Coloring simple hypergraphs

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Abstract

Fix an integer $k \geq 3$. A k-uniform hypergraph is simple if every two edges share at most one vertex. We prove that there is a constant c depending only on k such that every simple k-uniform hypergraph H with maximum degree Δ has chromatic number satisfying

$$\chi(H) < c \left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{k-1}}.$$

This implies a classical result of Ajtai-Komlós-Pintz-Spencer-Szemerédi and its strengthening due to Duke-Lefmann-Rödl. The result is sharp apart from the constant c.

1 Introduction

Hypergraph coloring has been studied for almost 50 years, since Erdős' seminal results on the minimum number of edges in uniform hypergraphs that are not 2-colorable. Some of the major tools in combinatorics have been developed to solve problems in this area, for example, the local lemma and the nibble or semirandom method. Consequently, the subject enjoys a prominent place among basic combinatorial questions.

Closely related to coloring problems are questions about the independence number of hypergraphs. An easy extension of Turán's graph theorem shows that a k-uniform hypergraph with n vertices and average degree d has an independent set of size at least $cn/d^{1/(k-1)}$, where c depends only on k. If we impose local constraints on the hypergraph, then this bound can be improved. An i-cycle in a k-uniform hypergraph is a collection of i distinct edges spanned by at most $i(k-1)$ vertices. Say that a k-uniform hypergraph has girth at least g if it contains no i-cycles for $2 \leq i < g$. Call a k-uniform hypergraph simple if it has girth at least 3. In other words, every two edges have at

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most one vertex in common. Throughout this paper we will assume that $k \geq 3$ is a fixed positive integer.

Ajtai-Komlós-Pintz-Spencer-Szemerédi [2] proved the following fundamental result that strengthened the bound obtained by Turán's theorem above.

Theorem 1 ([2]) *Let $H = (V, E)$ be a k-uniform hypergraph of girth at least 5 with maximum degree Δ. Then it has an independent set of size at least*

$$c\,n \left(\frac{\log \Delta}{\Delta} \right)^{1/(k-1)}$$

where c depends only on k.

Spencer conjectured that Theorem 1 holds even for simple hypergraphs, and this was later proved by Duke-Lefmann-Rödl [6]. Theorem 1 has proved to be a seminal result in combinatorics, with many applications. Indeed, the result was first proved for $k = 3$ by Komlós-Pintz-Szemerédi [13] to disprove the famous Heilbronn conjecture, that among every set of n points in the unit square, there are three points that form a triangle whose area is at most $O(1/n^2)$. For applications of Theorem 1 to coding theory or combinatorics, see [15] or [14], respectively.

The goal of this paper is to prove a result that is stronger than Theorem 1 (and also the accompanying result of [6]). Since the proof of our result does not use Theorem 1, it gives an alternative proof of all the applications of Theorem 1 as well. Our main result states that not only can one find an independent set of the size guaranteed by Theorem 1, but in fact that the entire vertex set can be partitioned into independent sets with this average size. Recall that the chromatic number $\chi(H)$ of H is the minimum number of colors needed to partition the vertex set so that no edge is monochromatic.

Theorem 2 *Fix $k \geq 3$. Let $H = (V, E)$ be a simple k-uniform hypergraph with maximum degree Δ. Then*

$$\chi(H) < c \left(\frac{\Delta}{\log \Delta} \right)^{\frac{1}{k-1}}$$

where c depends only on k.

It is shown in [5] that Theorem 2 is sharp apart from the constant c. In order to prove Theorem 2 we will first prove the following slightly weaker result. A triangle in a k-uniform hypergraph is a 3-cycle that contains no 2-cycle. In other words, it is a collection of three sets A, B, C such that every two of these sets have intersection of size one, and $A \cap B \cap C = \emptyset$.

Theorem 3 *Fix $k \geq 3$. Let $H = (V, E)$ be a simple triangle-free k-uniform hypergraph with maximum degree Δ. Then*

$$\chi(H) < c \left(\frac{\Delta}{\log \Delta} \right)^{\frac{1}{k-1}},$$

where c depends only on k.

The proof of Theorem 3 rests on several major developments in probabilistic combinatorics over the past 25 years. Our approach is inspired by Johansson's breakthrough result on graph coloring, which proved Theorem 3 for $k = 2$.

The proof technique, which has been termed the semi-random, or nibble method, was first used by Rödl (inspired by earlier work in [2, 13]) to confirm the Erdős-Hanani conjecture about the existence of asymptotically optimal designs. Subsequently, Kim [11] (see also Kahn [10]) proved Johansson's theorem for graphs with girth five and then Johansson proved his result. The approach used by Johansson for the graph case is to iteratively color a small portion of the (currently uncolored) vertices of the graph, record the fact that a color already used at v cannot be used in future on the uncolored neighbors of v, and continue this process until the graph induced by the uncolored vertices has small maximum degree. Once this has been achieved, the remaining uncolored vertices are colored using a new set of colors by the greedy algorithm.

In principle our method is the same, but there are several difficulties we encounter. The first, and most important, is that our coloring algorithm is not as simple. A proper coloring of a k-uniform hypergraph allows as many as $k-1$ vertices of an edge to have the same color, indeed, to obtain optimal results one must permit this. To facilitate this, we introduce a collection of $k-1$ different hypergraphs at each stage of the algorithm whose edges keep track of coloring restrictions. Keeping track of these hypergraphs requires controlling more parameters during the iteration and dealing with some more lack of independence and this makes the proof more complicated.

In an earlier paper [8], we had carried out this program for $k = 3$. Several technical ideas incorporated in the current proof can be found there. Because the notation in [8] is slightly simpler than that in the current paper, the reader interested in the technical details of our proof may want to familiarize him or herself with [8] first (although this paper is entirely self contained).

The implication Theorem 3 $\rightarrow$ Theorem 2 forms a much shorter (but still nontrivial) part of this paper (See Section 2). Our proof uses a recent concentration result of Kim and Vu [12] together with some additional ideas similar to those from Alon-Krivelevich-Sudakov [3]. The approach here is to partition the vertex set of a given simple hypergraph into some number of parts, where the hypergraph induced by each of the parts is triangle-free. Once this has been achieved, each of the parts is colored using Theorem 3.

Finally, we remark that our proof of Theorem 3 also gives the same upper bound for list chromatic number, although we phrase it only for chromatic number. On the other hand, we are not able to prove Theorem 2 for list chromatic number, since list chromatic number is not additive in the sense described in the previous paragraph. We end with a conjecture posed in [8], which states that we may replace the hypothesis "simple" with something much weaker.

Conjecture 4 ([8]) *Let F be a k-graph. There is a constant c_F depending only on F such that* *every F-free k-graph with maximum degree Δ has chromatic number at most $c_F(\Delta/\log\Delta)^{1/(k-1)}$.*

Conjecture 4 appears to be out of reach using current methods. For example, the special case $k = 2$ and $F = K_4$ remains open and would imply an old conjecture of [1].

Throughout this paper, we will assume that Δ is sufficiently large that all implied inequalities hold true. Any asymptotic notation is meant to be taken as $\Delta \to \infty$.

2 Simple hypergraphs

In this section we will prove that Theorem 3 $\to$ Theorem 2

Let $H = (V, E)$ be a simple k-uniform hypergraph. For $v \in V$ let its neighbor set $N_H(v)$ be defined by $N_H(v) = \{x : \exists S \in E \ s.t. \ \{v, x\} \subset S\}$. Let $d_H(v)$ denote the degree of v so that $|N_H(v)| = (k-1)d_H(v)$.

A pair $x, y \in N_H(v)$ is said to be *covered* if there exists $S \in E$ that contains both x and y but not v. Note that H simple implies that in this case no edge contains all of v, x, y.

Recall that k is a fixed. Let $\varepsilon = \varepsilon(k)$ be a sufficiently small positive constant depending only on k. Theorem 2 will follow from Theorem 3 and the following two lemmas:

Lemma 5 *Fix $k \geq 3$. Let $H = (V, E)$ be a simple k-uniform hypergraph with maximum degree Δ. Let $m = \left\lceil \Delta^{\frac{2}{3k-4}-\varepsilon} \right\rceil$. Then there exists a partition of V into subsets $V_1, V_2, \ldots, V_m$ such each induced subhypergraph $H_i = H[V_i]$, $i = 1, 2, \ldots, m$ has the following properties:*

(a) *The maximum degree Δ_i of H_i satisfies $\Delta_i \leq 2\Delta/m^{k-1}$.*

(b) *If $v \in V_i$ then its H_i-neighborhood $N_i(v)$ contains at most $k^2\Delta^2/m^{3k-4}$ covered pairs. (Here we mean covered w.r.t. H_i).*

Lemma 6 *Fix $k \geq 3$ and let δ be a sufficiently small positive constant depending on k. Let $L = (V, E)$ be a simple k-uniform hypergraph with maximum degree at most d. Suppose that each vertex neighborhood $N_L(v)$ contains at most d^δ covered pairs. Let $\ell = d^{\frac{1}{k-1}-\delta}$. Then there exists a partition of V into subsets $W_1, W_2, \ldots, W_{\ell_1}$, $\ell_1 = O(\ell)$ such that for each $1 \leq j \leq \ell_1$, the hypergraph $L_j = L[W_j]$ has the following properties:*

(a) *The maximum degree d_j of L_j satisfies $d_j \leq 2d/\ell^{k-1}$.*

(b) *L_j is triangle-free.*

□

2.1 Proof of Theorem 2

Our proof can be thought of as a nibble argument, involving two iterations, given by Lemmas 5 and 6. Suppose that H is a simple k-uniform hypergraph with maximum degree Δ. Apply Lemma 5 to obtain $H_1, \ldots, H_m$ that satisfy the conclusion of the lemma. Now fix $1 \leq i \leq m$ and let $L = H_i$. Lemma 5 part (a) implies that $\Delta(L) \leq 2\Delta/m^{k-1}$. Hence we may apply Lemma 6 to L with $d = 2\Delta/m^{k-1}$ and $\delta = \varepsilon(3k-4)^2/(k-2)$. By Lemma 5 part (b), each neighborhood $N_L(v)$ contains at most $k^2\Delta^2/m^{3k-4}$ covered pairs. Now

$$k^2 \frac{\Delta^2}{m^{3k-4}} \leq k^2 \Delta^{\varepsilon(3k-4)} = k^2 \Delta^{\frac{\delta(k-2)}{3k-4}} < d^{\delta}.$$

We may therefore apply Lemma 6 with $\ell = d^{1/(k-1)-\delta}$. Together with Theorem 3 we obtain

$$\chi(L) \leq \sum_{j=1}^{\ell} \chi(L_j) < O\left(\ell \left(\frac{d/\ell^{k-1}}{\log(d/\ell^{k-1})}\right)^{\frac{1}{k-1}}\right) = O\left(\left(\frac{d}{\log(d^{(k-1)\delta})}\right)^{\frac{1}{k-1}}\right) = O\left(\left(\frac{d}{\log d}\right)^{\frac{1}{k-1}}\right).$$

Since this holds for each H_i we obtain,

$$\chi(H) \leq \sum_{i=1}^{m} \chi(H_i) < O\left(m \left(\frac{d}{\log d}\right)^{\frac{1}{k-1}}\right) = O\left(m \left(\frac{\Delta/m^{k-1}}{\log(\Delta/m^{k-1})}\right)^{\frac{1}{k-1}}\right) = O\left(\left(\frac{\Delta}{\log \Delta}\right)^{\frac{1}{k-1}}\right).$$

2.2 Kim-Vu concentration

We will need the following very useful concentration inequality (1) from Kim and Vu [12]: Let $\Upsilon = (W, F)$ be a hypergraph of rank s, meaning that each $f \in F$ satisfies $|f| \leq s$. Let

$$Z = \sum_{f \in F} \prod_{i \in f} z_i$$

where the $z_i, i \in W$ are independent random variables taking values in $[0, 1]$. For $A \subseteq W, |A| \leq s$ let

$$Z_A = \sum_{\substack{f \in F \\ f \supseteq A}} \prod_{i \in f \setminus A} z_i.$$

Let $M_A = \mathsf{E}(Z_A)$ and $M_j = \max_{A,|A| \geq j} M_A$ for $j \geq 0$. There exist positive constants $a = a_s$ and $b = b_s$ such that for any $\lambda > 0$,

$$\mathsf{P}(|Z - \mathsf{E}(Z)| \geq a\lambda^s \sqrt{M_0 M_1}) \leq b|W|^{s-1} e^{-\lambda}. \tag{1}$$

2.3 Proof of Lemma 5

We will use the local lemma in the form below.

Theorem 7 (Local Lemma) *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be events in an arbitrary probability space. Suppose that each event $\mathcal{A}_i$ is mutually independent of a set of all the other events $\mathcal{A}_j$ but at most d, and that $P(\mathcal{A}_i) < p$ for all $1 \le i \le n$. If $ep(d+1) < 1$, then with positive probability, none of the events $\mathcal{A}_i$ holds.*

We will partition V randomly into m parts of size $\sim |V|/m$ and use the local lemma to show the existence of a partition. We make the partition by assigning a random number in $[m]$ to each $v \in V$.

Fix $v \in V$. To simplify notation, condition on $v \in V_1$. Let A_v be the event that (a) fails at v i.e. that v has degree greater than $2\Delta/m^{k-1}$ in the hypergraph H_1.

Let B_v be the event that its neighborhood in H_1 contains more than $k^2\Delta^2/m^{3k-4}$ covered pairs.

Each of these events is mutually independent of a set of all other events but at most $O(\Delta^4)$. We will show that
$\mathsf{P}(A_v), \mathsf{P}(B_v) = O(\Delta^{-5})$ and this is clearly sufficient for the application of the local lemma.

Let d_v be the degree of v in H_1. Then d_v has a distribution that is dominated by the binomial distribution $Bin(\Delta, 1/m^{k-1})$. It follows from the Chernoff bounds that

$$\mathsf{P}\left(d_v \ge 2\Delta/m^{k-1}\right) \le e^{-\Delta/(3m^{k-1})} = e^{-\Delta^{\frac{k-2}{3k-4}-(k-1)\varepsilon+o(1)}} \le \Delta^{-5}$$

and this disposes of A_v.

Our goal now is to bound $\mathsf{P}(B_v)$. For a vertex $x \in N_H(v)$, let $T_v(x)$ denote the unique $(k-1)$-set containing x such that $T_v(x) \cup \{v\} \in E$. A covered pair $x, y \in N_H(v)$ will remain as a covered pair in $N_{H_1}(v)$ iff $S_{x,y} = T_v(x) \cup T_v(y) \cup T \subseteq V_1$ where T is the unique $(k-2)$-set such that $T \cup \{x, y\} \in E$. Let $S_1, S_2, \ldots, S_r$, $r \le (k-1)^2\binom{\Delta}{2}$ be an enumeration of the $(3k-4)$-tuples $S_{x,y}$ as $\{x, y\}$ ranges over the covered pairs in $N_H(v)$.

We will use the concentration inequality (1). The edges of our hypergraph (W, F) are $S_1, S_2, \ldots, S_r$ and if $x \in W$ then z_x is an independent $\{0, 1\}$ Bernoulli random variable with $\mathsf{P}(z_x = 1) = 1/m$. Note that $|W| \le k\Delta^2$.

Let Z_v denote the number of covered pairs in $N_{H_1}(v)$. There is a 1-1 correspondence between covered pairs and the S_i. Therefore

$$\mu = \mathsf{E}(Z_v) = \frac{r}{m^{3k-4}} \le \frac{(k-1)^2\Delta^2}{2m^{3k-4}}. \tag{2}$$

We now have to estimate M_1.

For each set $A \subset W$, let Y_A denote the number of edges of F containing A.

Claim. $|Y_A| = O(\Delta)$ if $|A| \le k-1$ and $|Y_A| = O(1)$ if $|A| \ge k$.

Proof. Suppose that $A \subset S \in F$. Then S can be written as $T_v(x) \cup T_v(y) \cup T$, for some $x, y \in N_H(v)$. We will count the number of S containing A by the number of $T_v(x)$'s and T's.

First, the number of S where both $T_v(x)$ and $T_v(y)$ have a vertex in A is at most $5k^4$ by the following argument. There are $\binom{|A|}{2} < 5k^2$ choices for the two intersection points, these points uniquely determine $T_v(x)$ and $T_v(y)$, and there are at most $|T_v(y)||T_v(x)| = (k-1)^2$ possible covered pairs, each of which determines T uniquely (if T exists).

Now suppose that $A \cap T_v(y) = \emptyset$. If $|A| \geq k$, then A must contain a vertex from T and a (different) vertex from $T_v(x)$. There are at most $9k^2$ choices for these two vertices. For each of these choices, $T_v(x)$ is determined uniquely. For each vertex in $T_v(x)$ and the chosen vertex of $T \cap A$, there is at most one choice for T (since H is simple), hence the number of choices for T is at most $k-1$. Having chosen T, there are at most $k-1$ choices for $T_v(y)$. Altogether, there are at most $9k^4$ choices for S. We conclude that if $|A| \geq k$, then

$$|Y_A| \leq 5k^4 + 9k^4 = O(1).$$

If $|A| \leq k-1$, the argument above still applies unless either $A \subset T_v(x)$ or $A \subset T$. In either case, there are at most $k\Delta$ ways of choosing the other part of $S \setminus T_v(y)$ and at most k ways of choosing $T_v(y)$. Thus $|Y_A| = O(\Delta)$ as claimed. □

The probability of choosing each vertex in $S \setminus A$ is $1/m$, so for given A, the probability of a particular $S \supset A$ is $(1/m)^{3k-4-|A|}$. The Claim now implies that for $1 \leq |A| < 3k-4$,

$$M_A \leq \max\left\{O\left(\frac{\Delta}{m^{2k-3}}\right), O\left(\frac{1}{m}\right)\right\}.$$

By our choice of m, it follows that if ε is sufficiently small then

$$M_1 = O(\Delta^{-1/(3k)}).$$

The choice of m also gives

$$M_0 = \max\{\mu, M_1\} \leq \frac{k^2\Delta^2}{m^{3k-4}}.$$

It follows that if we take $a\lambda^{3k-4} = \frac{k^2\Delta^2}{m^{3k-4}(M_0M_1)^{1/2}}$ then $\lambda > \Delta^{\delta_k}$ where $\delta_k > 0$. Now (1) implies that

$$\mathsf{P}(B_v) \leq \mathsf{P}(Z_v - \mathsf{E}(Z_v) \geq k^2\Delta^2/m^{3k-4}) \leq b(k\Delta^2)^{3k}e^{-\lambda} \leq \Delta^{-5}.$$

This completes the proof of Lemma 5. □

2.4 Proof of Lemma 6

This part follows an approach taken in Alon, Krivelevich and Sudakov [3]. We will first partition V randomly into ℓ parts $V_1, V_2, \ldots, V_\ell$ of size $\sim |V|/\ell$ and use the local lemma to show the existence of a partition satisfying certain properties. To simplify notation, condition on $v \in V_1$.

For $v \in V$ let A_v be the event that (a) fails at v i.e. that v has degree greater than $2d/\ell^{k-1}$ in L_1.

Let B_v be the event that $N_{L_1}(v)$ contains at least $200k^2$ covered pairs w.r.t. L_1.

Each of these events is mutually independent of a set of all other events but at most $O(d^4)$. We will show that
$\mathsf{P}(A_v), \mathsf{P}(B_v) = O(d^{-5})$ and this is clearly sufficient for the application of the local lemma.

Let d_v be the degree of v in L_1. Then d_v has a distribution that is dominated by the binomial distribution $Bin(d, 1/\ell^{k-1})$. It follows from the Chernoff bounds that

$$\mathsf{P}\left(d_v \geq 2d/\ell^{k-1}\right) \leq e^{-d/(3\ell^{k-1})} = e^{-d^{(k-1)\delta}/3} \leq d^{-5}$$

and this disposes of the A_v.

If B_v fails then either

(i) There exists a vertex $w \in N_L(v)$ such that w is in at least $10k$ covered pairs of $N_{L_1}(v)$, or

(ii) $N_{L_1}(v)$ contains at least $10k$ pair-wise disjoint covered pairs.

Now

$$\begin{aligned} \mathsf{P}((i)) &\leq kd\binom{d^\delta}{10k}\ell^{-10k} \leq d^{-5} \\ \mathsf{P}((ii)) &\leq \binom{d^{2\delta}}{10k}\ell^{-20k} \leq d^{-5} \end{aligned}$$

and this disposes of the B_v.

So, assume that none of the events A_v, B_v occur. We show now that we can partition each V_j into at most $400k^2+1$ sets, each of which induces a triangle free hypergraph. Consider the digraph D_1 with vertex set V_1 and an edge directed from $v \in V_1$ to each vertex of each of the at most $200k^2$ covered pairs in $N_{L_1}(v)$. D_1 has maximum out-degree $400k^2$ and so its underlying graph G_1 is $400k^2$-degenerate and so it can be properly colored with $400k^2+1$ colors. Partition V_1 into color classes $W_1, W_2, \ldots, W_{400k^2+1}$. We claim that for each s, the hypergraph $L[W_s]$ induced by W_s is triangle-free. Suppose then that there is a triangle $v \cup T_v(x), v \cup T_v(y), T$ inside $L[W_s]$, where T contains both x and y. Then $\{x, y\}$ is a covered pair for v and by construction v and x are not in the same W_s, contradiction. □

3 Triangle-free hypergraphs

In this section, which forms the bulk of the paper, we will prove Theorem 3.

3.1 Local Lemma

The driving force of our upper bound argument, both in the semi-random phase and the final phase, is the Local Lemma. Note that the Local Lemma immediately implies that every k-graph with maximum degree Δ can be properly colored with at most $\lceil 4\Delta^{1/(k-1)} \rceil$ colors. Indeed, if we color each vertex randomly and independently with one of these colors, the probability of the event $\mathcal{A}_e$, that an edge e is monochromatic, is at most $\frac{1}{4^{k-1}\Delta}$. Moreover $\mathcal{A}_e$ is independent of all other events $\mathcal{A}_f$ unless $|f \cap e| > 0$, and the number of f satisfying this is less than $k\Delta$. We conclude that there is a proper coloring.

3.2 Coloring Procedure

In the rest of the paper, we will prove the upper bound in Theorem 3. Suppose that $k \geq 3$ is fixed and H is a simple triangle-free k-graph with maximum degree Δ.

Let V be the vertex set of H. As usual, we write $\chi(H)$ for the chromatic number of H. Let ε be a sufficiently small fixed positive constant (depending only on k). Let

$$\omega = \frac{\varepsilon^2 \log \Delta}{100 \times k^{2k+1}}$$

and set

$$q = \left\lceil \frac{\Delta^{1/(k-1)}}{\omega^{1/(k-1)}} \right\rceil.$$

Note that $q < c(\Delta/\log \Delta)^{1/(k-1)}$ where c depends only on k.

We color V with $2q$ colors and therefore show that

$$\chi(H) \leq 2c \left(\frac{\Delta}{\log \Delta} \right)^{1/(k-1)}.$$

We use the first q colors to color H in rounds and then use the second q colors to color any vertices not colored by this process.

Our algorithm for coloring in rounds is semi-random. At the beginning of a round certain parameters will satisfy certain properties, (9) − (14) below. We describe a set of random choices for the parameters in the next round and we use the local lemma to prove that there is a set of choices that preserves the required properties.

- $C = [q]$ denotes the set of available colors for the semi-random phase.
- $U^{(t)}$: The set of vertices which are currently uncolored. ($U^{(0)} = V$).
- $H^{(t)}$: The sub-hypergraph of H induced by $U^{(t)}$.
- $W^{(t)} = V \setminus U^{(t)}$: The set of vertices that have been colored. We use the notation κ to denote the color of an item e.g. $\kappa(w)$, $w \in W^{(t)}$ denotes the color permanently assigned to w.

- $H_i^{(t)}$, $2 \le i \le k-1$: An edge-colored i-graph with vertex set $U^{(t)}$. There is an edge $u_1u_2\cdots u_i \in H_i^{(t)}$ iff there are vertices $u_{i+1}, u_{i+2}, \ldots, u_k \in W^{(t)}$ and an edge $u_1u_2\cdots u_k \in H$ with $\kappa(u_{i+1}) = \kappa(u_{i+2}) = \cdots = \kappa(u_k)$. For a fixed $u_1u_2\cdots u_i$, this color is well defined because of the fact that H is simple. The edge $u_1u_2\cdots u_i$ is given the color $\kappa(u_{i+1})$. (These hypergraphs are used to keep track of coloring restrictions).

- $p_u^{(t)} \in [0,1]^C$ for $u \in U^{(t)}$: This is a vector of coloring probabilities. The cth coordinate is denoted by $p_u^{(t)}(c)$ and $p_u^{(0)} = (q^{-1}, q^{-1}, \ldots, q^{-1})$.

We can now describe the "algorithm" for computing $U^{(t+1)}, H_i^{(t+1)}, p_u^{(t+1)}$, given $U^{(t)}, H_i^{(t)}, p_u^{(t)}$, for $u \in U^{(t)}$: Let

$$\theta = \frac{\varepsilon}{\omega} = \frac{100 \times k^{2k+1}}{\varepsilon \log \Delta}$$

where we recall that ε is a sufficiently small positive constant.

For each $u \in U^{(t)}$ and $c \in C$ we *tentatively activate* c at u with probability $\theta p_u^{(t)}(c)$. A color c is lost at $u \in U^{(t)}$, $p_u^{(t+1)}(c) = 0$ and $p_u^{(t')}(c) = 0$ for $t' > t$ if either

(i) there is an edge $uu_2\cdots u_k \in H^{(t)}$ such that c is tentatively activated at $u_2, u_3, \ldots, u_k$ or

(ii) there is a $2 \le i \le k-1$ and an edge $e = uu_2\cdots u_i \in H_i^{(t)}$ such that $c = \kappa(e)$ and c is tentatively activated at $u_2, u_3, \ldots, u_i$.

The vertex $u \in U^{(t)}$ is given a permanent color if there is a color tentatively activated at u which is not lost due to the above reasons. If there is a choice, it is made arbitrarily. Then u is placed into $W^{(t+1)}$.

We fix

$$\hat{p} = \frac{1}{\Delta^{1/(k-1)-\varepsilon}}.$$

We keep

$$p_u^{(t)}(c) \le \hat{p}$$

for all t, u, c.

We let

$$B^{(t)}(u) = \left\{c : \ p_u^{(t)}(c) = \hat{p}\right\} \qquad \textit{for all } u \in V.$$

A color in $B^{(t)}(u)$ cannot be used at u. The role of $B^{(t)}(u)$ is clarified later.

Suppose that color c has not been lost at u prior to round t. Let us compute the probability that c is not lost at u in round t. Since for each $u \in U^{(t)}$ and $c \in C$, the tentative activation of c at u is done independently of all other tentative activations, the probability that c is not lost at u due to (i) is

$$\prod_{uu_2\cdots u_k \in H^{(t)}} \left(1 - \prod_{j=2}^{k} \theta p_{u_j}^{(t)}(c)\right) = \prod_{uu_2\cdots u_k \in H^{(t)}} \left(1 - \theta^{k-1} \prod_{j=2}^{k} p_{u_j}^{(t)}(c)\right).$$

Similarly the probability that c is not lost at u due to (ii) is

$$\prod_{i=2}^{k-1} \prod_{\substack{e=uu_2\cdots u_i \in H_i^{(t)} \\ \kappa(e)=c}} \left(1 - \theta^{i-1} \prod_{j=2}^{i} p_{u_j}^{(t)}(c)\right).$$

Consequently, the probability that c is not lost at u in round t, given that it wasn't lost in any prior round is

$$q_u^{(t)}(c) = \prod_{uu_2\cdots u_k \in H^{(t)}} \left(1 - \theta^{k-1} \prod_{j=2}^{k} p_{u_j}^{(t)}(c)\right) \prod_{i=2}^{k-1} \prod_{\substack{e=uu_2\cdots u_i \in H_i^{(t)} \\ \kappa(e)=c}} \left(1 - \theta^{i-1} \prod_{j=2}^{i} p_{u_j}^{(t)}(c)\right) \quad (3)$$

The parameter $q_u^{(t)}(c)$ is of great importance in our proof.

Coloring Procedure: Round t

Make tentative random color choices:

Independently, for all $u \in U^{(t)}$, $c \in C$, let

$$\gamma_u^{(t)}(c) = \begin{cases} 1 & \textit{Probability} \ = \ \theta p_u^{(t)}(c) \\ 0 & \textit{Probability} \ = \ 1 - \theta p_u^{(t)}(c) \end{cases} \quad (4)$$

$$\Theta^{(t)}(u) = \left\{c : \ \gamma_u^{(t)}(c) = 1\right\} \ = \ \text{the set of colors tentatively activated at } u.$$

Deal with color clashes:

$$L^{(t)}(u) = \left\{c : \ \exists uu_2 \cdots u_k \in H^{(t)}, \textit{ such that } c \in \bigcap_{j=2}^{k} \Theta^{(t)}(u_j)\right\} \cup$$
$$\left\{c : \ \exists 2 \leq i \leq k-1 \textit{ and } e = uu_2 \cdots u_i \in H_i^{(t)} \textit{ such that } \kappa(e) = c \in \bigcap_{j=2}^{i} \Theta^{(t)}(u_j)\right\}$$

is the set of colors *lost* at u in this round.

$$A^{(t)}(u) = A^{(t-1)}(u) \cup L^{(t)}(u).$$

Assign some permanent colors:

Let

$$\Psi^{(t)}(u) = \Theta^{(t)}(u) \setminus (A^{(t)}(u) \cup B^{(t)}(u)) = \text{set of activated colors that can be used at } u.$$

If $\Psi^{(t)}(u) \neq \emptyset$ then choose $c \in \Psi^{(t)}(u)$ arbitrarily. Let $\kappa(u) = c$.

Update parameters:

(a)

$$U^{(t+1)} = U^{(t)} \setminus \Big\{u:\ \Psi^{(t)}(u) \neq \emptyset\Big\}.$$

(b) $H_i^{(t+1)}$, $2 \le i \le k-1$ is the i-graph with vertex set $U^{(t+1)}$ and edge set

$$\{u_1 u_2 \cdots u_i:\ \exists u_{i+1}, \ldots, u_k \in W^{(t+1)} \text{ with } u_1 \cdots u_k \in H \text{ and } \kappa(u_{i+1}) = \cdots = \kappa(u_k) = c\}.$$

Edge $u_1 u_2 \cdots u_i$ has color c. (H simple implies that this color is well-defined).

(c) $p_u^{(t)}(c)$ is replaced by $p_u^{(t+1)}(c)$ which is either 0, $p_u^{(t)}(c)/q_u^{(t)}(c)$, or $\hat{p}$ (note that the last two are at least $p_u^{(t)}(c)$). Furthermore, if $u \in U^{(t)} \setminus U^{(t+1)}$ then by convention $p_u^{(t')} = p_u^{(t+1)}$ for all $t' > t$.

In order to decide which of these three values is taken by $p_u^{(t+1)}(c)$, we perform a random experiment, where we replace $p_u^{(t)}(c)$ by a random value $p'_u(c)$. Based on the outcome of this random experiment, we will decide on the value of $p_u^{(t+1)}(c)$. One of the key properties is

$$\mathsf{E}(p'_u(c)) = p_u^{(t)}(c). \tag{5}$$

The update rule is as follows: If $c \in A^{(t-1)}(u)$ then $p_u^{(t)}(c)$ remains unchanged at zero.

Otherwise, let $\eta_u^{(t)}(c)$ be a random variable with

$$\eta_u^{(t)}(c) \in \{0,1\} \text{ and } \mathsf{P}(\eta_u^{(t)}(c) = 1) = p_u^{(t)}(c)/\hat{p}, \text{ independently of other variables.}$$

Then

$$p'_u(c) = \begin{cases} \begin{cases} 0 & c \in L^{(t)}(u) \\ \frac{p_u^{(t)}(c)}{q_u^{(t)}(c)} & c \notin L^{(t)}(u) \end{cases} & \frac{p_u^{(t)}(c)}{q_u^{(t)}(c)} < \hat{p} \qquad \textbf{Case A} \\ \\ \eta_u^{(t)}(c)\hat{p} & \frac{p_u^{(t)}(c)}{q_u^{(t)}(c)} \ge \hat{p}. \qquad \textbf{Case B} \end{cases} \tag{6}$$

There will be

$$t_0 = \varepsilon^{-1} \log \Delta \log\log \Delta \textit{ rounds}.$$

Before getting into the main body of the proof, we check (5). First observe that $q_u^{(t)}(c)$ is the probability that $c \notin L^{(t)}(u)$ given that $c \notin A^{(t-1)}(u)$.

If $p_u^{(t)}(c)/q_u^{(t)}(c) < \hat{p}$ then

$$\mathsf{E}(p'_u(c)) = q_u^{(t)}(c) \frac{p_u^{(t)}(c)}{q_u^{(t)}(c)} = p_u^{(t)}(c).$$

If $p_u^{(t)}(c)/q_u^{(t)}(c) \geq \hat{p}$ then

$$\mathsf{E}(p'_u(c)) = \hat{p}\frac{p_u^{(t)}(c)}{\hat{p}} = p_u^{(t)}(c).$$

Note that once a color enters $B^{(t)}(u)$, it will be in $B^{(t')}(u)$ for all $t' \geq t$. This is because we update $p_u(c)$ according to Case B and now $\mathsf{P}(\eta_u^{(t)}(c) = 1) = 1$. We arrange things this way, because we want to maintain (5). Then because $p_u^{(t)}(c)$ cannot exceed $\hat{p}$, it must actually remain at $\hat{p}$. This could cause some problems for us if neighbors of u had been colored with c. There might be an edge $e = uu_2 \cdots u_k$ where $u_2, \ldots, u_k$ are (tentatively) colored c. We don't want to raise $p_u^{(t)}(c)$ and to keep it monotone, we can't allow it to drop to zero. This is why $B^{(t)}(u)$ is excluded in the definition of $\Psi^{(t)}(u)$ i.e. we cannot color u with $c \in B^{(t)}(u)$.

3.3 Correctness of the coloring

Observe that if color c enters $A^{(t)}(x)$ at some time t then $\kappa(x) \neq c$ since $A^{(i)}(x) \subseteq A^{(i+1)}(x)$ for all i. Suppose that some edge $u_1 u_2 \cdots u_k$ is improperly colored by the above algorithm. Suppose that $u_1, u_2, \ldots, u_k$ get colored at times $t_1 \leq t_2 \leq \cdots \leq t_k$ and that $\kappa(u_j) = c$ for $j = 1, 2, \ldots, k$. If $t_1 = t_2 = \cdots = t_{k-1} = t$ then $c \in L^{(t)}(u_k)$ and so $\kappa(u_k) \neq c$. If there exists $1 \leq i \leq k-2$ such that $t_i < t = t_{i+1} = \cdots = t_{k-1}$ then $u_{i+1}u_{i+2}\cdots u_k$ is an edge of $H_{k-i}^{(t)}$ and $\kappa(u_{i+1}u_{i+2}\cdots u_k) = c$ and so $c \in L^{(t)}(u_k)$ and again $\kappa(u_k) \neq c$.

3.4 Parameters for the problem

We will now drop the superscript (t), unless we feel it necessary. It will be implicit i.e. $p_u(c) = p_u^{(t)}(c)$ etcetera. Furthermore, we use a $'$ to replace the superscript $(t+1)$ i.e. $p'_u(c) = p_u^{(t+1)}(c)$ etcetera. The following are the main parameters that we need in the course of the proof:

In what follows $u_1 = u$ and $2 \leq i \leq k-1$:

$$\begin{aligned}
\Xi_e &= \sum_{c \in C} \prod_{j=1}^{k} p_{u_j}(c) \qquad \textit{for edge } e = u_1 u_2 \cdots u_k \textit{ of } H^{(t)}. \\
\Phi_{u,i} &= \sum_{c \in C} \sum_{\substack{e = uu_2 \cdots u_i \in H_i \\ \kappa(e) = c}} \prod_{j=1}^{i} p_{u_j}(c) \\
h_u &= -\sum_{c \in C} p_u(c) \log p_u(c). \\
d_i(u,c) &= |\{e : \ u \in e \in H_i \textit{ and } \kappa(e) = c\}| \\
d_i(u) &= \sum_{c \in C} d_i(u,c) = \textit{ degree of } u \textit{ in } H_i \\
d_{H^{(t)}}(u) &= |\{e : \ u \in e \in H^{(t)}\}| = \textit{ degree of } u \textit{ in } H^{(t)} \\
d(u) &= d_2(u) + d_3(u) + \cdots + d_{k-1} + d_{H^{(t)}}(u)
\end{aligned}$$

It will also be convenient to define the following auxiliary parameters:

$$\begin{aligned}
\Xi_e(c) &= \prod_{j=1}^{k} p_{u_j}(c) \qquad \textit{for edge } e = u_1 u_2 \cdots u_k \textit{ of } H^{(t)}. \\
\Xi_u &= \sum_{e=uu_2\cdots u_k \in H^{(t)}} \Xi_e \\
\Xi_u(c) &= \sum_{uu_2\cdots u_k \in H^{(t)}} \prod_{j=2}^{k} p_{u_j}(c) \\
\Phi_{u,i}(c) &= \sum_{\substack{e=uu_2\cdots u_i \in H_i \\ \kappa(e)=c}} \prod_{j=2}^{i} p_{u_j}(c)
\end{aligned}$$

This gives

$$\Xi_u = \sum_{c \in C} p_u(c) \Xi_u(c) \tag{7}$$

$$\Phi_{u,i} = \sum_{c \in C} p_u(c) \Phi_{u,i}(c). \tag{8}$$

3.5 Invariants

We define a set of properties such that if they are satisfied at time t then it is possible to extend our partial coloring and maintain these properties at time $t+1$. These properties are now listed. They are only claimed for $u \in U$ and they are easily verified for $t = 0$. In fact, the reason that q cannot be lowered (given Δ and ω) is the second inequality in (10) for $t = 0$.

$$\left| 1 - \sum_c p_u(c) \right| \leq t\Delta^{-\varepsilon}. \tag{9}$$

$$\begin{aligned}
\Xi_e &\leq \Xi_e^{(0)} + \frac{t}{\Delta^{1+\varepsilon}} \\
&\leq \frac{\omega}{\Delta} + \frac{t}{\Delta^{1+\varepsilon}}, \qquad \forall e \in H^{(t)}.
\end{aligned} \tag{10}$$

$$\Phi_{u,i} \leq k^{2k-2i} \omega (1 - \theta/3k)^t, \qquad 2 \leq i \leq k-1. \tag{11}$$

$$h_u \geq h_u^{(0)} - k^{2k} \varepsilon \sum_{\tau=0}^{t} (1 - \theta/3k)^\tau. \tag{12}$$

$$d(u) \leq (1 - \theta/2k)^t \, \Delta. \tag{13}$$

$$d_i(u,c) \leq (1 + 2k\theta)^t \Delta \hat{p}^{k-i}, \qquad 2 \leq i \leq k-1. \tag{14}$$

Equation (13) shows that after t_0 rounds we find that the maximum degree in the hypergraph induced by the uncolored vertices satisfies

$$\begin{aligned}\Delta(H^{(t_0)}) &\leq (1-\theta/2k)^{t_0}\,\Delta \\ &\leq e^{-\theta t_0/2k}\Delta \\ &< e^{-100k^{2k}\log\log\Delta/2\varepsilon^2}\Delta \\ &= \frac{\Delta}{(\log\Delta)^{100k^{2k}/2\varepsilon^2}}. \end{aligned} \tag{15}$$

and then the local lemma will show that the remaining vertices can be colored with a set of $4(\Delta/(\log\Delta)^{100k^{2k}/2\varepsilon^2})^{1/(k-1)}+1 < q$ new colors.

The above invariants allow us to prove the following bounds: By repeatedly using $(1-a)(1-b) \geq 1-a-b$ for $a, b \geq 0$ we see that

$$q_u(c) \geq 1 - \theta^{k-1}\Xi_u(c) - \sum_{i=2}^{k-1}\theta^{i-1}\Phi_{u,i}(c). \tag{16}$$

3.6 Dynamics

To prove (9) – (14) we show that we can find updated parameters such that

$$\left|\sum_c p'_u(c) - \sum_c p_u(c)\right| \leq \Delta^{-\varepsilon}. \tag{17}$$

$$\Xi'_e \leq \Xi_e + \Delta^{-1-\varepsilon}. \tag{18}$$

$$\begin{aligned}\Phi'_{u,i} - \Phi_{u,i} &\leq \binom{k-1}{i-1}\theta^{k-i}\Xi_u + \sum_{l=i+1}^{k-1}\binom{l-1}{i-1}\theta^{l-i}\Phi_{u,l} \\ &\quad -\theta(1-k^{2k}\varepsilon)\Phi_{u,i} + \Delta^{-\varepsilon}, \qquad 2 \leq i \leq k-1.\end{aligned} \tag{19}$$

$$h_u - h'_u \leq k^{2k}\varepsilon(1-\theta/3k)^t. \tag{20}$$

$$d'(u) \leq (1-\theta/k)d(u) + \Delta^{2/3}. \tag{21}$$

$$d'_i(u,c) \leq d_i(u,c) + 2k\theta(1+2k\theta)^t\Delta\hat{p}^{k-i}, \quad 2 \leq i \leq k-1. \tag{22}$$

3.7 (17)–(22) imply (9)–(14)

First let us show that (17)–(22) are enough to inductively prove that (9)–(13) hold throughout.

Property (9)**:** Trivial.

Property (10)**:** Trivial.

Property (11)**:** Fix u and note that (10) and (13) imply

$$\Xi_u \leq \left(\frac{\omega}{\Delta} + t\Delta^{-1-\varepsilon}\right) d(u) \leq \omega(1-\theta/2k)^t + \Delta^{-\varepsilon/2}. \tag{23}$$

Therefore,

$$\Phi'_{u,i} - \Phi_{u,i} \leq \binom{k-1}{i-1}\theta^{k-i}\omega(1-\theta/2k)^t + \sum_{l=i+1}^{k-1}\binom{l-1}{i-1}\theta^{l-i}\Phi_{u,l} - \theta(1-k^{2k}\varepsilon)\Phi_{u,i} + \Delta^{-\varepsilon/3}$$

from (20) and (23). Thus,

$$\begin{aligned}
\Phi'_{u,k-1} &\leq (k-1)\theta\omega(1-\theta/2k)^t + (1-\theta(1-k^{2k}\varepsilon))k^2\omega(1-\theta/3k)^t + \Delta^{-\varepsilon/3} \\
&\leq k^2(1-\theta/3k)^{t+1}\omega\left(\frac{\theta(1-\theta/2k)^t}{k(1-\theta/3k)^{t+1}} + \frac{1-\theta(1-k^{2k}\varepsilon)}{1-\theta/3k}\right) + \Delta^{-\varepsilon/3} \\
&\leq k^2(1-\theta/3k)^{t+1}\omega.
\end{aligned}$$

Now for $i \leq k-2$, using $\Phi_{u,l} \leq k^{2k-2l}\omega(1-\theta/3k)^t$,

$$\begin{aligned}
\Phi'_{u,i} \leq \binom{k-1}{i-1}\theta^{k-i}\omega(1-\theta/2k)^t + \frac{k^{2k}}{\theta^i}\omega(1-\theta/3k)^t \sum_{l=i+1}^{k-1}\binom{l-1}{i-1}\left(\frac{\theta}{k^2}\right)^l \\
+ (1-\theta(1-k^{2k}\varepsilon))k^{2k-2i}\omega(1-\theta/3k)^t + \Delta^{-\varepsilon/3}.
\end{aligned}$$

Factoring out the first term $\binom{i+1}{i-1}(\theta/k^2)^{i+1}$ in the sum above we are left with a sum that can be upper bounded by $\sum_{i=0}^{\infty} r^i$ where $0 < r < \theta(k-1)/k^2$. Since $\theta = O(1/\log\Delta)$, this geometric series is upper bounded by $1+\theta/k$. Consequently, $\Phi'_{u,i}$ is upper bounded by

$$\begin{aligned}
&\binom{k-1}{i-1}\theta^{k-i}\omega(1-\theta/2k)^t + \frac{k^{2k}}{\theta^i}\omega(1-\theta/3k)^t\binom{k}{2}\left(\frac{\theta}{k^2}\right)^{i+1}(1+\theta/k) \\
&\qquad + (1-\theta(1-k^{2k}\varepsilon))k^{2k-2i}\omega(1-\theta/3k)^t + \Delta^{-\varepsilon/3} \\
&\leq k^{2k-2i}(1-\theta/3k)^{t+1}\omega\left(\binom{k-1}{i-1}\left(\frac{\theta}{k^2}\right)^{k-i}\frac{(1-\theta/2k)^t}{(1-\theta/3k)^{t+1}} + \frac{\theta(1+\theta/k)+2(1-\theta(1-k^{2k}\varepsilon))}{2(1-\theta/3k)}\right) + \Delta^{-\varepsilon/3} \\
&\leq k^{2k-2i}(1-\theta/3k)^{t+1}\omega\left(\frac{2-\theta(1-2k^{2k}\varepsilon)+O(\theta^2)}{2(1-\theta/3k)}\right) + \Delta^{-\varepsilon/4} \\
&\leq k^{2k-2i}(1-\theta/3k)^{t+1}\omega.
\end{aligned}$$

Property (12)**:** Trivial.

Property (13)**:** If $d(u) \leq (1-\theta/2k)^t\Delta$ then from (21) we get

$$\begin{aligned} d'(u) &\leq \left(1-\frac{\theta}{k}\right)\left(1-\frac{\theta}{2k}\right)^t\Delta + \Delta^{2/3} \\ &= \left(1-\frac{\theta}{2k}\right)^{t+1}\Delta - \frac{\theta}{2k}\left(1-\frac{\theta}{2k}\right)^t\Delta + \Delta^{2/3} \\ &\leq \left(1-\frac{\theta}{2k}\right)^{t+1}\Delta. \end{aligned}$$

Property (14)**:**

$$d_i'(u,c) \leq (1+2k\theta)^t\Delta\hat{p}^{k-i} + 2k\theta(1+2k\theta)^t\Delta\hat{p}^{k-i} = (1+2k\theta)^{t+1}\Delta\hat{p}^{k-i}$$

To complete the proof it suffices to show that there are choices for $\gamma_u(c), \eta_u(c)$, $u \in U, c \in C$ such that (17)–(22) hold.

In order to help understand the following computations, the reader is reminded that quantities $\Xi_u, \Phi_{u,i}, \omega, \theta^{-1}$ can all be upper bounded by $\Delta^{o(1)}$. Note also that in (14), $(1+2k\theta)^{t_0} = \log^{O(1)}\Delta = \Delta^{o(1)}$.

3.8 Bad colors

We now put a bound on the weight of the colors in $B(u)$.

Assume that (9)–(13) hold. It follows from (12) that

$$h_u^{(0)} - h_u^{(t)} \leq k^{2k}\varepsilon\sum_{i=0}^{\infty}(1-\theta/3k)^i = 3k \times k^{2k}\omega = \frac{3\varepsilon^2\log\Delta}{100}. \tag{24}$$

Since $p_u^{(0)}(c) = 1/q$ for all u, c we have

$$\begin{aligned} h_u^{(0)} &= -\sum_c p_u^{(0)}(c)\log p_u^{(0)}(c) \\ &= -\sum_c p_u^{(t)}(c)\log p_u^{(0)}(c) - (\log 1/q)\sum_c(p_u^{(0)}(c) - p_u^{(t)}(c)) \\ &\geq -\sum_c p_u^{(t)}(c)\log p_u^{(0)}(c) - t\Delta^{-\varepsilon}\log\Delta. \end{aligned}$$

where the last inequality uses (9).

Plugging this lower bound on $h_u^{(0)}$ into (24) gives

$$\begin{aligned} \frac{3\varepsilon^2\log\Delta}{100} &\geq h_u^{(0)} - h_u^{(t)} \\ &\geq -\sum_c p_u^{(t)}(c)\log p_u^{(0)}(c) - t\Delta^{-\varepsilon}\log\Delta + \sum_c p_u^{(t)}(c)\log p_u^{(t)}(c) \\ &= \sum_c p_u^{(t)}(c)\log(p_u^{(t)}(c)/p_u^{(0)}(c)) - t\Delta^{-\varepsilon}\log\Delta. \end{aligned}$$

Thus,

$$\sum_c p_u^{(t)}(c)\log(p_u^{(t)}(c)/p_u^{(0)}(c)) \le \frac{3\varepsilon^2\log\Delta}{100} + \Delta^{-\varepsilon/2}. \tag{25}$$

Now, all terms in (25) are non-negative ($p_u^{(t)}(c) = 0$ or $p_u^{(t)}(c) \ge p_u^{(0)}(c)$). Thus after dropping the contributions from $c \notin B(u)$ we get

$$\begin{aligned}\frac{3\varepsilon^2\log\Delta}{100} + \Delta^{-\varepsilon/2} &\ge \sum_{c\in B(u)} p_u^{(t)}(c)\log(p_u^{(t)}(c)/p_u^{(0)}(c)) \\ &= \sum_{c\in B(u)} p_u^{(t)}(c)\log(\hat{p}q) = \sum_{c\in B(u)} p_u^{(t)}(c)\log(\Delta^{\varepsilon-o(1)}) \\ &\ge \frac{2}{3}\varepsilon p_u(B(u))\log\Delta.\end{aligned}$$

So,

$$p_u(B(u)) \le \frac{\varepsilon}{10}. \tag{26}$$

3.9 Verification of Dynamics

Let $\mathcal{E}_{17}(u)$ – $\mathcal{E}_{22}(u)$ be the events claimed in equations (17) – (22). Let $\mathcal{E}(u) = \mathcal{E}_{17}(u)\cap\cdots\cap\mathcal{E}_{22}(u)$. We have to show that $\bigcap_{u\in U}\mathcal{E}(u)$ has positive probability. We use the local lemma. Each of the above events depends only on the vertex u or its neighbors. Therefore, the dependency graph of the $\mathcal{E}(u)$, $u \in U$ has maximum degree $\Delta^{O(1)}$ and so it is enough to show that each event $\mathcal{E}_{17}(u), \dots, \mathcal{E}_{22}(u)$, $u \in U$ has failure probability $e^{-\Delta^{\Omega(1)}}$.

While parameters Ξ_u, Φ_u etc. are only needed for $u \in U$ we do not for example consider Ξ_u' conditional on $u \in U'$. We do not impose this conditioning and so we do not have to deal with it. Thus the local lemma will guarantee a value for Ξ_u, $u \in U \setminus U'$ and we are free to disregard it for the next round. (We will however face this conditioning for other reasons, see (36)).

In the following we will use two forms of Hoeffding's inequality for sums of bounded random variables: Suppose first that $X_1, X_2, \dots, X_m$ are independent random variables and $|X_i| \le a_i$ for $1 \le i \le m$. Let $X = X_1 + X_2 + \cdots + X_m$. Then, for any $t > 0$,

$$\max\{\mathsf{P}(X - \mathsf{E}(X) \ge t), \mathsf{P}(X - \mathsf{E}(X) \le -t)\} \le \exp\left\{-\frac{2t^2}{\sum_{i=1}^m a_i^2}\right\}. \tag{27}$$

We will also need the following version in the special case that $X_1, X_2, \dots, X_m$ are independent [0,1] random variables. For $\alpha > 1$ we have

$$\mathsf{P}(X \ge L) \le (3/\alpha)^L \tag{28}$$

for any $L \ge \alpha\mathsf{E}(X)$. (We replace e by 3 as the symbol e is over-used in the paper).

For proofs, see for example Alon and Spencer [4], Appendix A and Lugosi [16].

3.9.1 Dependencies

In our random experiment, we start with the $p_u(c)$'s and then we instantiate the independent random variables $\gamma_u(c), \eta_u(c), u \in U$, $c \in C$ and then we compute the $p'_u(c)$ from these values. Observe first that $p'_u(c)$ depends only on $\gamma_v(c), \eta_v(c)$ for $v = u$ or v a neighbor of u in H. So $p'_u(c)$ and $p'_v(c^*)$ are independent if $c \neq c^*$, even if $u = v$. We call this *color independence*.

Let

$$N_i(u) = \{\{u_2, u_3, \ldots, u_i\} \subseteq U : \exists e \in H \ s.t.\ \{u, u_2, \ldots, u_i\} \subseteq e\}.$$

We shorten $N_2(u)$ to $N(u)$.

If $f = \{u_2, u_3, \ldots, u_i\} \in N_i(u)$ and $l > i$, then $E_{u,f,l} = \{e \in N_l(u) : e \supseteq f \cup \{u\}, e \in H_l\}$.

Next let

$$N_{i,l}(u) = \{f = \{u_2, \ldots, u_i\} \subseteq U :\ f \in N_i(u) \ and\ E_{u,f,l} \neq \emptyset\}, \quad 2 \leq i < l \leq k.$$

In words, $N_{i,l}$ is the collection of i-sets containing u that are subsets of edges of H_l.

In these definitions $H_k = H^{(t)}$.

For each $v \in N(u)$ we let

$$C_u(v) = \{c \in C : \gamma_u(c) = 1\} \cup L(v) \cup B(v).$$

Note that while the first two sets in this union depend on the random choices made in this round, the set $B(v)$ is already defined at the beginning of the round.

We will later use the fact that if $c^* \notin C_u(v)$ and $\gamma_v(c^*) = 1$ then this is enough to place c^* into $\Psi(v)$ and allow v to be colored. Indeed, we only need to check that $c^* \notin A^{(t-1)}(v)$ as it will then follow that $c^* \notin A(v)$. However, $\gamma_v(c^*) = 1$ implies that $p_v(c^*) \neq 0$ from which it follows that $c^* \notin A^{(t-1)}(v)$.

Let $Y_v = \sum_c p_v(c) 1_{c \in C_u(v)} = p_v(C_u(v))$. $C_u(v)$ is a random set and Y_v is the sum of q independent random variables each one bounded by $\hat{p}$. Then by (7), (8) and (16),

$$\begin{aligned} \mathsf{E}(Y_v) &\leq \sum_{c \in C} p_v(c) \mathsf{P}(\gamma_u(c) = 1) + \sum_{c \in C} p_v(c)(1 - q_v(c)) + p_v(B(v)) \\ &\leq \theta \sum_{c \in C} p_u(c) p_v(c) + \theta^{k-1} \Xi_v + \sum_{i=2}^{k-1} \theta^{i-1} \Phi_{v,i} + p_v(B(v)). \end{aligned}$$

Now let us bound each term separately:

$$\theta \sum_{c \in C} p_u(c) p_v(c) \leq \theta q \hat{p}^2 < \theta \Delta^{1/(k-1)} \Delta^{2\varepsilon - 2/(k-1)} < \frac{\varepsilon}{3}.$$

Using (10) we obtain

$$\theta^{k-1} \Xi_v < \omega \theta^{k-1} + t\theta^{k-1} \Delta^{-\varepsilon} \leq \varepsilon \theta^{k-2} + t\theta^{k-1} \Delta^{-\varepsilon} < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}.$$

Using (11) we obtain

$$\sum_{i=2}^{k-1}\theta^{i-1}\Phi_{v,i} \le \sum_{i=2}^{k-1}\theta^{i-1}k^{2k-2i}\omega(1-\theta/3k)^t \le k^{2k-1}\varepsilon.$$

Together with $\mathsf{P}(B(v)) \le \varepsilon/10$ we get

$$\mathsf{E}(Y_v) \le (k^{2k-1}+1)\varepsilon.$$

Hoeffding's inequality then gives

$$\mathsf{P}(Y_v \ge \mathsf{E}(Y_v)+\rho) \le \exp\left\{-\frac{2\rho^2}{q\hat{p}^2}\right\} < e^{-2\rho^2\Delta^{1/k}}.$$

Taking $\rho = \Delta^{-1/2k}$ say, it follows that

$$\mathsf{P}(p_v(C_u(v)) \ge (k^{2k-1}+2)\varepsilon) = \mathsf{P}(Y_v \ge (k^{2k-1}+2)\varepsilon) \le e^{-\Delta^{1/2k}}. \tag{29}$$

Let $\mathcal{E}_{(29)}$ be the event $\left\{p_v(C_u(v)) \le (k^{2k-1}+2)\varepsilon\right\}$.

Now consider some fixed vertex $u \in U$. It will sometimes be convenient to condition on the values $\gamma_x(c), \eta_x(c)$ for all $c \in C$ and all $x \notin N(u)$ and for $x = u$. This conditioning is needed to obtain independence. We let $\mathcal{C}$ denote these conditional values.

Remark 8 *Note that $\mathcal{C}$ determines the set $C_u(v)$, and hence it also determines whether or not $\mathcal{E}_{(29)}$ occurs. Indeed, if $\gamma_u(c) = 1$, then $c \in C_u(v)$. On the other hand, if $\gamma_u(c) = 0$ then whether or not $c \in L(v)$ depends only on colors tentatively assigned to vertices not in $N(u)$. This uses the simplicity and triangle-freeness of H.*

Given the conditioning $\mathcal{C}$, simplicity and triangle freeness imply that the events $\{v \notin U'\}$, $\{w \notin U'\}$ for $v, w \in N(u)$ are independent provided u, v, w are not part of an edge of H. Indeed, triangle-freeness implies that in this case, there is no edge containing both v and w. Therefore the random choices at w will not affect the coloring of v (and vice versa). Thus random variables $p'_v(c), p'_w(c)$ will become (conditionally) independent under these circumstances. We call this *conditional neighborhood independence.*

3.9.1.1 Some expectations

Let us fix a color c and an edge $u_1u_2\cdots u_k \in H$ (here we mean H and not $H^{(t)}$). In this subsection we will estimate the expectation of $\mathsf{E}\left(\prod_{j=1}^{i} p'_{u_j}(c)\right)$ for $2 \le i \le k$ in two distinct situations.

Case 1: $i = k$ and $u_1u_2\cdots u_k \in H^{(t)}$.

Our goal is to prove that

$$\mathsf{E}\left(\prod_{j=1}^{k} p'_{u_j}(c)\right) \le (1+2k\theta^2\hat{p}^2)\prod_{j=1}^{k} p_{u_j}(c) \tag{30}$$

If $c \in \bigcup_{j=1}^k A^{(t-1)}(u_j)$ then $\prod_{j=1}^k p'_{u_j}(c) = 0$. Assume then that $c \notin \bigcup_{j=1}^k A^{(t-1)}(u_j)$. If Case B of (6) occurs for any of $u_1, u_2, \ldots, u_k$ e.g. u_k then

$$\mathsf{E}\left(\prod_{j=1}^k p'_{u_j}(c) \middle| Case\ B\ for\ k\right) = \mathsf{E}\left(\prod_{j=1}^{k-1} p'_{u_j}(c)\right) p_{u_k}(c). \tag{31}$$

This is because in Case B the value of $\eta_{u_k}(c)$ is independent of all other random variables and we may use (5). One can see then that we have to prove something slightly more general than (30). So we now aim to show that

$$\mathsf{E}\left(\prod_{j=1}^i p'_{u_j}(c)\right) \leq (1 + 2k\theta^2\hat{p}^2) \prod_{j=1}^i p_{u_j}(c) \tag{32}$$

assuming that $1 \leq i \leq k$ and that there is an edge $u_1 u_2 \cdots u_k \in H^{(t)}$ and that Case A of (6) happens for u_j, c, $1 \leq j \leq i$. The case $i = 1$ follows from (5) and so we assume that $i \geq 2$. By simplicity, $u_{i+1}, \ldots, u_k$ are determined by $u_1, u_2, \ldots, u_i$.

Now $\prod_{j=1}^i p'_{u_j}(c) = 0$ unless $c \notin \bigcup_{j=1}^i L(u_j)$. Consequently,

$$\mathsf{E}\left(\prod_{j=1}^i p'_{u_j}(c)\right) = \prod_{j=1}^i \frac{p_{u_j}(c)}{q_{u_j}(c)} \times \mathsf{P}\left(c \notin \bigcup_{j=1}^i L(u_j)\right). \tag{33}$$

Furthermore,

$$\begin{aligned}
\mathsf{P}\left(c \notin \bigcup_{j=1}^i L(u_j) \middle| \gamma_{u_j}(c) = 0, 1 \leq j \leq k\right) &= \prod_{j=1}^i \left(q_{u_j}(c)\left(1 - \theta^{k-1} \prod_{j' \neq j} p_{u_{j'}}(c)\right)^{-1}\right) \\
&\leq (1 + 2k\theta^{k-1}\hat{p}^{k-1}) \prod_{j=1}^i q_{u_j}(c) \\
&\leq (1 + 2k\theta^2\hat{p}^2) \prod_{j=1}^i q_{u_j}(c).
\end{aligned}$$

On the other hand we will show that

$$\mathsf{P}\left(c \notin \bigcup_{j=1}^i L(u_j) \middle| \exists 1 \leq j \leq k : \gamma_{u_j}(c) = 1\right) \leq \mathsf{P}\left(c \notin \bigcup_{j=1}^i L(u_j) \middle| \gamma_{u_j}(c) = 0, 1 \leq j \leq k\right). \tag{34}$$

This is intuitively clear, since color c is at least as likely to be lost at u_j if it is tentatively activated at some $u_{j'}$. Indeed, to see this formally, partition the probability space Ω of outcomes of the γ's and η's into the sets $\Omega_{\varepsilon_1, \ldots, \varepsilon_k}$ in which $\gamma_{u_j}(c) = \varepsilon_j \in \{0, 1\}$ for $1 \leq j \leq k$. Let $\Omega'_{\varepsilon_1, \ldots, \varepsilon_k}$ be the set of outcomes in $\Omega_{\varepsilon_1,\dots,\varepsilon_k}$ in which $c \notin \bigcup_{j=1}^{i} L(u_j)$. Now consider the map $f : \Omega_{\varepsilon_1,\dots,\varepsilon_k} \to \Omega'_{0,\dots,0}$ which just sets $\gamma_{u_j}(c)$ to 0 for $1 \le j \le k$. Then if $\pi_j^1 = \theta p_{u_j}(c)$ and $\pi_j^0 = 1 - \pi_j^1$

$$\frac{\mathsf{P}(\Omega_{\varepsilon_1,\dots,\varepsilon_k})}{\mathsf{P}(\Omega_{0,\dots,0})} = \frac{\prod_{j=1}^{k} \pi_j^{\varepsilon_j}}{\prod_{j=1}^{k} \pi_j^{0}} = \frac{\mathsf{P}(\Omega'_{\varepsilon_1,\dots,\varepsilon_k})}{\mathsf{P}(f(\Omega'_{\varepsilon_1,\dots,\varepsilon_k}))}.$$

If $c \notin \bigcup_{j=1}^{i} L(u_j)$ and $\exists 1 \le j \le k$ such that $\gamma_{u_j}(c) = 1$, then we still have $c \notin \bigcup_{j=1}^{i} L(u_j)$ if we change $\gamma_{u_j}(c)$ to 0 for $1 \le j \le k$ and make no other changes. Consequently, $f(\Omega'_{\varepsilon_1,\dots,\varepsilon_k}) \subseteq \Omega'_{0,\dots,0}$ and we have

$$\frac{\mathsf{P}(\Omega'_{0,\dots,0})}{\mathsf{P}(\Omega_{0,\dots,0})} \ge \frac{\mathsf{P}(f(\Omega'_{\varepsilon_1,\dots,\varepsilon_k}))}{\mathsf{P}(\Omega_{\varepsilon_1,\dots,\varepsilon_k})} \cdot \frac{\mathsf{P}(\Omega_{\varepsilon_1,\dots,\varepsilon_k})}{\mathsf{P}(\Omega_{0,\dots,0})} = \frac{\mathsf{P}(\Omega'_{\varepsilon_1,\dots,\varepsilon_k})}{\mathsf{P}(\Omega_{\varepsilon_1,\dots,\varepsilon_k})},$$

which is (34).

It follows that

$$\mathsf{P}\left(c \notin \bigcup_{j=1}^{i} L(u_j)\right) \le (1 + 2k\theta^2\hat{p}^2) \prod_{j=1}^{i} q_{u_j}(c). \tag{35}$$

and in combination with (33) this proves (32) and hence (30).

Case 2: $e = u_1 u_2 \cdots u_i \in H_i$, $\kappa(e) = c$.

Our goal is now to prove

$$\mathsf{E}\left(\prod_{j=1}^{i} p'_{u_j}(c) \times 1_{u_1,u_2,\dots,u_i \in U'}\right) \le (1 + k\theta\hat{p}) \prod_{j=1}^{i} p_{u_j}(c). \tag{36}$$

Suppose that $p'_{u_j}(c)$ is determined by Case A of (6) for $1 \le j \le l$ and by Case B otherwise. We factor out $\prod_{j=l+1}^{i} p_{u_j}(c)$ as in (31) and concentrate on bounding

$$\mathsf{E}\left(\prod_{j=1}^{l} p'_{u_j}(c) \times 1_{u_1,u_2,\dots,u_i \in U'}\right)$$

$$= \prod_{j=1}^{l} \frac{p_{u_j}(c)}{q_{u_j}(c)} \times \mathsf{P}(c \notin L(u_1) \cup \cdots L(u_l) \wedge u_1, \dots, u_i \in U')$$

$$\le \prod_{j=1}^{l} \frac{p_{u_j}(c)}{q_{u_j}(c)} \times \mathsf{P}(c \notin L(u_1) \cup \cdots L(u_l))$$

$$\le \prod_{j=1}^{l} \frac{p_{u_j}(c)}{q_{u_j}(c)} \times \mathsf{P}(c \notin L(u_1) \cup \cdots L(u_l) \mid \gamma_{u_1}(c) = \cdots = \gamma_{u_l}(c) = 0) \tag{37}$$

$$\le \prod_{j=1}^{l} p_{u_j}(c) \times (1 - \theta\hat{p})^{-l}. \tag{38}$$

$$\le (1 + k\theta\hat{p}) \prod_{j=1}^{l} p_{u_j}(c) \tag{39}$$

and (36) follows.

Explanation: Equation (37) follows as for (34). Equation (38) now follows because the events $c \notin L(u_j)$ become conditionally independent. And then $\mathsf{P}(c \notin L(u_j) \mid \gamma_{u_j}(c) = 0)$ gains a factor $\left(1 - \theta^{i-1} \prod_{j' \neq j} p_{u_j}(c)\right)^{-1} \leq (1 - \theta\hat{p})^{-1}$.

3.9.2 Proof of (17)

Given the $p_u(c)$ we see that if $Z' = \sum_{c \in C} p'_u(c)$ then $\mathsf{E}(Z') = \sum_{c \in C} p_u(c)$. This follows on using (5). By color independence Z' is the sum of q independent non-negative random variables each bounded by $\hat{p}$. Applying (27) we see that

$$\mathsf{P}(|Z' - \mathsf{E}(Z')| \geq \rho) \leq 2 \exp\left\{-\frac{2\rho^2}{q\hat{p}^2}\right\} = 2e^{-2\rho^2 \Delta^{1/(k-1) - 2\varepsilon - o(1)}}.$$

We take $\rho = \Delta^{-\varepsilon}$ to see that $\mathcal{E}_{17}(u)$ holds **whp**[1].

3.9.3 Proof of (18)

Given the $p_u(c)$ we see that by (30), Ξ'_e has expectation no more than $\Xi_e(1 + 2k\theta^2\hat{p}^2)$ and is the sum of q independent non-negative random variables, each of which is bounded by $\hat{p}^k$. We have used color independence again here. Applying (27) we see that

$$\mathsf{P}(\Xi'_e \geq \Xi_e(1 + 2k\theta^2\hat{p}^k) + \rho/2) \leq \exp\left\{-\frac{\rho^2}{2q\hat{p}^{2k}}\right\} \leq e^{-\rho^2 \Delta^{(2k-1)/(k-1) - 2k\varepsilon - o(1)}}.$$

We also have

$$k\Xi_e\theta^2\hat{p}^2 \leq k\left(\frac{\omega}{\Delta} + \frac{t}{\Delta^{1+\varepsilon}}\right)\theta^2\hat{p}^2 < \frac{1}{2\Delta^{1+\varepsilon}}.$$

We take $\rho = \Delta^{-1-\varepsilon}$ to obtain

$$\mathsf{P}(\Xi'_e \geq \Xi_e + \Delta^{-1-\varepsilon}) \leq e^{-\Delta^{\Omega(1)}}$$

and so $\mathcal{E}_{18}(u)$ holds **whp**.

3.9.4 Proof of (20)

Throughout this section $u_1 = u$. Recall that

$$\Phi_{u,i} = \sum_{c \in C} \sum_{e = uu_2 \cdots u_i \in H_i} 1_{\kappa(e) = c} \prod_{j=1}^{i} p_{u_j}(c).$$

[1]By **whp**, *with high probability*, we mean with probability $1 - e^{-\Delta^{\Omega(1)}}$.

If $\{u_2,\ldots,,u_i\} \in N_i(u)$ and $e = uu_2\cdots u_i \notin H_i$ then $\kappa(e)$ is defined to be $0 \notin C$. Now

$$\Phi'_{u,i} - \Phi_{u,i} = \sum_{c\in C}\left(\sum_{e=uu_2\cdots u_i\in H'_i} 1_{\kappa'(e)=c}\prod_{j=1}^{i} p'_{u_j}(c) - \sum_{e=uu_2\cdots u_i\in H_i} 1_{\kappa(e)=c}\prod_{j=1}^{i} p_{u_j}(c)\right)$$

If $e \in H_i$ and $\kappa(e) = c$ then $e \in H'_i$ is equivalent to $\kappa'(e) = c$. If $\kappa(e) \neq c$ but $\kappa'(e) = c$ then the edge of H containing $u, u_2, \ldots, u_i$ has some other vertices in U that will be colored with c in the current round. Consequently, the above expression is

$$D_1 + \sum_{l=i+1}^{k} D_{2,l}$$

where

$$\begin{aligned}
D_1 &= \sum_{c\in C}\sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c}}\left(1_{\kappa'(e)=c}\prod_{j=1}^{i} p'_{u_j}(c) - \prod_{j=1}^{i} p_{u_j}(c)\right)\\
D_{2,k} &= \sum_{c\in C}\sum_{\{u_2,\ldots u_i\}\in N_{i,k}(u)} 1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i} p'_{u_j}(c)\\
D_{2,l} &= \sum_{c\in C}\sum_{\substack{\{u_2,\ldots u_i\}\in N_{i,l}(u)\\ \kappa(uu_2\cdots u_l)=c}} 1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i} p'_{u_j}(c) \qquad i+1 \leq l \leq k-1.
\end{aligned}$$

Here D_1 accounts for the contribution from edges leaving H_i and $D_{2,i+1},\ldots,D_{2,k}$ account for the contribution from edges entering H_i.

We bound $\mathsf{E}(D_1)$ and $\mathsf{E}(D_{2,i+1}),\ldots,\mathsf{E}(D_{2,k})$ separately.

$\mathsf{E}(D_1)$:

$$\begin{aligned}
D_1 &= \sum_{c\in C}\sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c}}\left(1_{\kappa'(e)=c}\prod_{j=1}^{i} p'_{u_j}(c) - \prod_{j=1}^{i} p_{u_j}(c)\right)\\
&= -\sum_{c\in C}\sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c\\ \kappa'(e)\neq c}}\prod_{j=1}^{i} p_{u_j}(c) + \sum_{c\in C}\sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c\\ \kappa'(e)=c}}\left(\prod_{j=1}^{i} p'_{u_j}(c) - \prod_{j=1}^{i} p_{u_j}(c)\right).
\end{aligned}$$

Now suppose that $\exists j \geq 2 : u_j \notin U'$. This means that u_j has been colored in the current round and so $e \notin H'_i$. So $\kappa'(e) \neq c$ is implied by $\exists j \geq 2 : u_j \notin U'$ and more simply it is implied by

$u_2 \notin U'$. Conversely, if $e \in H_i'$ then $u_1, u_2, \ldots, u_i \in U'$. Therefore the prior expression is bounded from above by

$$-D_{1,1} + D_{1,2}$$

where

$$\begin{aligned} D_{1,1} &= \sum_{c \in C} \sum_{\substack{e = uu_2 \cdots u_i \in H_i \\ \kappa(e) = c}} \prod_{j=1}^{i} p_{u_j}(c) 1_{u_2 \notin U'} \\ D_{1,2} &= \sum_{c \in C} \sum_{\substack{e = uu_2 \cdots u_i \in H_i \\ \kappa(e) = c}} \left(\left(\prod_{j=1}^{i} p'_{u_j}(c) - \prod_{j=1}^{i} p_{u_j}(c) \right) \times 1_{u_1, u_2, \ldots, u_i \in U'} \right). \end{aligned}$$

Suppose that $e = uu_2 \cdots u_k \in H$ and $u_{i+1}, u_{i+2}, \ldots, u_k \notin U$ and $\kappa(u_{i+1}) = \cdots = \kappa(u_k) = c$. Let $v = u_2$. Recall that

$$C_u(v) = \{c \in C : \gamma_u(c) = 1\} \cup L(v) \cup B(v).$$

If there is a tentatively activated color c^* at v (i.e. $\gamma_v(c^*) = 1$) that lies outside $C_u(v) \cup \{c\}$, then $c^* \in \Psi(v)$ and v will be colored in this round (recall that we had argued earlier that $c^* \notin A^{(t-1)}(v)$). Therefore

$$\mathsf{P}(v \notin U' \mid \mathcal{C}) \geq \mathsf{P}(\exists c^* \notin C_u(v) \cup \{c\} : \ \gamma_v(c^*) = 1 \mid \mathcal{C}).$$

We have introduced the conditioning $\mathcal{C}$ because we will need it later when we prove concentration.

So by inclusion-exclusion and the independence of the $\gamma_v(c^*)$ we can write

$$\begin{aligned} \mathsf{E}\left(1_{v \notin U'} \mid \mathcal{C}\right) &\geq \mathsf{P}(\exists c^* \notin C_u(v) \cup \{c\} : \ \gamma_v(c^*) = 1 \mid \mathcal{C}) \\ &\geq \sum_{c^* \notin C_u(v) \cup \{c\}} \mathsf{P}(\gamma_v(c^*) = 1 \mid \mathcal{C}) - \frac{1}{2} \sum_{c_1^* \neq c_2^* \notin C_u(v) \cup \{c\}} \mathsf{P}(\gamma_v(c_1^*) = \gamma_v(c_2^*) = 1 \mid \mathcal{C}) \\ &\geq \sum_{c^* \notin C_u(v) \cup \{c\}} \theta p_v(c^*) - \frac{1}{2} \left(\sum_{c^* \notin C_u(v) \cup \{c\}} \theta p_v(c^*) \right)^2 \end{aligned}$$

Now

$$\begin{aligned} \sum_{c^* \notin C_u(v) \cup \{c\}} \theta p_v(c^*) &= \sum_{c^* \in C} \theta p_v(c^*) - \sum_{c^* \in C_u(v)} \theta p_v(c^*) - \theta p_v(c) \\ &\geq \theta((1 - t\Delta^{-\varepsilon}) - p_v(C_u(v)) - \hat{p}) \\ &> \theta(1 - p_v(C_u(v)) - \varepsilon/2) \end{aligned}$$

where we have used (9). Also by (9) and the definition of $\hat{p}$ we have

$$\sum_{c \neq c^*} p_v(c^*) \leq 1 + (t+1)\Delta^{-\varepsilon} < 1.1.$$

Consequently

$$\frac{1}{2}\left(\sum_{c^*\notin C_u(v)\cup\{c\}}\theta p_v(c^*)\right)^2=\frac{\theta^2}{2}\left(\sum_{c^*\notin C_u(v)\cup\{c\}}p_v(c^*)\right)^2\le\frac{2\theta^2}{3}<\frac{\theta\varepsilon}{2}.$$

Putting these facts together yields

$$\mathsf{E}\left(1_{v\notin U'}\mid\mathcal{C}\right)\ge\theta(1-p_v(C_u(v))-\varepsilon).$$

Therefore

$$\mathsf{E}(D_{1,1}\mid\mathcal{C})\ge\theta(1-p_v(C_u(v))-\varepsilon)\sum_{c\in C}\sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c}}\prod_{j=1}^{i}p_{u_j}(c)=\theta(1-p_v(C_u(v))-\varepsilon)\Phi_{u,i}.$$

Given $\mathcal{C}$, Remark 8 implies that $\mathcal{E}_{(29)}$ either holds for all outcomes in $\mathcal{C}$, or fails for all outcomes in $\mathcal{C}$. So,

$$\mathsf{E}(D_{1,1}\mid\mathcal{C})\ge\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i},\qquad\textit{for }\mathcal{C}\textit{ such that }\mathcal{E}_{(29)}\textit{ occurs.}\tag{40}$$

We now consider $D_{1,2}$. It follows from (11) that $\Phi_{u,i}<k^{2k-2i}\omega$. Together with (36), this gives

$$\mathsf{E}(D_{1,2})\le k\Phi_{u,i}\theta\hat{p}\le k^{2k}\varepsilon\hat{p}.\tag{41}$$

$\mathsf{E}(D_{2,k})$:
Recall that

$$D_{2,k}=\sum_{c\in C}\sum_{\{u_2,\dots,u_i\}\in N_{i,k}(u)}1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i}p'_{u_j}(c).$$

Instead of summing over sets in $N_{i,k}(u)$, we may sum over edges $uu_2\cdots u_k\in H^{(t)}$, and then over subsets of these edges that lie in $N_{i,k}(u)$. Thus

$$D_{2,k}=\sum_{c\in C}\sum_{uu_2\cdots u_k\in H^{(t)}}\sum_{f\subset\{u_2,\dots,u_k\},|f|=i-1}1_{\kappa'(f\cup\{u\})=c}\prod_{u_j\in f\cup\{u_1\}}p'_{u_j}(c).$$

Fix an edge $uu_2\cdots u_k\in H^{(t)}$. If $u_{i+1},\dots,u_k$ are colored with c in this round, then certainly c must have been tentatively activated at these vertices. Therefore,

$$\begin{aligned}
\mathsf{E}\left(1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i}p'_{u_j}(c)\right) &\le \mathsf{E}\left(\prod_{j=i+1}^{k}\gamma_{u_j}(c)\prod_{j=1}^{i}p'_{u_j}(c)\right)\\
&\le \theta^{k-i}\prod_{j=i+1}^{k}p_{u_j}(c)\prod_{j=1}^{i}\frac{p_{u_j}(c)}{q_{u_j}(c)}\mathsf{P}\left(c\notin\bigcup_{j=1}^{i}L(u_j)\,\middle|\,\bigwedge_{j=i+1}^{k}(\gamma_{u_j}(c)=1)\right)\\
&\le \theta^{k-i}\prod_{j=i+1}^{k}p_{u_j}(c)\prod_{j=1}^{i}\frac{p_{u_j}(c)}{q_{u_j}(c)}\mathsf{P}\left(c\notin\bigcup_{j=1}^{i}L(u_j)\right) &&(42)\\
&\le \theta^{k-i}\prod_{j=1}^{k}p_{u_j}(c)(1+2k\theta^2\hat{p}^2). &&(43)
\end{aligned}$$

We use the argument for (34) to obtain (42) and (35) to obtain (43).

It follows that

$$\mathsf{E}(D_{2,k}) \le \binom{k-1}{i-1}\theta^{k-i}\Xi_u(1+2k\theta^2\hat{p}^2). \tag{44}$$

$\mathsf{E}(D_{2,l})$, $l < k$:
Recall that

$$D_{2,l} = \sum_{c\in C}\sum_{\substack{\{u_2,u_3,\dots u_i\}\in N_{i,l}(u)\\ \kappa(uu_2\cdots u_l)=c}} 1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i} p'_{u_j}(c).$$

Fix an edge $uu_2\cdots u_k \in H$ with $uu_2\cdots u_l \in H_l$. Then arguing as we did for (43) we have

$$\mathsf{E}\left(1_{\kappa'(uu_2\cdots u_i)=c}\prod_{j=1}^{i} p'_{u_j}(c)\right) \le \theta^{l-i}\prod_{j=1}^{l} p_{u_j}(c)(1+2k\theta^2\hat{p}^2).$$

It follows that

$$\mathsf{E}(D_{2,l}) \le \binom{l-1}{i-1}\theta^{l-i}\Phi_{u,l}(1+2k\theta^2\hat{p}^2). \tag{45}$$

3.9.4.1 Concentration

We first deal with $D_{1,1}$. For this we condition on the values $\gamma_w(c),\eta_w(c)$ for all $c\in C$ and all $w\notin N(u)$ and for $w=u$. Then by conditional neighborhood independence $D_{1,1}$ is the sum of at most $d_i(u)$ independent random variables of value at most $\hat{p}^i$. By (14), we have $d_i(u)\le q(1+2k\theta)^t\Delta\hat{p}^{k-i} = \Delta^{1+1/(k-1)+o(1)}\hat{p}^{k-i}$. So, for $\rho>0$,

$$\mathsf{P}(D_{1,1}-\mathsf{E}(D_{1,1}\mid\mathcal{C})\le -\rho\mid\mathcal{C}) \le \exp\left\{-\frac{2\rho^2}{d_i(u)\hat{p}^{2i}}\right\} \le \exp\left\{-\frac{2\rho^2}{\Delta^{1+1/(k-1)+o(1)}\hat{p}^{k-i}\hat{p}^{2i}}\right\} \le e^{-\rho^2\Delta^{i/k}}.$$

So, by (40),

$$\begin{aligned}
&\mathsf{P}(D_{1,1}\le\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i}-\Delta^{-1/2k})\\
&=\sum_{\mathcal{C}}\mathsf{P}(D_{1,1}\le\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i}-\Delta^{-1/2k}\mid\mathcal{C})\mathsf{P}(\mathcal{C})\\
&\le\sum_{\mathcal{C}:\mathcal{E}_{(29)}\ occurs}\mathsf{P}(D_{1,1}\le\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i}-\Delta^{-1/2k}\mid\mathcal{C})\mathsf{P}(\mathcal{C})+\mathsf{P}(\neg\mathcal{E}_{(29)})\\
&\le\sum_{\mathcal{C}:\mathcal{E}_{(29)}\ occurs}\mathsf{P}(D_{1,1}\le\mathsf{E}(D_{1,1}\mid\mathcal{C})-\Delta^{-1/2k}\mid\mathcal{C})\mathsf{P}(\mathcal{C})+\mathsf{P}(\neg\mathcal{E}_{(29)})\\
&\le e^{-\Delta^{1/k}}+e^{-\Delta^{1/2k}}\\
&=e^{-\Delta^{\Omega(1)}}.
\end{aligned} \tag{46}$$

Now consider the sum $D_{1,2}$. Let

$$Y_c = \sum_{\substack{e=uu_2\cdots u_i\in H_i\\ \kappa(e)=c\\ \kappa'(e)=c}}\left(\prod_{j=1}^{i}p'_{u_j}(c)-\prod_{j=1}^{i}p_{u_j}(c)\right).$$

$D_{1,2}$ is the sum of q independent random variables Y_c satisfying $0 \leq Y_c \leq d_c\hat{p}^i$ where $d_c = d_i(u,c)$. Note that (14) implies $d_c \leq \Delta^{1+o(1)}\hat{p}^{k-i}$.

So, for $\rho > 0$,

$$\mathsf{P}(D_{1,2} - \mathsf{E}(D_{1,2}) \geq \rho) \leq \exp\left\{-\frac{2\rho^2}{\sum_c d_c^2\hat{p}^{2i}}\right\} \leq \exp\left\{-\frac{2\rho^2}{\Delta^{2+1/(k-1)+o(1)}\hat{p}^{2k}}\right\} \leq e^{-\rho^2\Delta^{1/(k-1)-2k\varepsilon+o(1)}}.$$

We take $\rho = \Delta^{-1/2k}$ to see that $\mathsf{P}(D_{1,2} \geq 2\Delta^{-1/2k}) \leq e^{-\Delta^\varepsilon}$. Combining this with (46) we see that

$$\begin{aligned}
&\mathsf{P}(D_1 \geq -\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i} + 3\Delta^{-1/2k}) \\
&\leq \mathsf{P}(D_{1,1} \leq \theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i} + \Delta^{-1/2k}) + \mathsf{P}(D_{1,2} \geq 2\Delta^{-1/2k}) \\
&\leq e^{-\Delta^{\Omega(1)}}.
\end{aligned} \tag{47}$$

We now deal with the $D_{2,l}$. There is a minor problem in that the $D_{2,l}$ are sums of random variables for which we do not have a sufficiently small absolute bound. These variables do however have a small bound which holds with high probability. There are several ways to use this fact. We proceed as follows: First assume $l \leq k-1$ and let

$$D_{2,l,c} = \sum_{\substack{\{u_2,u_3,\dots u_i\}\in N_{i,l}(u) \\ \kappa(uu_2\cdots u_l)=c}} 1_{\kappa'(uu_2\cdots u_i)=c} \prod_{j=1}^{i} p'_{u_j}(c)$$

which we re-write as

$$D_{2,l,c} = \sum_{\substack{e=uu_2\cdots u_l\in H_l \\ \kappa(e)=c}} Z_e,$$

where

$$Z_{uu_2\cdots u_l} = \sum_{\substack{S\subset\{u_2,u_3,\dots u_l\} \\ |S|=i-1}} 1_{\kappa'(S\cup\{u\})=c} \prod_{u_j\in S\cup\{u_1\}} p'_{u_j}(c).$$

Then we let

$$\hat{D}_{2,l} = \sum_{c\in C} \min\left\{(1+2k\theta)^t\Delta\hat{p}^k, D_{2,l,c}\right\}.$$

Observe that $\hat{D}_{2,l}$ is the sum of q independent random variables each bounded by $(1+2k\theta)^t\Delta\hat{p}^k$. So, for $\rho > 0$,

$$\mathsf{P}(\hat{D}_{2,l} - \mathsf{E}(\hat{D}_{2,l}) \geq \rho) \leq \exp\left\{-\frac{2\rho^2}{\Delta^{2+o(1)}\hat{p}^{2k}}\right\} \leq e^{-\rho^2\Delta^{1/(k-1)-2k\varepsilon}}.$$

We take $\rho = \Delta^{-1/2k}$ to see that

$$\mathsf{P}(\hat{D}_{2,l} \geq \mathsf{E}(\hat{D}_{2,l}) + \Delta^{-1/2k}) \leq e^{-\Delta^\varepsilon}. \tag{48}$$

We must of course compare $D_{2,l}$ and $\hat{D}_{2,l}$. Now $D_{2,l} \neq \hat{D}_{2,l}$ only if there exists c such that $D_{2,l,c} > (1+2k\theta)^t \Delta \hat{p}^k$. For each c, $D_{2,l,c}$ is the sum of the $d_l(u,c) \leq (1+2k\theta)^t \Delta \hat{p}^{k-l}$ variables Z_e, $e \in H_l$. Each Z_e is bounded above by $\binom{l-1}{i-1}\hat{p}^i$ and $\mathsf{E}(Z_e) \leq \binom{l-1}{i-1}\theta^{l-i}\hat{p}^l$. This is because Z_e is bounded by the sum of $\binom{l-1}{i-1}$ variables $Z_{e,S}$, each taking the value 0 or $\hat{p}^i$. Here $Z_{e,S}$ corresponds to some $S = \{u_2, u_3, \ldots u_i\} \subseteq \{u_2, u_3, \ldots u_l\}$. Furthermore, $\mathsf{P}(Z_{e,S} = \hat{p}^i) \leq (\theta\hat{p})^{l-i}$ because this will happen only if the vertices in S tentatively choose c.

H being simple and triangle free, if we condition on $\mathcal{C}$ then the random variables Z_e become independent.

Now put $X_e = Z_e/(\binom{l-1}{i-1}\hat{p}^i)$ and $X = \sum_e X_e$. We see that $0 \leq X_e \leq 1$ and $\mathsf{E}(X_e) \leq (\theta\hat{p})^{l-i}$.

We now use (28) with $\alpha = 1/(\binom{l-1}{i-1}\theta^{l-i})$ and $\mathsf{E}(X) \leq (1+2k\theta)^t \Delta \hat{p}^{k-l} \times (\theta\hat{p})^{l-i} = \frac{(1+2k\theta)^t \Delta \hat{p}^k}{\alpha\binom{l-1}{i-1}\hat{p}^i}$. This gives

$$\mathsf{P}(D_{2,l,c} \geq (1+2k\theta)^t \Delta \hat{p}^k \mid \mathcal{C}) \leq \mathsf{P}\left(X \geq \frac{(1+2k\theta)^t \Delta \hat{p}^k}{\binom{l}{i}\hat{p}^i} \middle| \mathcal{C}\right) \leq \left(3\binom{l}{i}\theta^{l-i}\right)^{(1+2k\theta)^t \Delta \hat{p}^{k-i}/\binom{l}{i}}$$

Therefore

$$\begin{aligned}
\mathsf{P}(D_{2,l} \neq \hat{D}_{2,l}) &\leq \sum_{\mathcal{C}} \mathsf{P}(\exists c:\ D_{2,l,c} \geq (1+2k\theta)^t \Delta \hat{p}^k \mid \mathcal{C})\mathsf{P}(\mathcal{C}) \\
&\leq \sum_{\mathcal{C}} q\left(3\binom{l}{i}\theta^{l-i}\right)^{(1+2k\theta)^t \Delta \hat{p}^{k-i}/\binom{l}{i}} \mathsf{P}(\mathcal{C}) \\
&\leq e^{-\Delta^\varepsilon}. \qquad (49)
\end{aligned}$$

It follows from (49) and $\hat{D}_{2,l} \leq D_{2,l} \leq \Delta$ that

$$|\mathsf{E}(D_{2,l}) - \mathsf{E}(\hat{D}_{2,l})| \leq \Delta \mathsf{P}(D_{2,l} \neq \hat{D}_{2,l}) \leq \Delta q e^{-\Delta^\varepsilon} < \Delta^{-1/2k}.$$

Applying (48) and (49) we see that

$$\mathsf{P}(D_{2,l} \geq \mathsf{E}(D_{2,l}) + 2\Delta^{-1/2k}) \leq \mathsf{P}(\hat{D}_{2,l} \geq \mathsf{E}(\hat{D}_{2,l}) + \Delta^{-1/2k}) + \mathsf{P}(D_{2,l} \neq \hat{D}_{2,l}) \leq 2e^{-\Delta^\varepsilon}. \qquad (50)$$

We must now deal with the case of $l = k$ i.e.

$$D_{2,k,c} = \sum_{\{u_2,u_3,\ldots u_i\} \in N_{i,k}(u)} 1_{\kappa'(uu_2\cdots u_i)=c} \prod_{j=1}^{i} p'_{u_j}(c)$$

and

$$\hat{D}_{2,k} = \sum_{c \in C} \min\left\{\Delta\hat{p}^k, D_{2,k,c}\right\}.$$

We re-write

$$D_{2,k,c} = \sum_{S \in N_{i,k}(u)} W_S$$

where for $S = \{u_2, u_3, \dots u_i\}$,

$$W_S = \sum_{e \supseteq S, e \in H^{(t)}} 1_{\kappa'(uu_2 \cdots u_i) = c} \prod_{j=1}^{i} p'_{u_j}(c).$$

Now we view $D_{2,k,c}$ as the sum of at most Δ random variables, each of which is bounded by $\binom{k}{i}\hat{p}^i$ and has expectation bounded by $\binom{k}{i}\theta^{k-i}\hat{p}^k$. We now simply follow the argument for $l < k$ by taking $l = k$ to show that

$$\mathsf{P}(D_{2,k} \geq \mathsf{E}(D_{2,k}) + 2\Delta^{-1/2k}) \leq 2e^{-\Delta^{\varepsilon}}. \tag{51}$$

Indeed, (48) holds with $l = k$. Then

$$\mathsf{P}(D_{2,k} \neq \hat{D}_{2,k}) \leq q\mathsf{P}\left(Bin\left(\Delta, (\theta\hat{p})^{k-i}\right) \geq \Delta\hat{p}^{k-i}\right) \leq q\left(3\theta^{k-i}\right)^{\Delta\hat{p}^{k-i}} \leq e^{-\Delta^{\varepsilon}}.$$

Combining (50) and (51) with (47) we see that **whp**,

$$\begin{aligned}
\Phi'_{u,i} - \Phi_{u,i} &\leq -\theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i} + \binom{k-1}{i-1}\theta^{k-i}\Xi_u + \sum_{l=i+1}^{k-1}\binom{l}{i-1}\theta^{l-i}\Phi_{u,l} + \\
&\quad +2k\Delta^{-1/2k} + 2k\theta^2\hat{p}^2\left(\binom{k-1}{i-1}\theta^{k-i}\Xi_u + \sum_{l=i+1}^{k-1}\binom{l}{i-1}\theta^{l-i}\Phi_{u,l}\right) \\
&\leq \binom{k-1}{i-1}\theta^{k-i}\Xi_u + \sum_{l=i+1}^{k-1}\binom{l}{i-1}\theta^{l-i}\Phi_{u,l} - \theta(1-(k^{2k-1}+3)\varepsilon)\Phi_{u,i} + \Delta^{-\varepsilon}.
\end{aligned}$$

This confirms (20).

3.9.5 Proof of (20)

Fix c and write $p' = p'_u(c) = p\beta$. We consider two cases, but in both cases $\mathsf{E}(\beta) = 1$ and β takes two values, 0 and $1/\mathsf{P}(\beta > 0)$. Then we have

$$\mathsf{E}(-p'\log p') = -p\log p - p\log(1/\mathsf{P}(\beta > 0)).$$

(i) $p = p_u(c)$ and $\beta = \gamma_u(c)/q_u(c)$ and $\gamma_u(c)$ is a $\{0,1\}$ random variable with $\mathsf{P}(\beta > 0) = q_u(c)$.

(ii) $p = p_u(c) = \hat{p}$ and β is a $\{0,1\}$ random variable with $\mathsf{P}(\beta > 0) = p_u(c)/\hat{p} \geq q_u(c)$.

Thus in both cases

$$\mathsf{E}(-p'\log p') \geq -p\log p - p\log 1/q_u(c).$$

Observe next that $0 \leq a, b \leq 1$ implies that $(1-ab)^{-1} \leq (1-a)^{-b}$ and $-\log(1-x) \leq x + x^2$ for $0 \leq x \ll 1$. So, from (3),

$$\begin{aligned} \log 1/q_u(c) &\leq -\Xi_u(c)\log(1-\theta^{k-1}) - \sum_{i=2}^{k-1} \Phi_{u,i}(c)\log(1-\theta^{i-1}) \\ &\leq (\theta^{k-1}+\theta^{2k-2})\Xi_u(c) + \sum_{i=2}^{k-1}(\theta^{i-1}+\theta^{2i-2})\Phi_{u,i}(c). \end{aligned}$$

Now

$$\begin{aligned} \mathsf{E}(h_u - h'_u) &\leq -\sum_c p_u(c)\log p_u(c) - \mathsf{E}\left(\sum_c -p'_u(c)\log p'_u(c)\right) \\ &\leq \sum_c -p_u(c)\log p_u(c) - \left(\sum_c -p_u(c)\log p_u(c) - p_u(c)\log 1/q_u(c)\right) \\ &= \sum_c p_u(c)\log 1/q_u(c) \\ &\leq (\theta^{k-1}+\theta^{2k-2})\sum_c p_u(c)\Xi_u(c) + \sum_c \sum_{i=2}^{k-1}(\theta^{i-1}+\theta^{2i-2})p_u(c)\Phi_{u,i}(c) \\ &= (\theta^{k-1}+\theta^{2k-2})\Xi_u + \sum_{i=2}^{k-1}(\theta^{i-1}+\theta^{2i-2})\Phi_{u,i} \\ &\leq (\theta^{k-1}+\theta^{2k-2})(\omega + t\Delta^{-\varepsilon})(1-\theta/2k)^t + \sum_{i=2}^{k-1}(\theta^{i-1}+\theta^{2i-2})k^{2k-2i}\omega(1-\theta/3k)^t \\ &\leq k^{2k-3}\varepsilon(1-\theta/3k)^t. \end{aligned}$$

Given the $p_u(c)$ we see that h'_u is the sum of q independent non-negative random variables with values bounded by $-\hat{p}\log\hat{p} \leq \Delta^{-1/(k-1)+\varepsilon+o(1)}$. Here we have used color independence. So,

$$\mathsf{P}(h_u - h'_u \geq k^{2k-3}\varepsilon(1-\theta/3k)^t + \rho) \leq \exp\left\{-\frac{2\rho^2}{q(\hat{p}\log\hat{p})^2}\right\} = e^{-2\rho^2\Delta^{1/k}}.$$

We take $\rho = \varepsilon(1-\theta/3k)^t \geq (\log\Delta)^{-O(1)}$ to see that $h_u - h'_u \leq k^{2k}\varepsilon(1-\theta/3k)^t$ holds **whp**.

3.9.6 Proof of (21)

Fix u and condition on the values $\gamma_w(c), \eta_w(c)$ for all $c \in C$ and all $w \notin N(u)$ and for $w = u$. Now write $u \sim v$ to mean that $\{u, v\}$ lies in an edge of $H^{(t)}$ or some H_i. Then write

$$Z_u = d(u) - d'(u) \geq \frac{1}{k-1}\sum_{u\sim v} Z_{u,v} \text{ where } Z_{u,v} = 1_{v\notin U'}.$$

Now, for $e = uu_2 \cdots u_k \in H^{(t)}$ let $Z_{u,e} = \sum_{j=2}^{k} Z_{u,u_j}$ and if $e = uu_2 \cdots u_i \in H_i$ let $Z_{u,e} = \sum_{j=2}^{i} Z_{u,u_j}$. Conditional neighborhood independence implies that the collection $Z_{u,e}$ constitute an independent set of random variables. Applying (27) to $Z_u = \sum_e Z_{u,e}$ we see that

$$\mathsf{P}(Z_u \leq \mathsf{E}(Z_u) - \Delta^{2/3}) \leq \exp\left\{-\frac{2\Delta^{4/3}}{(k-1)^2\Delta}\right\} = e^{-2\Delta^{1/3}/(k-1)^2}. \tag{52}$$

and so we only have to estimate $\mathsf{E}(Z_u)$.

Fix $v \sim u$. Let $C_u(v)$ be as in (29). Condition on $\mathcal{C}$. The vertex v is a member of U' if none of the colors $c \notin C_u(v))$ are tentatively activated. The activations we consider are done independently and so

$$\begin{aligned} \mathsf{P}(v \in U' \mid \mathcal{C}) &\leq \prod_{c \notin C_u(v)} (1 - \theta p_v(c)) \qquad (53) \\ &\leq \exp\left\{-\sum_{c \notin C_u(v)} \theta p_v(c)\right\} \\ &\leq \exp\left\{-\theta(1 - t\Delta^{-\varepsilon}) + \theta p_v(C_u(v))\right\} \end{aligned}$$

If $\mathcal{E}_{(29)}$ occurs then $p_v(C_u(v)) \leq k^{2k}\varepsilon$. Consequently,

$$\mathsf{P}(v \notin U') \geq \sum_{\mathcal{C}:\mathcal{E}_{(29)}\ occurs} \left(1 - \exp\left\{-\theta(1 - \Delta^{-\varepsilon}) + k^{2k}\varepsilon\theta\right\}\right) \mathsf{P}(\mathcal{C}),$$

where the sum is well-defined due to Remark 8. Since $\theta \to 0$ as $\Delta \to \infty$, and ε is sufficiently small,

$$1 - \exp\left\{-\theta(1 - \Delta^{-\varepsilon}) + k^{2k}\varepsilon\theta\right\} > 1 - e^{-\theta(1-2k^{2k}\varepsilon)} > \theta(1 - 3k^{2k}\varepsilon) > \theta(1 - 1/k^2).$$

Recall that (29) shows that

$$\mathsf{P}(\mathcal{E}_{(29)}\ fails) \leq e^{-\Delta^{1/2k}} < 1/k^2.$$

Therefore

$$\mathsf{P}(v \notin U') \geq \theta(1 - 1/k^2) \sum_{\mathcal{C}:\mathcal{E}_{(29)}\ occurs} \mathsf{P}(\mathcal{C}) > \theta(1 - 1/k^2)^2 > \theta(1 - 1/k).$$

This gives

$$\mathsf{E}(Z_u) \geq \frac{1}{k}\theta d(u)$$

and (21).

3.9.7 Proof of (22)

Observe that if $e = uu_2 \cdots u_i \in H_i' \setminus H_i$ and $\kappa'(e) = c$ then either

(i) there exists $1 \le j \le k-i-1$ and vertices $u_{i+1}, \ldots, u_{i+j}$ and an edge $uu_2 \cdots u_{i+j} \in H_{i+j}$ such that $u_{i+1}, \ldots, u_{i+j}$ get colored in Step t with c and so $\gamma_{u_{i+1}}(c) = \cdots = \gamma_{u_{i+j}}(c) = 1$ or

(ii) there exists $uu_2 \cdots u_k \in H^{(t)}$ such that $u_{i+1}, \ldots, u_k$ all receive the color c and so $\gamma_{u_{i+1}}(c) = \cdots = \gamma_{u_k}(c) = 1$. Hence,

$$d_i'(u) - d_i(u) \le \sum_{j=1}^{k-i} Z_j$$

where for $j \le k-i-1$, $Z_j \le Bin((1+2k\theta)^t \Delta \hat{p}^{k-i-j}, \binom{k-i}{j}(\theta\hat{p})^j)$ and $Z_{k-i} \le Bin(\Delta, \binom{k}{i}(\theta\hat{p})^{k-i})$. If $i < k-1$, then the Chernoff bound

$$Pr(Bin(n,p) \ge 2np) \le e^{-np/3}$$

implies that for $1 \le j < k-i$,

$$\mathsf{P}\left(Z_j \ge 2(1+2k\theta)^t \binom{k-i}{j} \theta^j \Delta \hat{p}^{k-i}\right) \le e^{-\Delta^\varepsilon}.$$

Similarly,

$$\mathsf{P}(Z_{k-i} \ge 2(1+2k\theta)^t \theta^{k-i} \Delta \hat{p}^{k-i}) \le e^{-\Delta^\varepsilon}.$$

Therefore **whp**

$$\begin{aligned} d_i'(u,c) - d_i(u,c) &\le 2(1+2k\theta)^t \Delta \hat{p}^{k-i} \sum_{j=1}^{k-i} \binom{k-i}{j} \theta^j \\ &= 2(1+2k\theta)^t \Delta \hat{p}^{k-i}((1+\theta)^{k-i} - 1) \le 2k\theta(1+2k\theta)^t \Delta \hat{p}^{k-i}. \end{aligned}$$

3.10 List Coloring

Here we describe the small modifications needed to our argument to prove the same result for list colorings. Each vertex $v \in V$ starts with a set A_v of $2q$ available colors. Choose for each v a set $B_v \subseteq A_v$ where $|B_v| = q$. Let now $C = \bigcup_{v \in V} B_v$. We initialise $p_v(c) = q^{-1} 1_{c \in B_v}$ and follow the main argument as before. When the semi-random procedure finishes, the local lemma can be used to show that the lists $A_v \setminus B_v$ can be used to color the vertices that remain uncolored.

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