Coloring with three-colored subgraphs

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Abstract

Let \( f(n) \) be the minimum number of colors required to color the edges of \( K_{n,n} \) such that every copy of \( K_{3,3} \) receives at least three colors on its edges. We prove that

\[
(0.62 + o(1))\sqrt{n} < f(n) < (1 + o(1))\sqrt{n},
\]

where the upper bound is obtained by an explicit edge-coloring. This complements earlier results of Axenovich, Füredi, and the author [1].

1 Introduction

We continue the study of the generalized Ramsey parameter \( r(G, H, q) \), defined as the minimum number of colors needed to edge-color \( G \) giving every copy of \( H \) in \( G \) at least \( q \) colors. Initiated by Erdős [4], and subsequently developed in [1, 5], this general parameter has given rise to many interesting open problems (the classical Ramsey numbers for multicolorings are the special case \( G = K_n, H = K_p, q = 2 \)).

The case \( G = K_{n,n} \) and \( H = K_{p,p} \) was investigated in [1]. Below is a summary of the results from [1] for \( p = 2 \) and \( 3 \). In the chart, “\(<\lt f(n)\)” means “\( O(f(n)) \)”, and “\( >\> g(n) \)” means “\( \Omega(g(n)) \)”.

<table>
<thead>
<tr>
<th>( q )</th>
<th>( r(K_{n,n}, C_4, q) )</th>
<th>( r(K_{n,n}, K_{3,3}, q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \sqrt{n(1 + o(1))} )</td>
<td>( n^{1/3}(1 + o(1)) )</td>
</tr>
<tr>
<td>3</td>
<td>( \geq (2/3)n ) ( \leq n + 1 )</td>
<td>( \ll n^{4/7} )</td>
</tr>
<tr>
<td>4</td>
<td>( n^2 )</td>
<td>( \ll n^{2/3} )</td>
</tr>
<tr>
<td>5</td>
<td>( - - - )</td>
<td>( \ll n^{4/5} )</td>
</tr>
<tr>
<td>6</td>
<td>( - - - )</td>
<td>( &gt; n/4, \ll cn )</td>
</tr>
<tr>
<td>7</td>
<td>( - - - )</td>
<td>( &gt; n, \ll n^{4/3} )</td>
</tr>
<tr>
<td>8</td>
<td>( - - - )</td>
<td>( \left[ \frac{n}{2} \left( \frac{2n}{3} \right) \right] )</td>
</tr>
<tr>
<td>9</td>
<td>( - - - )</td>
<td>( n^2 )</td>
</tr>
</tbody>
</table>

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In this note we improve the upper bound for $r(K_{n,n}, K_{3,3}, 3)$ given in the table above. We also observe that results from [3, 7, 11] imply a lower bound with the same order of magnitude.

**Theorem 1.** Let $\alpha$ be the positive root of $x^2 - x - 4 = 0$, let $\beta = 1/\sqrt{\alpha} = 0.62481...$ Then

$$(\beta + o(1)) \sqrt{n} < r(K_{n,n}, K_{3,3}, 3) < (1 + o(1)) \sqrt{n}.$$ 

2 Proof of Theorem 1

For the lower bound we need the following result of [11], which is a slight modification of ideas from [7]. For $l \leq r$, we write $H(r,l)$ for the graph obtained from $K_{r,l}$ by deleting the edges of a copy of $K_{l,l}$. We write $\text{ex}(G, H)$ to denote the maximum number of edges in a subgraph of $G$ containing no copy of $H$.

**Theorem 2.** If $\alpha$ is the positive root of $x^2 - x - 4 = 0$, and $\alpha' = \sqrt{\alpha} = 1.60048...$, then

$$\text{ex}(K_{n,n}, H(3,1)) \leq \alpha' n^{3/2} + n.$$ 

The lower bound in Theorem 1 now follows by observing that in an edge-coloring of $K_{n,n}$ with every copy of $K_{3,3}$ receiving at least three colors, each color class is $H(3,1)$-free.

The construction we use for the upper bound in Theorem 1 was first used in [10] to give lower bounds on the Ramsey number $r_k(C_4)$. It is also a special case of the "Projective Norm graphs" of [2, 9]: For $q$ an odd prime power not congruent to three, and $n = q^2$, we construct an edge-coloring of $K_{n,n}$ with $q$ colors that gives every copy of $K_{3,3}$ at least three colors. Then the upper bound for $r(K_{n,n}, K_{3,3}, 3)$ follows from the fact that there is a prime not congruent to three between $n$ and $n + o(n)$ (see [8]).

Let $F_q$ be the finite field on $q$ elements, and let $V(K_{n,n})$ have parts $P$ and $Q$, where both are identified with $F_q \times F_q$. Color the edge $(a,b)(x,y)$ with the field element $ax - b - y$.

For the following two Claims, we suppose that $(a,b),(a',b'),(x,y),(x',y')$ are four distinct vertices (see Figure 1).

**Claim 1:** If the two edges $(a,b)(x,y),(a',b')(x,y)$ receive the same color, and the two edges $(a,b)(x',y'),(a',b')(x',y')$ receive the same color, then $x = x'$ (and $a \neq a'$).

**Proof:** We have $ax - b - y = a'x - b' - y$ and $ax' - b - y' = a'x' - b' - y'$. These yield

$$x(a-a') = b-b' = x'(a-a').$$

Consequently, $a = a'$ or $x = x'$. Since $a = a'$ implies that $(a,b) = (a',b')$, which is excluded, the result follows. ☐

**Claim 2:** If the two edges $(a,b)(x,y),(a',b')(x',y')$ receive the same color, and the two edges $(a,b)(x',y'),(a',b')(x,y)$ receive the same color, then $(a-a')(x+x') = 2(b-b')$.

**Proof:** We have $ax - b - y = a'x' - b' - y'$ and $ax' - b - y' = a'x - b' - y$. Adding these two equations and simplifying yields the result. ☐
Claim 1

Claim 2

Figure 1: Claims 1 and 2

Note that Claim 1 implies that there is no monochromatic four-cycle. To see this, suppose that \((a, b)(x, y), (a', b')(x, y), (a, b)(x', y'), (a', b')(x', y')\) is such a cycle. Then Claim 1 applied twice yields \(x = x'\) and \(a = a'\), a contradiction.

Now suppose that there is a copy \(H\) of \(K_{3,3}\) with at most two colors and parts \(A = \{A_i = (a_i, b_i) : i = 1, 2, 3\}\) and \(X = \{X_i = (x_i, y_i) : i = 1, 2, 3\}\). Since there are no monochromatic four-cycles this leaves, up to symmetries, three possibilities for the distribution of colors on \(E(H)\).

Figure 2: The 2-colored \(K_{3,3}'s\)

**Configuration 1:** \(A_i X_j\) is red if \(i = j\) and blue otherwise.

The equations defining colors yield, for \(i, j, k\) distinct, \(a_i x_k - b_i - y_k = a_j x_k - b_j - y_k\). Consequently,

\[
x_k (a_i - a_j) = b_i - b_j.
\]

(1)

This implies that \(a_i \neq a_j\), and by symmetry that \(x_i \neq x_j\). Claim 2 applied to the four-cycle \(A_i, X_i, A_j, X_j\) yields

\[
(a_i - a_j)(x_i + x_j) = 2(b_i - b_j).
\]

(2)
Now (1) and (2) imply that either $a_i = a_j$ or $2x_k = x_i + x_j$. We have already observed that the former cannot occur, so we may assume that $2x_k = x_i + x_j$. By symmetry, we also have $2x_i = x_j + x_k$. Together these two equations yield

$$4x_k - 2x_j = 2x_i = x_j + x_k.$$  

This simplifies to $3x_k = 3x_j$ which by the choice of $q$ implies that $x_k = x_j$, a contradiction.  

**Configuration 2:** $A_1X_1, A_2X_1, A_2X_3, A_3X_3$ are all red and all other edges are blue.

The edges $A_2X_1$ and $A_2X_3$ yield $a_2x_3 - b_2 - y_3 = a_2x_1 - b_2 - y_1$. This simplifies to $a_2(x_3 - x_1) = y_3 - y_1$. Claim 1 applied to the four-cycle $A_1, X_1, A_2, X_2$ yields $x_1 = x_2$, and Claim 1 applied to the four-cycle $A_2, X_2, A_3, X_3$ yields $x_2 = x_3$. Consequently we have $x_1 = x_3$, which by our earlier observation gives $X_1 = X_3$, a contradiction.  

**Configuration 3:** $A_1X_2, A_1X_3, A_2X_3, A_3X_1$ are all red and all other edges are blue.

Claim 1 applied to the four cycle $A_1, X_2, A_3, X_3$ yields $a_1 = a_3$. Claim 2 applied to the four-cycle $A_1, X_1, A_3, X_3$ yields $(a_1 - a_3)(x_1 + x_3) = 2(b_1 - b_3)$. Since $a_1 = a_3$, we obtain $A_1 = A_3$, a contradiction.

### 3 Concluding Remarks

- The construction in [1] (which comes from Füredi [6]) that yields $r(K_{n,n}, C_4, 2) \leq (1 + o(1))\sqrt{n}$ does not give every copy of $K_{3,3}$ more than two colors. In that construction, we again let $V(K_{n,n})$ have parts $P$ and $Q$, where each is identified with $(F_q - \{0\}) \times (F_q - \{0\})$. The edge $(a, b)(x, y)$ is colored with the field element $ax + by$. Then the edges that received color 0 are recolored inductively.

  If we consider the vertices $(1, 1), (-1, -1), (2y - 1, -2x - 1)$ in one part, and the vertices $(x, y), (-x, -y), (-x - 1, -y + 1)$ in the other part, for any $x, y \in F_q$, then it is easy to see that all edges between these two sets receive color $x + y$ or $-x - y$. Clearly we can choose $x$ and $y$ so that $x + y \neq 0$ as well as $(x, y) \neq (-x - 1, -y + 1)$ and $(2y - 1, -2x - 1) \neq \pm (1, 1)$, so this is a copy of $K_{3,3}$ with precisely two nonzero colors on its edges. It therefore seems that the Projective Norm graphs are really needed for this construction.

- Easy modifications of the results in this paper yield

$$(1/\sqrt{3} - o(1))\sqrt{n} < r(K_n, K_{3,3}, 3) < (1 + o(1))\sqrt{n}.$$  

The lower bound follows from [7], where it is shown that $\text{ex}(K_n, H(3, 1)) \leq (\sqrt{3}/2 + o(1))n^{3/2}$. The upper bound follows by associating $V(K_n)$ with $F_q \times F_q$ and coloring the edge $(a, b)(x, y)$ with the field element $ax - b - y$. The proof of Theorem 1 then applies to show that this coloring gives every copy of $K_{3,3}$ at least three colors.

### 4 Acknowledgments

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References


