Coloring with three-colored subgraphs

Dhruv Mubayi *

August 10, 2002

Abstract

Let $f(n)$ be the minimum number of colors required to color the edges of $K_{n,n}$ such that every copy of $K_{3,3}$ receives at least three colors on its edges. We prove that

$$(0.62 + o(1))\sqrt{n} < f(n) < (1 + o(1))\sqrt{n},$$

where the upper bound is obtained by an explicit edge-coloring. This complements earlier results of Axenovich, Füredi, and the author [1].

1 Introduction

We continue the study of the generalized Ramsey parameter $r(G, H, q)$, defined as the minimum number of colors needed to edge-color G giving every copy of H in G at least q colors. Initiated by Erdős [4], and subsequently developed in [1, 5], this general parameter has given rise to many interesting open problems (the classical Ramsey numbers for multicolorings are the special case $G = K_n, H = K_p, q = 2$).

The case $G = K_{n,n}$ and $H = K_{p,p}$ was investigated in [1]. Below is a summary of the results from [1] for $p = 2$ and 3. In the chart, "$<< f(n)$" means "$O(f(n))$", and "$>> g(n)$" means "$\Omega(g(n))$".

q	$r(K_{n,n}, C_4, q)$	$r(K_{n,n}, K_{3,3}, q)$
2	$\sqrt{n}(1 + o(1))$	$n^{1/3}(1 + o(1))$
3	$> \lfloor (2/3)n \rfloor, \quad \leq n + 1$	$<< n^{4/7}$
4	n^2	$<< n^{2/3}$
5	—	$<< n^{4/5}$
6	—	$> n/4, \quad < cn$
7	—	$>> n, \quad << n^{4/3}$
8	—	$\lceil \frac{n}{2} \lceil \frac{3n}{2} \rceil \rceil$
9	—	n^2

*Microsoft Research, One Microsoft Way, Redmond, WA 98052-6399
Current address: Department of Mathematics, Statistics, & Computer Science (m/c 249), University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045; Email: mubayi@math.uic.edu

In this note we improve the upper bound for $r(K_{n,n}, K_{3,3}, 3)$ given in the table above. We also observe that results from [3, 7, 11] imply a lower bound with the same order of magnitude.

Theorem 1. *Let α be the positive root of $x^2 - x - 4 = 0$, let $\beta = 1/\sqrt{\alpha} = 0.62481\ldots$ Then*

$$(\beta + o(1))\sqrt{n} < r(K_{n,n}, K_{3,3}, 3) < (1 + o(1))\sqrt{n}.$$

2 Proof of Theorem 1

For the lower bound we need the following result of [11], which is a slight modification of ideas from [7]. For $l \le r$, we write $H(r,l)$ for the graph obtained from $K_{r,r}$ by deleting the edges of a copy of $K_{l,l}$. We write $\mathrm{ex}(G,H)$ to denote the maximum number of edges in a subgraph of G containing no copy of H.

Theorem 2. *If α is the positive root of $x^2 - x - 4 = 0$, and $\alpha' = \sqrt{\alpha} = 1.60048..$, then*

$$ex(K_{n,n}, H(3,1)) \le \alpha' \ n^{3/2} + n.$$

The lower bound in Theorem 1 now follows by observing that in an edge-coloring of $K_{n,n}$ with every copy of $K_{3,3}$ receiving at least three colors, each color class is $H(3,1)$-free.

The construction we use for the upper bound in Theorem 1 was first used in [10] to give lower bounds on the Ramsey number $r_k(C_4)$. It is also a special case of the "Projective Norm graphs" of [2, 9]: For q an odd prime power not congruent to three, and $n = q^2$, we construct an edge-coloring of $K_{n,n}$ with q colors that gives every copy of $K_{3,3}$ at least three colors. Then the upper bound for $r(K_{n,n}, K_{3,3}, 3)$ follows from the fact that there is a prime not congruent to three between n and $n + o(n)$ (see [8]).

Let $\mathbf{F}_q$ be the finite field on q elements, and let $V(K_{n,n})$ have parts P and Q, where both are identified with $\mathbf{F}_q \times \mathbf{F}_q$. Color the edge $(a,b)(x,y)$ with the field element $ax - b - y$.

For the following two Claims, we suppose that $(a,b), (a',b'), (x,y), (x',y')$ are four distinct vertices (see Figure 1).

Claim 1: If the two edges $(a,b)(x,y), (a',b')(x,y)$ receive the same color, and the two edges $(a,b)(x',y'), (a',b')(x',y')$ receive the same color, then $x = x'$ (and $a \ne a'$).
Proof: We have $ax - b - y = a'x - b' - y$ and $ax' - b - y' = a'x' - b' - y'$. These yield

$$x(a - a') = b - b' = x'(a - a').$$

Consequently, $a = a'$ or $x = x'$. Since $a = a'$ implies that $(a,b) = (a',b')$, which is excluded, the result follows. □

Claim 2: If the two edges $(a,b)(x,y), (a',b')(x',y')$ receive the same color, and the two edges $(a,b)(x',y'), (a',b')(x,y)$ receive the same color, then $(a - a')(x + x') = 2(b - b')$.
Proof: We have $ax - b - y = a'x' - b' - y'$ and $ax' - b - y' = a'x - b' - y$. Adding these two equations and simplifying yields the result. □

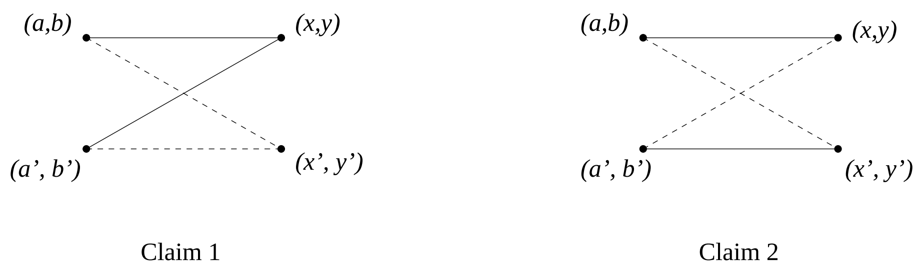

Figure 1: Claims 1 and 2

Note that Claim 1 implies that there is no monochromatic four-cycle. To see this, suppose that $(a,b)(x,y), (a',b')(x,y), (a,b)(x',y'), (a',b')(x',y')$ is such a cycle. Then Claim 1 applied twice yields $x = x'$ and $a = a'$, a contradiction.

Now suppose that there is a copy H of $K_{3,3}$ with at most two colors and parts $A = \{A_i = (a_i, b_i) : i = 1,2,3\}$ and $X = \{X_i = (x_i, y_i) : i = 1,2,3\}$. Since there are no monochromatic four-cycles this leaves, up to symmetries, three possibilities for the distribution of colors on $E(H)$.

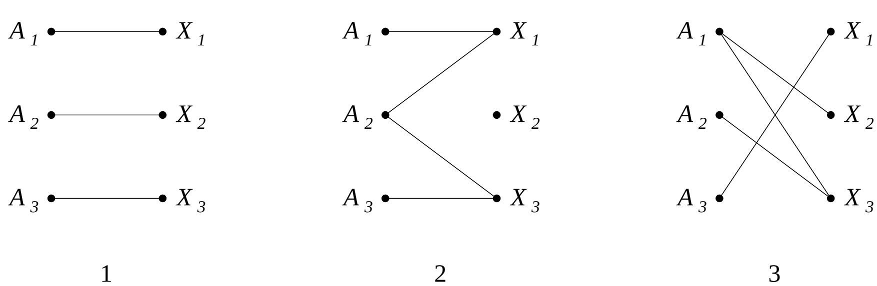

Figure 2: The 2-colored $K_{3,3}$'s

Configuration 1: A_iX_j is red if $i = j$ and blue otherwise.
The equations defining colors yield, for i,j,k distinct, $a_ix_k - b_i - y_k = a_jx_k - b_j - y_k$. Consequently,

$$x_k(a_i - a_j) = b_i - b_j. \tag{1}$$

This implies that $a_i \neq a_j$, and by symmetry that $x_i \neq x_j$. Claim 2 applied to the four-cycle A_i, X_i, A_j, X_j yields

$$(a_i - a_j)(x_i + x_j) = 2(b_i - b_j). \tag{2}$$

Now (1) and (2) imply that either $a_i = a_j$ or $2x_k = x_i + x_j$. We have already observed that the former cannot occur, so we may assume that $2x_k = x_i + x_j$. By symmetry, we also have $2x_i = x_j + x_k$. Together these two equations yield

$$4x_k - 2x_j = 2x_i = x_j + x_k.$$

This simplifies to $3x_k = 3x_j$ which by the choice of q implies that $x_k = x_j$, a contradiction. $\square$

Configuration 2: $A_1X_1, A_2X_1, A_2X_3, A_3X_3$ are all red and all other edges are blue.
The edges A_2X_1 and A_2X_3 yield $a_2x_3 - b_2 - y_3 = a_2x_1 - b_2 - y_1$. This simplifies to $a_2(x_3 - x_1) = y_3 - y_1$. Claim 1 applied to the four-cycle A_1, X_1, A_2, X_2 yields $x_1 = x_2$, and Claim 1 applied to the four-cycle A_2, X_2, A_3, X_3 yields $x_2 = x_3$. Consequently we have $x_1 = x_3$, which by our earlier observation gives $X_1 = X_3$, a contradiction. $\square$

Configuration 3: $A_1X_2, A_1X_3, A_2X_3, A_3X_1$ are all red and all other edges are blue.
Claim 1 applied to the four cycle A_1, X_2, A_3, X_3 yields $a_1 = a_3$. Claim 2 applied to the four-cycle A_1, X_1, A_3, X_3 yields $(a_1 - a_3)(x_1 + x_3) = 2(b_1 - b_3)$. Since $a_1 = a_3$, we obtain $A_1 = A_3$, a contradiction. $\square$

3 Concluding Remarks

• The construction in [1] (which comes from Füredi [6]) that yields $r(K_{n,n}, C_4, 2) \leq (1 + o(1))\sqrt{n}$ does not give every copy of $K_{3,3}$ more than two colors. In that construction, we again let $V(K_{n,n})$ have parts P and Q, where each is identified with $(\mathbf{F}_q - \{0\}) \times (\mathbf{F}_q - \{0\})$. The edge $(a,b)(x,y)$ is colored with the field element $ax+by$. Then the edges that received color 0 are recolored inductively.

If we consider the vertices $(1,1), (-1,-1), (2y-1, -2x-1)$ in one part, and the vertices $(x,y), (-x,-y), (-x-1, -y+1)$ in the other part, for any $x, y \in \mathbf{F}_q$, then it is easy to see that all edges between these two sets receive color $x+y$ or $-x-y$. Clearly we can choose x and y so that $x + y \neq 0$ as well as $(x,y) \neq (-x-1, -y+1)$ and $(2y-1, -2x-1) \neq \pm(1,1)$, so this is a copy of $K_{3,3}$ with precisely two nonzero colors on its edges. It therefore seems that the Projective Norm graphs are really needed for this construction.

• Easy modifications of the results in this paper yield

$$(1/\sqrt{3} - o(1))\sqrt{n} < r(K_n, K_{3,3}, 3) < (1 + o(1))\sqrt{n}.$$

The lower bound follows from [7], where it is shown that $\mathrm{ex}(K_n, H(3,1)) \leq (\sqrt{3}/2 + o(1))n^{3/2}$. The upper bound follows by associating $V(K_n)$ with $\mathbf{F}_q \times \mathbf{F}_q$ and coloring the edge $(a,b)(x,y)$ with the field element $ax - b - y$. The proof of Theorem 1 then applies to show that this coloring gives every copy of $K_{3,3}$ at least three colors.

4 Acknowledgments

I am grateful to Z. Füredi and the referees for useful comments.

References

[1] M. Axenovich, Z. Füredi, and D. Mubayi, On generalized Ramsey theory: the bipartite case, J. Combin. Theory Ser. B 79 (2000), no. 1, 66–86

[2] N. Alon, L. Rónyai and T. Szabó, Norm-graphs: variations and applications, J. Combin. Theory, Ser. B, 76, no. 2, (1999) 280–290.

[3] P. Erdős, On an extremal problem in graph theory, Colloq. Math. 13 (1964/1965) 251–254

[4] P Erdős, Solved and unsolved problems in combinatorics and combinatorial number theory, Congressus Numerantum, 32 (1981), 49-62

[5] P. Erdős and A. Gyárfás, A variant of the classical Ramsey problem, Combinatorica, 17 (1997), 459–467

[6] Z. Füredi, New asymptotics for bipartite Turán numbers, Journal of Combinatorial Theory, Ser. A **75** (1996), 141–144.

[7] Z. Füredi and D. B. West, Ramsey theory and bandwidth of graphs, Graphs and Combinatorics, (to appear)

[8] M. N. Huxley and H. Iwaniec, Bombieri's theorem in short intervals, Mathematika, 22 (1975), 188–194.

[9] J. Kollár, L. Rónyai and T. Szabó, Norm-Graphs and Bipartite Turán Numbers, Combinatorica, 16, (3), (1996), 399–406.

[10] F. Lazebnik, A. J. Woldar, New lower bounds on the multicolor Ramsey numbers $r_k(C_4)$, J. Comb. Theory, Ser. B **79** (2000), 172–176.

[11] D. Mubayi, D. B. West, On Restricted Edge-Colorings of Bicliques, Discrete Math. (to appear)