{2} + 2\binom{(1-\alpha)n}{2} + \alpha(1-\alpha)n^2 + n \\
&\leq (\alpha^2 + 2(1-\alpha)^2 + 2\alpha(1-\alpha))\binom{n-1}{2} + 2c(2n-1) - 2n \\
&\leq 2\alpha\binom{n-1}{2} + 2c(2n-1) - 2n.
\end{aligned} \tag{7}$$

The last inequality holds since the quadratic $z^2 + 2(1-z) = 2z$ has roots $z = 2 \pm \sqrt{2}$ and opens upward. Thus one of the graphs G_x, or G_y has at most $\alpha\binom{n-1}{2} + c(2n-1) - n$ edges, and the vertex corresponding to this link graph has degree in $\mathcal{H}$ at most $\alpha\binom{n-1}{2} + c(2n-1)$. This completes the proof of the Theorem. $\square$

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