

On a two-sided Turán problem

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Abstract

Given positive integers n, k, t , with $2 \leq k \leq n$, and $t < 2^k$, let $m(n, k, t)$ be the minimum size of a family \mathcal{F} of nonempty subsets of $[n]$ such that every k -set in $[n]$ contains at least t sets from \mathcal{F} , and every $(k-1)$ -set in $[n]$ contains at most $t-1$ sets from \mathcal{F} . Sloan et. al [11] determined $m(n, 3, 2)$ and Füredi et. al [6] studied $m(n, 4, t)$ for $t = 2, 3$. We consider $m(n, 3, t)$ and $m(n, 4, t)$ for all the remaining values of t and obtain their exact values except for $k = 4$ and $t = 6, 7, 11, 12$. For example, we prove that $m(n, 4, 5) = \binom{n}{2} - 17$ for $n \geq 160$. The values of $m(n, 4, t)$ for $t = 7, 11, 12$ are determined in terms of well-known (and open) Turán problems for graphs and hypergraphs. We also obtain bounds of $m(n, 4, 6)$ that differ by absolute constants.

Keywords: Extremal problem, Hypergraph, Turán number, set system

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1 Introduction

We consider an extremal problem for set systems. Given integers n, k, t , with $2 \leq k \leq n$, and $t < 2^k$, a family $\mathcal{F} \subset 2^{[n]} \setminus \{\emptyset\}$ is a (k, t) -system of $[n]$ if every k -set in $[n]$ contains at least t sets

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from \mathcal{F} , and every $(k-1)$ -set in $[n]$ contains at most $t-1$ sets from \mathcal{F} . Let $m(n, k, t)$ denote the minimum size of a (k, t) -system of $[n]$. This threshold function first arose in problems on computer science [10, 11] (although the notation $m(n, k, t)$ was not used until [6]). It was shown in [11] that $m(n, k, t) = \Theta(n^{k-1})$ for $1 < t < k$ and $m(n, 3, 2) = \binom{n-1}{2} + 1$. In [6], $m(n, 4, 3)$ was determined exactly for large n and it was shown that for fixed k , $m(n, k, 2) = (1 + o(1))T_{k-1}(n, k, 2)$, where $T_r(n, k, t)$ is the generalized Turán number. For fixed k and $t < 2^k$, the order of magnitude of $m(n, k, t)$ was determined in [9]. A special case of this result is the following proposition, where $\binom{a}{\leq b} = \sum_{i=1}^b \binom{a}{i}$.

Proposition 1. [9] $m(n, k, 1) = \binom{n}{k}$, $m(n, k, k) = n$, $m(n, k, 2^k - 2) = \binom{n}{\leq k-1}$ and $m(n, k, 2^k - 1) = \binom{n}{\leq k}$.

In this paper we study $m(n, k, t)$ for $k = 3, 4$. The case $k = 3$ is not very difficult: Proposition 1 determines $m(n, 3, t)$ for $t \in \{1, 3, 6, 7\}$ and [11] shows that $m(n, 3, 2) = \binom{n-1}{2} + 1$. The remaining cases $t = 4$ and $t = 5$ are covered below.

Proposition 2.

$$m(n, 3, t) = \begin{cases} n + \binom{n}{2} - \lfloor n^2/4 \rfloor & t = 4, \\ n + \binom{n}{2} - \lfloor n/2 \rfloor & t = 5. \end{cases}$$

The main part of this paper is devoted to $m(n, 4, t)$, a problem which is substantially more difficult than the case $k = 3$. As mentioned above, both $m(n, 4, 2)$ and $m(n, 4, 3)$ were studied in [6]. It was shown in [11] how these two functions apply to frequent sets of Boolean matrices, a concept used in knowledge discovery and data mining. Perhaps the determination of $m(n, 4, t)$ for other t will have similar applications.

The cases $t = 1, 4, 14, 15$ are answered by Proposition 1 immediately. In this paper we obtain the exact values of $m(n, 4, t)$ for $t = 5, 8, 9, 10, 13$. Our bounds for $m(n, 4, 6)$ differ only by an absolute constant. For $t = 7, 11, 12$, we determine $m(n, 4, t)$ exactly in terms of well-known (and open) Turán problems in extremal graph and hypergraph theory. Perhaps this connection provides additional motivation for investigating $m(n, k, t)$ (the first connection between $m(n, k, t)$ and Turán numbers was shown in [6] via $m(n, k, 2) = (1 + o(1))T_{k-1}(n, k, 2)$).

For a family of r -uniform hypergraphs \mathcal{H} , let $\text{ex}(n, \mathcal{H})$ be the maximum number of edges in an n vertex r -uniform hypergraph \mathcal{G} containing no member of \mathcal{H} as a subhypergraph. The (2-uniform) cycle of length l is written C_l . The complete 3-uniform hypergraph on four points is $K_4^{(3)}$, and the 3-uniform hypergraph on four points with three edges is $H(4, 3)$.

An $(n, 3, 2)$ -packing is a 3-uniform hypergraph on n vertices such that every pair of vertices is contained in at most one edge. The packing number $P(n, 3, 2)$ is the size of a maximal $(n, 3, 2)$ -packing. Note that the maximal packing is a Steiner system when $n \equiv 1$ or $3 \pmod{6}$.

Theorem 3 (Main Theorem).

$$m(n, 4, 5) = \binom{n}{2} - 17,$$

when $n \geq 160$ and

$$\binom{n}{2} - 190 < m(n, 4, 6) \leq \binom{n}{2} - 5,$$

when $n \geq 8$. Furthermore,

$$\begin{aligned} m(n, 4, 7) &= n + \binom{n}{2} - ex(n, \{C_3, C_4\}), \\ m(n, 4, 8) &= n + \binom{n}{2} - 2n/3, \\ m(n, 4, 9) &= n + \binom{n}{2} - 1, \\ m(n, 4, 10) &= n + \binom{n}{2}, \\ m(n, 4, 11) &= n + \binom{n}{2} + \binom{n}{3} - ex(n, K_4^{(3)}), \\ m(n, 4, 12) &= n + \binom{n}{2} + \binom{n}{3} - ex(n, H(4, 3)), \\ m(n, 4, 13) &= n + \binom{n}{2} + \binom{n}{3} - P(n, 3, 2). \end{aligned}$$

It is worth recalling the known results for the three Turán numbers and the packing number $P(n, 3, 2)$ in Theorem 3 above.

- It is known that $(\frac{1}{2\sqrt{2}} + o(1))n^{3/2} \leq ex(n, \{C_3, C_4\}) \leq (\frac{1}{2} + o(1))n^{3/2}$ (Erdős-Rényi [3], Kővari-Sós-Turán [7]). Erdős and Simonovits [4] conjectured that $ex(n, \{C_3, C_4\}) = (\frac{1}{2\sqrt{2}} + o(1))n^{3/2}$.
- It is known that $(5/9)\binom{n}{3} \leq ex(n, K_4^{(3)}) \leq (0.592 + o(1))\binom{n}{3}$ (Turán [14], Chung-Lu [2]). It was conjectured [14] that the lower bound is correct (Erdős offered \$1000 for a proof).
- It is known $(2/7 + o(1))\binom{n}{3} \leq ex(n, H(4, 3)) \leq (1/3 - 10^{-6} + o(1))\binom{n}{3}$ (Frankl-Füredi [5], Mubayi [8]). It was conjectured [8] that $ex(n, H(4, 3)) = (2/7 + o(1))\binom{n}{3}$.
- Spencer [12] determine $P(n, 3, 2)$ exactly:

$$P(n, 3, 2) = \begin{cases} \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor - 1 & \text{if } n \equiv 5 \pmod{6}, \\ \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor & \text{otherwise.} \end{cases}$$

This paper is organized as follows. In Section 2 we describe the main idea in the proofs and prove Propositions 2. The Main Theorem (Theorem 3) is proved in Section 3.

Most of our notations are standard: $[n] = \{1, 2, \dots, n\}$. For a set system \mathcal{F} , let \mathcal{F}_t denote the family of t -sets in \mathcal{F} , let $\mathcal{F}_{\leq t} = \cup_{i \leq t} \mathcal{F}_i$ and $\mathcal{F}_{\geq t} = \cup_{i \geq t} \mathcal{F}_i$. If $a \in \mathcal{F}$ and $b \notin \mathcal{F}$, we simply write $\mathcal{F} - a$ for $\mathcal{F} \setminus \{a\}$ and $\mathcal{F} + b$ for $\mathcal{F} \cup \{b\}$. Given a set X and an integer a , let $2^X = \{S : S \subseteq X\}$, $\binom{X}{a} = \{S \subset X : |S| = a\}$, $\binom{X}{\leq a} = \{S \subset X : 1 \leq |S| \leq a\}$ and $\binom{X}{\geq a} = \{S \subset X : |S| \geq a\}$. We write $\mathcal{F}(X)$ for $\mathcal{F} \cap 2^X$. An r -graph on X is a (hyper)graph whose edges are r -subsets of X . All sets or subsets considered in this paper are nonempty unless specified differently.

2 Ideas in the proofs and $m(n, 3, t)$

In this section we make some basic observations on $m(n, k, t)$ and prove Propositions 2.

Recall that a (k, t) -system $\mathcal{F} \subseteq 2^{[n]} \setminus \emptyset$ satisfies the following two conditions:

Property D (DENSE): Every k -set in $[n]$ contains at least t sets from \mathcal{F} ,

Property S (SPARSE): Every $(k - 1)$ -set in $[n]$ contains at most $t - 1$ sets from \mathcal{F} .

The main idea in our proofs is to work with optimal (k, t) -systems which are defined as follows.

Definition 4. *Suppose that \mathcal{F} is a (k, t) -system of $[n]$. We say that \mathcal{F} is optimal if $|\mathcal{F}| = m(n, k, t)$ and $\sum_{S \in \mathcal{F}} |S|$ is minimal among all (k, t) -system of $[n]$ with size $m(n, k, t)$.*

The advantage of considering optimal (k, t) -systems \mathcal{F} is that it allows us to assume certain structure on \mathcal{F} : if \mathcal{F} does not have such a structure, we always modify \mathcal{F} to \mathcal{F}' such that \mathcal{F}' is a (k, t) -system with $\sum_{S \in \mathcal{F}'} |S| < \sum_{S \in \mathcal{F}} |S|$, a contradiction to the optimality of \mathcal{F} . A typical modification of \mathcal{F} is replacing a set in \mathcal{F} by one of its subsets. Because the new system still satisfies Property **D**, we only need to check Property **S** in this case.

For example, if \mathcal{F} is an optimal (k, t) -system for $t \geq 2^{k-1}$, then we may assume that

$$A \in \mathcal{F} \Rightarrow (2^A \setminus \emptyset) \subset \mathcal{F}. \quad (\star)$$

Indeed, if $A \in \mathcal{F}$ has a nonempty subset $B \notin \mathcal{F}$, then $\mathcal{F}' = \mathcal{F} - A + B$ is also a (k, t) -system, because Property **S** holds trivially (any $(k - 1)$ -set of $[n]$ has at most $2^{k-1} - 1 \leq t - 1$

nonempty subsets). Since $\sum_{S \in \mathcal{F}'} |S| < \sum_{S \in \mathcal{F}} |S|$, this contradicts the optimality of \mathcal{F} .

Now we consider $m(n, 3, t)$ for $3 \leq t \leq 7$. Applying Proposition 1 directly, we have $m(n, 3, 3) = n$, $m(n, 3, 6) = \binom{n}{\leq 2}$ and $m(n, 3, 7) = \binom{n}{\leq 3}$.

Proof of Proposition 2. We determine $m(n, 3, t)$ exactly for $t = 4, 5$. Recall that $\mathcal{F}(S) = \mathcal{F} \cap 2^S$ for a set system \mathcal{F} and a set S .

Let \mathcal{F} be an optimal $(3, t)$ -system with $4 \leq t \leq 5$. Since $t \geq 4 \geq 2^2$, we may assume that (\star) holds in \mathcal{F} . First, we claim that $\binom{[n]}{1} \subset \mathcal{F}$. Suppose instead, that there exists some $a \in [n]$ such that $\{a\} \notin \mathcal{F}$. Pick a 3-set $T = \{a, b, c\}$. Since $\{a\} \notin \mathcal{F}$, by (\star) , we know that \mathcal{F} does not contain $\{a, b\}, \{a, c\}$ and T as well. Thus $|\mathcal{F}(T)| \leq 3$, a contradiction to Property **D**. Second, we claim that $\mathcal{F} \subset \binom{[n]}{\leq 2}$. Suppose instead, that there exists a set $T \in \mathcal{F}_3$. Then $|\mathcal{F}(T)| = 7$ by (\star) and consequently $\mathcal{F}' = \mathcal{F} - T$ is a $(3, t)$ -system of cardinality $|\mathcal{F}| - 1$, contradicting the optimality of \mathcal{F} .

When $t = 4$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on n vertices in which every set of 3 vertices has at least one edge, *i.e.*, $\overline{\mathcal{F}_2}$, the complement of \mathcal{F}_2 is a K_3 -free graph. Thus $|\mathcal{F}_2| \geq \binom{n}{2} - \text{ex}(n, K_3) = \binom{n}{2} - \lfloor n^2/4 \rfloor$. Consequently $m(n, 3, 4) = n + |\mathcal{F}_2| \geq n + \binom{n}{2} - \lfloor n^2/4 \rfloor$. On the other hand, $\binom{[n]}{1} \cup E(G)$ is a $(3, 4)$ -system, where G is a complete bipartite graph with two color classes of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Consequently $m(n, 3, 4) = n + \binom{n}{2} - \lfloor n^2/4 \rfloor$.

When $t = 5$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on n vertices in which every 3 vertices have at least two edges. Therefore $\overline{\mathcal{F}_2}$ is a matching M and $|\mathcal{F}_2| = \binom{n}{2} - |M| \geq \binom{n}{2} - \lfloor n/2 \rfloor$. Consequently $m(n, 3, 5) \geq n + \binom{n}{2} - \lfloor n/2 \rfloor$ and equality holds for the $(3, 5)$ -system $\mathcal{F} = \binom{[n]}{1} \cup E(G)$, where G is a complete graph except for a matching of size $\lfloor n/2 \rfloor$. \square

3 The values of $m(n, 4, t)$

Applying Proposition 1, we obtain that $m(n, 4, 1) = \binom{n}{4}$, $m(n, 4, 4) = n$, $m(n, 4, 14) = \binom{n}{\leq 3}$ and $m(n, 4, 15) = \binom{n}{\leq 4}$. In this section we prove Theorem 3, *i.e.*, determine $m(n, 4, t)$ for $5 \leq t \leq 13$. We consider the cases $7 \leq t \leq 13$ in Section 3.1. The more difficult cases $t = 5, 6$ are studied in Section 3.2 and 3.3, respectively.

3.1 The cases $7 \leq t \leq 13$

Our proof is facilitated by the following four lemmas, whose proofs are postponed to the end of this section.

In Lemmas 5 - 8, \mathcal{F} is an optimal $(4, t)$ -system.

Lemma 5. *If $2 \leq t \leq 14$, then $\mathcal{F}_4 = \emptyset$.*

Lemma 6. *If $7 \leq t \leq 10$, then $\mathcal{F}_3 = \emptyset$.*

Lemma 7. *If $7 \leq t \leq 14$, then $\binom{[n]}{1} \subset \mathcal{F}$.*

Lemma 8. *If $11 \leq t \leq 14$, then $\binom{[n]}{2} \subset \mathcal{F}$.*

Proof of Theorem 3 for $7 \leq t \leq 13$:

By Lemmas 5, 6 and 7, we conclude that

$$\binom{[n]}{1} \subset \mathcal{F} \subset \binom{[n]}{\leq 2} \quad \text{for } 7 \leq t \leq 10.$$

Clearly, when $t = 10$, $\mathcal{F} = \binom{[n]}{\leq 2}$ and consequently $m(n, 4, 10) = \binom{n}{\leq 2}$.

When $t = 9$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 5 edges. Then there is at most one edge absent from \mathcal{F}_2 , or $|\mathcal{F}_2| \geq \binom{n}{2} - 1$. Consequently $m(n, 4, 9) \geq n + \binom{n}{2} - 1$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 2} \setminus e$ for some $e \in \binom{[n]}{2}$.

When $t = 8$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 4 edges. Therefore, $\overline{\mathcal{F}_2}$ contains no K_3 , S_3 (a star with 3 leaves), or P_3 (a path of length 3). Thus all connected components of $\overline{\mathcal{F}_2}$ have size at most 3 and each component is either an edge or P_2 . So $|\overline{\mathcal{F}_2}| \leq \lfloor 2n/3 \rfloor$ and $|\mathcal{F}| \geq n + \binom{n}{2} - \lfloor 2n/3 \rfloor$. Consequently $m(n, 4, 8) = n + \binom{n}{2} - \lfloor 2n/3 \rfloor$ and the optimal system is $\binom{[n]}{\leq 2} \setminus E(G)$, where G is the union of disjoint copies of P_2 and P_1 covering $[n]$ with maximum copies of P_2 .

When $t = 7$, $\mathcal{F}_2 = \mathcal{F} \setminus \binom{[n]}{1}$ is the edge set of a graph on $[n]$ in which every 4-set has at least 3 edges. Let G be a graph on $[n]$ with $E(G) = \overline{\mathcal{F}_2}$. Then G contains no copies of C_4 or C_3^+ (C_3 plus an edge). If C_3 is also absent in G , then $e(G) \leq ex(n, \{C_3, C_4\})$. Otherwise, assume that G contains $t (\geq 1)$ copies of C_3 on a vertex-set T . Because G is C_3^+ -free, the copies of C_3 must be vertex-disjoint and

$$e(G) = 3t + e(G \setminus T) \leq 3t + ex(n - 3t, \{C_3, C_4\}) \leq ex(n, \{C_3, C_4\}),$$

where the last inequality is an easy exercise. Consequently $m(n, 4, 7) \geq n + \binom{n}{2} - ex(n, \{C_3, C_4\})$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 2} \setminus E(G)$, where G is an extremal graph without C_3 or C_4 .

By Lemma 5, 7 and 8, we conclude that

$$\binom{[n]}{\leq 2} \subset \mathcal{F} \subset \binom{[n]}{\leq 3} \quad \text{for } 11 \leq t \leq 13.$$

When $t = 11$, $\mathcal{F}_3 = \mathcal{F} \setminus \binom{[n]}{\leq 2}$ is the edge set of a 3-graph in which every 4-set has at least one hyper-edge. In other words, the 3-graph $([n], \overline{\mathcal{F}_3})$ contains no $K_4^{(3)}$ and therefore $|\overline{\mathcal{F}_3}| \leq ex(n, K_4^{(3)})$. Consequently $m(n, 4, 11) \geq \binom{n}{\leq 3} - ex(n, K_4^{(3)})$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 3} \setminus \mathcal{H}$, where \mathcal{H} is the edge set of an extremal 3-graph without $K_4^{(3)}$.

By a similar argument, we obtain that $m(n, 4, 12) \geq \binom{n}{\leq 3} - ex(n, H(4, 3))$ and equality holds when $\mathcal{F} = \binom{[n]}{\leq 3} \setminus \mathcal{H}$, where \mathcal{H} is the edge set of an extremal 3-graph without $H(4, 3)$.

Finally, when $t = 13$, $\overline{\mathcal{F}_3}$ is an $(n, 3, 2)$ -packing since every 4-set of $[n]$ contains at most one hyper-edge of $\overline{\mathcal{F}_3}$. Since $|\overline{\mathcal{F}_3}| \leq P(n, 3, 2)$, we have $m(n, 4, 13) \geq \binom{n}{\leq 3} - P(n, 3, 2)$ and equality holds when $\overline{\mathcal{F}_3}$ is a maximal $(n, 3, 2)$ -packing. \square

Before verifying Lemma 5, we start with a technical lemma, which is very useful in the cases $5 \leq t \leq 7$.

Lemma 9. *Suppose that $t \in \{5, 6, 7\}$ and \mathcal{F} is an optimal $(4, t)$ -system. Fix a set $P \in \binom{[n]}{\leq 2} \setminus \mathcal{F}$ and let*

$$\mathcal{T} = \{T \in \binom{[n]}{3} : T \supset P, |\mathcal{F}(T)| = t - 1\}. \quad (1)$$

If $\mathcal{T} \subset \mathcal{F}$, then $T \notin \mathcal{F}$ for every 3-set $T \supset P$.

Proof. Suppose instead, that there exists a 3-set $T_0 \supset P$ and $T_0 \in \mathcal{F}$. If $\mathcal{T} = \emptyset$, then let $\mathcal{F}' = \mathcal{F} - T_0 + P$. It is clear that \mathcal{F}' satisfies Property **D**. \mathcal{F}' also satisfies Property **S** because $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| + 1 \leq t - 2 + 1 = t - 1$ for every 3-set $Y \supset P$. Therefore \mathcal{F}' is a $(4, t)$ -system, a contradiction to the optimality of \mathcal{F} .

Now assume that $\mathcal{T} \neq \emptyset$. We claim that $\mathcal{F}' = \mathcal{F} - \mathcal{T} + P$ is a $(4, t)$ -system, contradicting the optimality of \mathcal{F} . To check Property **D**, we only need to consider those 4-sets S which contain two members T_1, T_2 of \mathcal{T} (because $|\mathcal{F}'(Q)| = |\mathcal{F}(Q)|$ for every 4-set Q that contains at most one member of \mathcal{T}). Since $|\mathcal{F}'(S)| \geq |\mathcal{F}(T_1)| + |\mathcal{F}(T_2)| - |\mathcal{F}(P)| \geq 2(t - 1) - 2 = 2t - 4 \geq t + 1$

(using the assumption that $t \geq 5$), we have $|\mathcal{F}'(S)| \geq t + 1 - 2 + 1 = t$. On the other hand, \mathcal{F}' also satisfies Property **S** since for every 3-set $Y \supset P$, $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| = t - 1$ if $Y \in \mathcal{T}(\subset \mathcal{F})$, otherwise $|\mathcal{F}'(Y)| = |\mathcal{F}(Y)| + 1 \leq t - 2 + 1 = t - 1$. \square

Proof of Lemma 5. We are to show that $\mathcal{F}_4 = \emptyset$ for $2 \leq t \leq 14$.

When $8 \leq t \leq 14$, (\star) holds in \mathcal{F} (since $t \geq 2^3$). We may thus assume that \mathcal{F} contains no 4-set, otherwise removing these 4-sets results in a smaller $(4, t)$ -system, a contradiction to the optimality of \mathcal{F} .

Let $2 \leq t \leq 7$. Suppose to the contrary, that there exists a set $S \in \mathcal{F}_4$. We may assume that $|\mathcal{F}(S)| = t$, otherwise S could be removed from \mathcal{F} . Let $\mathcal{T} = \binom{S}{3} \setminus \mathcal{F}$.

Case 1. $\mathcal{T} \neq \emptyset$.

Suppose that $T_0 \in \mathcal{T}$ has the minimal value of $|\mathcal{F}(T)|$ among all $T \in \mathcal{T}$. We claim that $|\mathcal{F}(T_0)| \leq t - 2$. Suppose instead, that $|\mathcal{F}(T_0)| \geq t - 1$. If $|\mathcal{T}| < 4$, then there exists $T_1 \in \binom{S}{3} \cap \mathcal{F}$. Because $T_1, S \in \mathcal{F}$, we have $|\mathcal{F}(S)| \geq |\mathcal{F}(T_0)| + 2 \geq t - 1 + 2 > t$, a contradiction to the assumption that $|\mathcal{F}(S)| = t$. If $|\mathcal{T}| = 4$, then for every $T \in \binom{S}{3}$, we have $|\mathcal{F}(T)| \geq t - 1$ and $T \notin \mathcal{F}$. Since $|\cup_{T \in \binom{S}{3}} \mathcal{F}(T)| = |\mathcal{F}(S) \setminus S| = t - 1$, we have $\mathcal{F}(T_1) = \mathcal{F}(T_2) \neq \emptyset$ for every $T_1, T_2 \in \binom{S}{3}$. But this is impossible because $\cap_{i=1}^4 T_i = \emptyset$. Now let $\mathcal{F}' = \mathcal{F} - S + T_0$. Trivially \mathcal{F}' satisfies Property **D** and because $|\mathcal{F}'(T_0)| \leq t - 1$, \mathcal{F}' satisfies Property **S** as well. Thus \mathcal{F}' is a $(4, t)$ -system, a contradiction to the optimality of \mathcal{F} .

Case 2. $\mathcal{T} = \emptyset$, i.e., $\binom{S}{3} \subset \mathcal{F}$.

Note that this case does not exist for $t = 2, 3, 4$, because it implies that $|\mathcal{F}(S)| \geq 4 + 1$, a contradiction to the assumption $|\mathcal{F}(S)| = t$.

When $t = 5$, we know that $\mathcal{F}(S) = \{S\} \cup \binom{S}{3}$. Pick any two elements $a, b \in S$ and consider $\mathcal{T} = \{\{a, b, c\} : |\mathcal{F}(\{a, b, c\})| = 4\}$. Since $\mathcal{F}(\{a, b\}) = \emptyset$, it must be the case that $\mathcal{F}(T) = \{\{c\}, \{c, a\}, \{c, b\}, \{c, a, b\}\}$ for every $T = \{a, b, c\} \in \mathcal{T}$. In particular, $\mathcal{T} \subset \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$ for every 3-set $T \subset \{a, b\}$. This is a contradiction to the assumption that $T \in \mathcal{F}$ for all $T \in \binom{S}{3}$.

When $t = 6, 7$, since $|\mathcal{F}(S)| \leq 7$ and $\binom{S}{\geq 3} \subset \mathcal{F}$, we have $|\mathcal{F} \cap \binom{S}{\leq 2}| \leq 2$. Consequently there exist $a, b \in S$ such that $\mathcal{F}(\{a, b\}) = \emptyset$. Since $\mathcal{T} = \{\{a, b, c\} : |\mathcal{F}(\{a, b, c\})| = t\} = \emptyset$, we may again apply Lemma 9 and derive a contradiction as in the previous paragraph. \square

Proof of Lemma 6. We are to show that $\mathcal{F}_3 = \emptyset$ for $7 \leq t \leq 10$. Suppose to the contrary, that there exists a set $T \in \mathcal{F}_3$. We now separate the case $t = 7$ and the cases $t = 8, 9, 10$.

Case 1. $t = 7$.

Since $|\mathcal{F}(T)| < 7$ (by Property **S**), there exists a set $P \in \binom{T}{\leq 2} \setminus \mathcal{F}$. Define \mathcal{T} as in (1), trivially $\mathcal{T} \subset \mathcal{F}$. We may apply Lemma 9 to conclude that $T \notin \mathcal{F}$, a contradiction.

Case 2. $t = 8, 9, 10$.

Since $t \geq 2^3$, we may assume that (\star) holds in \mathcal{F} . In particular, if $T \in \mathcal{F}_3$, then $|\mathcal{F}(T)| = 7$.

Let $\mathcal{D} = \{S \in \binom{[n]}{4} : S \supset T, |\mathcal{F}(S)| = t\}$. If $\mathcal{D} = \emptyset$, then $\mathcal{F}' = \mathcal{F} - T$ satisfies Property **D** and is thus a $(4, t)$ -system of size $|\mathcal{F}| - 1$, a contradiction. Now suppose that $|\mathcal{D}| = 1$ and $\{a\} \cup T$ is the only element of \mathcal{D} . Since $t < 11$, at least one of $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}$, say $\{a\}$, is not contained in \mathcal{F} . Let $\mathcal{F}' = \mathcal{F} - T + \{a\}$. \mathcal{F}' satisfies Property **S** trivially. Consider a 4-set $S \supset T$ of $[n]$. If $S \neq \{a\} \cup T$, then $|\mathcal{F}(S)| \geq t + 1$ and $|\mathcal{F}'(S)| \geq t$. If $S = \{a\} \cup T$, then $|\mathcal{F}'(S)| = |\mathcal{F}(S)| = t$. This means that \mathcal{F}' satisfies Property **D** and consequently \mathcal{F}' is a $(4, t)$ -system, a contradiction.

Now we assume that there exist $a_1, a_2 \in [n]$ such that $\{a_i\} \cup T \in \mathcal{D}$ for $i = 1, 2$. We will show that when $8 \leq t \leq 10$, there are two vertices $v_1, v_2 \in T$ such that $|\mathcal{F}(\{a_1, a_2, v_1, v_2\})| < t$, contradicting Property **D**.

Define $\mathcal{F}_{\{a_i\}}(T) = \mathcal{F}(\{a_i\} \cup T) - \mathcal{F}(T)$ for $i = 1, 2$. Since $|\mathcal{F}(T)| = 7$, we have $|\mathcal{F}_{\{a_i\}}(T)| = 1, 2, 3$ for $t = 8, 9, 10$, respectively. Using (\star) , we thus know that $\{a_i\} \subseteq \mathcal{F}_{\{a_i\}}(T) \subset \mathcal{F}_{\leq 2}$ for every $t \in \{8, 9, 10\}$.

- When $t = 8$, we have $\mathcal{F}_{\{a_i\}}(T) = \{\{a_i\}\}$ for $i = 1, 2$. Thus $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 6 < 8$ for any $b \neq c \in T$.
- When $t = 9$, we have $\mathcal{F}_{\{a_1\}}(T) = \{\{a_1\}, \{a_1, c\}\}$ and $\mathcal{F}_{\{a_2\}}(T) = \{\{a_2\}, \{a_2, d\}\}$, for not necessarily distinct $c, d \in T$. Consequently $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 8 < 9$ for some $b \in T \setminus \{c, d\}$.
- When $t = 10$, we may assume that $\mathcal{F}_{\{a_1\}}(T) = \{\{a_1\}, \{a_1, b\}, \{a_1, d\}\}$ and $\mathcal{F}_{\{a_2\}}(T) = \{\{a_2\}, \{a_2, c\}, \{a_2, d\}\}$, where $c, b \in T$ are not necessarily distinct. If $c \neq b$, then $|\mathcal{F}(\{a_1, a_2, b, c\})| \leq 8 < 10$. Otherwise, $|\mathcal{F}(\{a_1, a_2, b, w\})| \leq 8 < 10$, where $w = T \setminus \{c, d\}$. □

Proof of Lemma 7. Let $7 \leq t \leq 14$. We are to show that $\binom{[n]}{1} \subset \mathcal{F}$. Suppose instead, say, that $\{n\} \notin \mathcal{F}$.

For $t \geq 8$, consider a set $S \in \binom{[n]}{4}$ and $S \ni n$. We know that no set from $\mathcal{F}(S)$ contains n (otherwise (\star) forces $\{n\} \in \mathcal{F}$). Thus $|\mathcal{F}(S)| \leq 7 < t$, a contradiction to Property **D**.

For $t = 7$, consider a set $T \in \binom{[n-1]}{3}$. By Property **S** and Property **D**, we have $|\mathcal{F}(T)| \leq 6$ and $|\mathcal{F}(\{n\} \cup T)| \geq 7$. Then there exists a set $P \in \mathcal{F}(\{n\} \cup T)$ such that $P \supset n$. Let $\mathcal{F}' = \mathcal{F} - P + \{n\}$. For any $Y \in \binom{[n]}{3}$ and $n \in Y$, we have $|\mathcal{F}(Y)| \leq 5$ (because $\{n\}, Y \notin \mathcal{F}$). Therefore \mathcal{F}' satisfies Property **S** and is thus a $(4, t)$ -system, a contradiction. \square

Proof of Lemma 8. We are to show that $\binom{[n]}{2} \subset \mathcal{F}$ for $11 \leq t \leq 13$. Suppose to the contrary, that there exist $a, b \in [n]$ such that $\{a, b\} \notin \mathcal{F}$. Pick any two elements $v_1, v_2 \in [n] \setminus \{a, b\}$ and consider $D = \{a, b, v_1, v_2\}$. Since (\star) holds, we have $\{a, b, v_1\}, \{a, b, v_2\} \notin \mathcal{F}$ (otherwise $\{a, b\} \in \mathcal{F}$). Together with $\{a, b\}$ and D , this gives us four members of $(2^D \setminus \emptyset) \setminus \mathcal{F}$. Consequently $|\mathcal{F}(D)| \leq 11$, which contradicts Property **D** when $t = 12, 13$. Now assume that $t = 11$. Then $|\mathcal{F}(D)| = 11$ and $|\mathcal{F}(\{a, v_1, v_2\})| = |\mathcal{F}(\{b, v_1, v_2\})| = 7$. Let $\mathcal{F}' = \mathcal{F} - \{a, v_1, v_2\} + \{a, b\}$. \mathcal{F}' satisfies Property **S** trivially. To check Property **D**, we consider all the 4-sets S containing $\{a, v_1, v_2\}$. If $S = \{a, b, v_1, v_2\}$, then $|\mathcal{F}'(S)| = |\mathcal{F}(S)| > 11$. Otherwise, $S = \{a, v_1, v_2, v_3\}$ for some $v_3 \in [n] \setminus \{a, b, v_1, v_2\}$. Since $|\mathcal{F}(\{a, v_i, v_j\})| = 7$ for any $i \neq j$, only S and $\{v_1, v_2, v_3\}$ could be absent from $\mathcal{F}(S)$ and consequently $|\mathcal{F}(S)| \geq 13$. We thus have $|\mathcal{F}'(S)| = |\mathcal{F}(S)| - 1 \geq 13 - 1 > 11$. Therefore \mathcal{F}' is a $(4, 11)$ -system, a contradiction to the optimality of \mathcal{F} . \square

3.2 $m(n, 4, 5)$

In this section we prove that $m(n, 4, 5) = \binom{n}{2} - 17$. Before the proof, we introduce the following extensions of the Turán number:

Definition 10. A family $\mathcal{G} \in \binom{[n]}{i}$ is called a *Turán- $i(n, k, t)$ -system* if every k -set of $[n]$ contains at least t members of \mathcal{G} . The generalized Turán number $T_i(n, k, t)$ is defined as the minimum size of a *Turán- $i(n, k, t)$ -system*.

Replacing all the instances of i by $\geq i$ in the previous paragraph, we obtain the non-uniform Turán number $T_{\geq i}(n, k, t)$.

In the proof we will consider $T_3(k, 4, 1) = \binom{k}{3} - \text{ex}(k, K_4^{(3)})$. Turán [14] conjectured that

$T_3(k, 4, 1)$ is achieved by the following 3-graph \mathcal{H}_k (referred to as *Turán's 3-graph*). Partition $[k]$ into $A_1 \cup A_2 \cup A_3$, where $\lfloor k/3 \rfloor \leq |A_i| \leq \lceil k/3 \rceil$. The edges of \mathcal{H}_k are 3-sets which are either contained in some A_i or contain two vertices of A_i and one of $A_{i+1 \pmod{3}}$. It is known [13] that Turán's conjecture holds for $k \leq 13$. For larger k , the following lower bound of de Caen [1] suffices for our purpose:

$$T_3(k, 4, 1) \geq \frac{k(k-1)(k-3)}{18}. \quad (2)$$

We also need the following simple lemma on $T_{\geq 1}(n, k, t)$.

Lemma 11. [9] $T_{\geq 1}(n, k, t) = n - k + t$ for $1 \leq t \leq k$.

Let \mathcal{F} be an optimal $(4, 5)$ -system with $A = \{a : \{a\} \in \mathcal{F}\}$, $B = [n] - A$ and assume $|A| = k$. By Lemma 5, we may assume that \mathcal{F} contains no 4-sets. In order to show that $|\mathcal{F}| \geq \binom{n}{2} - 17$, our proof consists of three stages described in Section 3.2.1–3.2.3. The proof leads to a construction achieving this bound, which we present in Section 3.2.3 as well.

3.2.1 Stage 1

We start with Claim 12 which reflects a rough picture of \mathcal{F} and in turn implies a (weak) lower bound (4) for $|\mathcal{F}|$.

Given two disjoint sets $C, D \in [n]$, we write $\mathcal{F}(C, D) = \{S \in \mathcal{F} : S \cap C \neq \emptyset \text{ and } S \cap D \neq \emptyset\}$.

Claim 12. 1. $(\mathcal{F}(A))_2$ is a matching in A .

2. $\overline{(\mathcal{F}(B))_2}$ contains no matching of size 2 or star with 3 edges.

3. $|\mathcal{F}(A, B)| \geq (n - k)(k - 2) + |\mathcal{F}_{1,2}(A, B)|$, where $\mathcal{F}_{1,2}(A, B) = \{T \in \mathcal{F}_3 : |T \cap A| = 1, |T \cap B| = 2\}$.

4. $|(\mathcal{F}(A))_3| \geq k(k - 2)(k - 4)/24$.

Proof. Part 1: Property **S** prevents $\mathcal{F}(A)$ from containing two adjacent (graph) edges. Thus $(\mathcal{F}(A))_2$ is a matching.

Part 2: We first claim that

$$\text{If } P \in \binom{B}{2} \setminus \mathcal{F} \text{ and } P \subset T, |T| = 3, \text{ then } T \notin \mathcal{F}. \quad (3)$$

In fact, if Y is 3-set of $[n]$ such that $Y \supset P$ and $|\mathcal{F}(Y)| = 4$, then $Y \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$.

If there are $a, b, c, d \in B$ such that $\{a, b\}, \{c, d\} \notin \mathcal{F}$, then $(\mathcal{F}(\{a, b, c, d\}))_3 = \emptyset$ by (3). Consequently $|\mathcal{F}(\{a, b, c, d\})| \leq 4$, a contradiction to Property **D**. Therefore, $\overline{\mathcal{F}(B)}$ contains no two vertex-disjoint (graph) edges. A similar argument shows that $\overline{\mathcal{F}(B)}$ contains no star with 3 edges.

Part 3: Consider a vertex $b \in B$ and a 3-subset T of A . Since $\{b\} \notin \mathcal{F}$, $|\mathcal{F}(T)| \leq 4$ and $|\mathcal{F}(\{b\} \cup T)| \geq 5$, we have $|\mathcal{F}(\{b\}, T)| \geq 1$. Define $\mathcal{G}_b = \{Y \setminus \{b\} : Y \in \mathcal{F}(\{b\}, A)\}$ for every $b \in B$. Then \mathcal{G}_b is a set system of $\binom{A}{\leq 2}$ such that every 3-set in A contains at least one member of \mathcal{G}_b , in other words, \mathcal{G}_b is a Turán- $\geq 1(k, 3, 1)$ -system. By Lemma 11, we have $|\mathcal{H}_b| \geq T_{\geq 1}(k, 3, 1) = k - 2$. Repeating this for all $b \in B$, we have

$$|\{S \in \mathcal{F}(A, B) : |S \cap B| = 1\}| = \sum_{b \in B} |\mathcal{G}_b| \geq (n - k)(k - 2).$$

Consequently $|\mathcal{F}(A, B)| \geq (n - k)(k - 2) + |\mathcal{F}_{1,2}(A, B)|$.

Part 4. Now we give a crude lower bound for $(\mathcal{F}(A))_3$. From Part 1, we know that $(\mathcal{F}(A))_2$ is a matching $M = \{\{x_i, y_i\}\}_{i=1}^m$. Let

$$\mathcal{D} = \{S \in \binom{A}{4} : |S \cap \{x_i, y_i\}| \leq 1 \text{ for every } \{x_i, y_i\} \in M\}.$$

By Property **D**, every 4-set in \mathcal{D} contains at least one member of $(\mathcal{F}(A))_3$. Since \mathcal{D} is minimal when $m = \lfloor k/2 \rfloor$, we may assume that $m = \lfloor k/2 \rfloor$ when estimating $(\mathcal{F}(A))_3$ from below. The usual averaging arguments thus give the following lower bound (for even k , the case when k is odd yields an even larger bound):

$$(\mathcal{F}(A))_3 \geq \frac{|\mathcal{D}|}{k - 6} = \frac{k(k - 2)(k - 4)(k - 6)}{4!(k - 6)} = \frac{k(k - 2)(k - 4)}{24}. \quad \square$$

The consequence of Claim 12 is the following lower bound.

$$\begin{aligned} |\mathcal{F}| &\geq |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_3| + |\mathcal{F}(A, B)| + |(\mathcal{F}(B))_2| + |(\mathcal{F}(B))_3| \\ &\geq k + \frac{k(k - 2)(k - 4)}{24} + (n - k)(k - 2) + |\mathcal{F}_{1,2}(A, B)| \\ &\quad + \binom{n - k}{2} - 2 + |(\mathcal{F}(B))_3|. \end{aligned} \tag{4}$$

3.2.2 Stage 2

In this stage we first prove Claim 13, $(\mathcal{F}(A))_2 = \emptyset$, which not only implies that $(\mathcal{F}(A))_{\geq 2}$ is a Turán- $_3(k, 4, 1)$ -system, but also makes it possible to find more details about $\mathcal{F}(A, B)$ and $\mathcal{F}(B)$, which are summarized in Claim 14. Claim 13 and 14 together describe the fine structure of \mathcal{F} . This leads to an improved lower bound (5) for $|\mathcal{F}|$.

Let us first sketch the idea behind the proof of Claim 13. Suppose that $\{a_1, a_2\} \in (\mathcal{F}(A))_2$. Then at least one of $B_i = \{b \in B : \{a_i, b\} \notin \mathcal{F}\}$, $i = 1, 2$ has size $|B|/2$ and consequently either $|\mathcal{F}_{1,2}(A, B)|$ or $|(\mathcal{F}(B))_3|$ is at least $3(n - k)$. But because of (4), $|\mathcal{F}|$ is larger than the trivial upper bound $\binom{n}{2}$, which is a contradiction.

Claim 13. $(\mathcal{F}(A))_2 = \emptyset$ provided that $n \geq 160$.

Proof. Note that (4) and $|\mathcal{F}| \leq \binom{n}{2}$ imply that $k = O(n^{1/3})$ as $n \rightarrow \infty$ (in particular, when $n \geq 20$, $k < n/2$).

Suppose instead, that $\{a_1, a_2\} \in (\mathcal{F}(A))_2$. Pick a vertex $b \in B$. By Property **S**, at most one of $\{a_1, b\}$ and $\{a_2, b\}$ is contained in \mathcal{F} . Without loss of generality, we may assume that B has a subset B_1 of size $\frac{n-k}{2}$, such that $\{a_1, b\} \notin \mathcal{F}$ for every $b \in B_1$. Consider $\mathcal{T}_{a_1} = \{T \in \mathcal{F}_3 : a_1 \in T, |T \cap B_1| = 2\}$. If $|\mathcal{T}_{a_1}| \geq 3(n - k)$, then (4) implies that (when $n \geq 30$),

$$\begin{aligned} |\mathcal{F}| &\geq k + \frac{k(k-2)(k-4)}{24} + (n-k)(k-2) + \binom{n-k}{2} - 2 + |\mathcal{T}_{a_1}| \\ &\geq \binom{n}{2} + (n-k) + \frac{k(k-2)(k-4)}{24} - \binom{k}{2} + k - 2 \\ &\geq \binom{n}{2} + n - 29 > \binom{n}{2}, \end{aligned}$$

a contradiction to the trivial upper bound that $|\mathcal{F}| \leq \binom{n}{2}$, where the third inequality follows from the fact

$$\min_{k \geq 0} \frac{k(k-2)(k-4)}{24} - \binom{k}{2} = -26.125 \quad (\text{achieved by } k = 11).$$

We may therefore assume that $|\mathcal{T}_{a_1}| < 3(n - k)$. Let $\mathcal{P} = \{P \in \binom{B_1}{2} : \{a_1\} \cup P \in \mathcal{T}_{a_1}\}$ and

$\mathcal{T} = \{T \in \binom{B_1}{3} : \binom{T}{2} \cap P = \emptyset\}$. Then $|\mathcal{P}| = |\mathcal{T}_{a_1}|$, and therefore

$$\begin{aligned} |\mathcal{T}| &\geq \binom{(n-k)/2}{3} - |\mathcal{P}| \left(\frac{n-k}{2} - 2 \right) \\ &> \frac{(n-k)(n-k-2)(n-k-4)}{48} - \frac{3}{2}(n-k)^2 \\ &\geq 3(n-k), \quad \text{when } n-k \geq 80 \text{ or } n \geq 160. \end{aligned}$$

On the other hand, we have $T \in \mathcal{F}$ for every $T \in \mathcal{T}$ because $|\mathcal{F}(\{a_1\} \cup T)| \geq 5$ and $\mathcal{F}(\{a_1\}, T) = \emptyset$. Consequently $|(\mathcal{F}(B))_3| \geq |\mathcal{T}| > 3(n-k)$. Using this lower bound for $|(\mathcal{F}(B))_3|$ in (4), we obtain $|\mathcal{F}| \geq k(k-2)(k-4)/24 + \binom{n}{2} + n - 2 > \binom{n}{2}$, a contradiction. \square

Note that we make no effort to optimize the constant 160 in Claim 13.

With the help of Claim 13, we are able to see the fine structure of \mathcal{F} as follows.

Claim 14. 1. $(\mathcal{F}(A, B))_3 = \emptyset$.

2. $(\mathcal{F}(B))_3 = \emptyset$ and $|\mathcal{F}(B)| = |(\mathcal{F}(B))_2| \geq \binom{|B|}{2} - 1$.

3. For every $a \in A$, we have $|\{b \in B : \{a, b\} \in \mathcal{F}\}| \geq n - k - 2$. Consequently $|(\mathcal{F}(A, B))_2| \geq k(n - k - 2)$.

Proof. Part 1. Let T_0 be a 3-set of $[n]$ with $T_0 \cap A \neq \emptyset$ and $T \cap B \neq \emptyset$.

If $T_0 \cap B = \{b_1, b_2\} \notin \mathcal{F}$, then $T_0 \notin \mathcal{F}$ by (3). If $T_0 \supset \{a, b\} \notin \mathcal{F}$ for some $a \in A$ and $b \in B$, then we consider all the 3-sets $T \supset P$ such that $|\mathcal{F}(T)| = 4$. If $|T \cap B| = 2$, then it must be the case that $T \in \mathcal{F}$. Otherwise, assume that $T \cap A = \{a, a'\}$. Since $\{a, a'\} \notin \mathcal{F}$ by Claim 13, we also have $T \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T_0 \notin \mathcal{F}$.

Finally we assume that $P \in \mathcal{F}$ for every $P \in \binom{T_0}{2}$ and $P \not\subset A$. Either $|T_0 \cap B| = 2$ or $|T_0 \cap A| = 2$, we always have $|(\mathcal{F}(T_0))_{\leq 2}| = 4$. Therefore $T_0 \notin \mathcal{F}$ by Property **S**.

We thus conclude that $(\mathcal{F}(A, B))_3 = \emptyset$.

Part 2. Suppose instead, that there exists $T \in (\mathcal{F}(B))_3$. Then we know that $\binom{T}{2} \subset \mathcal{F}$ by (3). Let $\mathcal{D} = \{S \in \binom{[n]}{4} : T \subset S, |\mathcal{F}(S)| = 5\}$. We may assume that $\mathcal{D} \neq \emptyset$, otherwise $|\mathcal{F}(S)| \geq 6$ for every $S \supset T$, and T could be removed from \mathcal{F} without hurting Property **D**. Consider a set $S = \{a\} \cup T \in \mathcal{D}$. Either $a \in A$ or $a \in B$. If $a \in A$, then $\mathcal{F}(\{a\}, T) = \emptyset$; if $a \in B$, then $|\mathcal{F}(\{a\}, T)| = 1$.

We claim that $|\mathcal{D}| = 1$. Suppose instead, that \mathcal{D} contains two members $\{a_1\} \cup T$ and $\{a_2\} \cup T$. If $a_1, a_2 \in A$, then we consider $S_0 = \{a_1, a_2, b, c\}$ for any two vertices $b, c \in T$. From Part 1 we know that $(\mathcal{F}(S_0))_3 = \emptyset$. We also have $\{a_1, a_2\} \notin \mathcal{F}$ by Claim 13. Consequently $|\mathcal{F}(S_0)| = 3 < 5$, a contradiction to Property **D**. If $a_1, a_2 \in B$, then there are two vertices $b, c \in T$ such that $\{a_1, b\}, \{a_2, c\} \notin \mathcal{F}$. This already contradicts Claim 12 Part 2. Finally, assume that $a_1 \in A$, $a_2 \in B$ and $\{a_2, d\} \in \mathcal{F}$ for some $d \in T$. Consider $S_0 = \{a_1, a_2, b, c\}$, where $\{b, c\} = T \setminus \{d\}$. We know that $(\mathcal{F}(S_0))_3 = \emptyset$ from Part 1 and (3), $(\mathcal{F}(S_0))_2 \subseteq \{\{a_1, a_2\}, \{b, c\}\}$ from our assumption. Consequently $|\mathcal{F}(S_0)| = 3 < 5$, again a contradiction.

Now assume that $S_0 = \{a\} \cup T$ is the unique element of \mathcal{D} . Let $\mathcal{F}' = \mathcal{F} - T + \{a, b, c\}$ for any two vertices $b, c \in T$. \mathcal{F}' satisfies Property **S** since $|\mathcal{F}'(\{a, b, c\})| \leq 3$. \mathcal{F}' also satisfies Property **D** because $|\mathcal{F}'(S)| \geq 6 - 1 = 5$ for every $S \in \binom{[n]}{4} \setminus S_0$ and $|\mathcal{F}'(S_0)| = |\mathcal{F}(S_0)| = 5$. \mathcal{F}' is thus another optimal $(4, 5)$ -system. However, if $a \in A$, then \mathcal{F}' contradicts Part 1. If $a \in B$, then \mathcal{F}' contradicts (3), because $\{a, b, c\} \in \mathcal{F}'$ and at least one of $\{a, b\}$ and $\{a, c\}$ is not in \mathcal{F}' .

We thus conclude that $(\mathcal{F}(B))_3 = \emptyset$.

Now it is easy to see why there are no three vertices $a, b, c \in B$ such that $\{a, b\}, \{a, c\} \notin \mathcal{F}$. In such a case, since $(\mathcal{F}(B))_3 = \emptyset$, we have $|\mathcal{F}(\{a, b, c, d\})| \leq 4$ for any $d \in B \setminus \{a, b, c\}$, a contradiction to Property **D**. Together with Claim 12 Part 2, we conclude that $\mathcal{F}(B)$ misses at most one edge on B .

Part 3. Suppose instead, that there exists an $a \in A$ and $b_1, b_2, b_3 \in B$ such that $\{a, b_i\} \notin \mathcal{F}$ for all i . Part 1 and Part 2 together imply that $S = \{a, b_1, b_2, b_3\}$ contains no 3-set from \mathcal{F} . Thus, $|\mathcal{F}(S)| \leq 4$, contradicting Property **D**. \square

We now refine (4) by applying Claims 13 and 14:

$$\begin{aligned}
|\mathcal{F}| &= |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_3| + |(\mathcal{F}(A, B))_2| + |(\mathcal{F}(B))_2| \\
&\geq k + T_3(k, 4, 1) + k(n - k - 2) + \binom{n - k}{2} - 1 \\
&= g(k) + \binom{n}{2} - 1,
\end{aligned} \tag{5}$$

where $g(k) = T_3(k, 4, 1) - k - \binom{k}{2}$.

3.2.3 Stage 3

In this stage, we complete the proof that $m(n, 4, 5) = \binom{n}{2} - 17$ by analyzing (5).

Since $T_3(k, 4, 1)$ is known for $k \leq 13$, we compute $g(k)$ exactly for $0 \leq k \leq 11$ and obtain that $\min_{0 \leq k \leq 11} g(k) = g(7) = g(8) = -16$. For $k \geq 12$, using the inequality (2), we have $g(k) \geq k(k-1)(k-3)/18 - k - \binom{k}{2} \geq -12$. Putting these together, we have

$$\min_{k \geq 0} g(k) = g(7) = g(8) = -16 \quad (6)$$

Applying (6) to (5), we obtain that $|\mathcal{F}| \geq \binom{n}{2} - 17$.

Claims 13, 14 and (6) lead us to the following construction, which gives a $(4, 5)$ -system of cardinality $\binom{n}{2} - 17$.

Construction 1: Partition $[n]$ into $A \cup B$, where $|A| = k = 7$ or 8 . Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, where \mathcal{F}_1 , \mathcal{F}_2 and \mathcal{F}_3 are defined as follows:

- $\mathcal{F}_1 = \{\{a\} : a \in A\}$.
- Let M be a union of k disjoint copies of P_2 (paths of length 2) whose middle vertices are in A and end vertices make up a subset B_0 of B . Let $\mathcal{F}_2 = \binom{B}{2} \setminus \{e\} \cup (A \times B) \setminus M$ for some $e \in B \setminus B_0$.
- \mathcal{F}_3 is the edge set of the Turán 3-graph \mathcal{H}_k on A .

We thus conclude that $m(n, 4, 5) = \binom{n}{2} - 17$. □

3.3 $m(n, 4, 6)$

Let \mathcal{F} be an optimal $(4, 6)$ -system. Define A , B , $\mathcal{F}(A)$, $\mathcal{F}(B)$, $\mathcal{F}(A, B)$ as in Section 3.2 and assume that $|A| = k$. We first define another threshold function.

Definition 15. A $\widehat{(4, 2)}$ -system of $[n]$ is a set system $\mathcal{G} \subseteq \binom{[n]}{2} \cup \binom{[n]}{3}$ such that every 4-set of $[n]$ contains at least two members of \mathcal{G} and every 3-set of $[n]$ contains at most two members of \mathcal{G} . Let $\hat{m}(n, 4, 2)$ denote the minimum size of such a set system.

The lower bound $|\mathcal{F}| \geq \binom{n}{2} - 190$ in Theorem 3 is the consequence of the following claim, whose proof is postponed to the end of this section.

Claim 16. \mathcal{F} has the following properties.

1. There exists no $T \in \mathcal{F}_3$ which contains $a \in A$ and $b \in B$ such that $\{a, b\} \notin \mathcal{F}$.
2. $\mathcal{F}(B) = \binom{B}{2}$.
3. $|\mathcal{F}(A, B)| \geq k(n - k) - 4$.
4. $|\mathcal{F}_{\geq 2}(A)| \geq \hat{m}(k, 4, 2) \geq 2\binom{k}{7}/\binom{k-3}{4} = k(k-1)(k-2)/105$.

Claim 16 also suggests a general way to construct $(4, 6)$ -systems of $[n]$: Partition $[n]$ into $A \cup B$ with $|A| = k$ for any $0 \leq k \leq n$. Let $\mathcal{F} = \mathcal{F}(A) \cup \mathcal{F}(A, B) \cup \mathcal{F}(B)$, with $\mathcal{F}(B) = \binom{B}{2}$, $\mathcal{F}(A, B) = A \times B$ and $\mathcal{F}(A) = (\mathcal{F}(A))_1 \cup (\mathcal{F}(A))_{\geq 2}$, where $(\mathcal{F}(A))_1 = \{\{a\} : a \in A\}$ and $(\mathcal{F}(A))_{\geq 2}$ is a $\widehat{(4, 2)}$ -system on A .

In particular, the following construction gives a $(4, 6)$ -systems of $[n]$ of size $\binom{n}{2} - 5$.

Construction 2: Let $k = 8$ and $A, B, \mathcal{F}(B), \mathcal{F}(A, B)$ and $(\mathcal{F}(A))_1$ are defined as above. We construct $(\mathcal{F}(A))_{\geq 2}$ as follows.

- Suppose that $A = A_1 + A_2$, where $A_1 = \{u_1, u_2, u_3, u_4\}$ and $A_2 = \{v_1, v_2, v_3, v_4\}$. Let $E_0 = \{\{u_1, v_1\}, \{u_1, v_2\}, \{u_2, v_3\}, \{u_3, v_3\}, \{u_4, v_4\}\}$. Let $(\mathcal{F}(A))_2 = (A_1 \times A_2) - E_0$.
- $(\mathcal{F}(A))_3 = \{\{u_1, u_2, u_3\}, \{u_2, u_3, u_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}\}$.

Proof of Theorem 3 for $t = 6$. The upper bound $m(n, 4, 6) \leq \binom{n}{2} - 5$ follows from Construction 2. Claim 16 gives the lower bound for $|\mathcal{F}| = m(n, 4, 6)$ as follows.

$$\begin{aligned}
|\mathcal{F}| &= |(\mathcal{F}(A))_1| + |(\mathcal{F}(A))_{\geq 2}| + |\mathcal{F}(A, B)| + |\mathcal{F}(B)| \\
&\geq k + \frac{k(k-1)(k-2)}{105} + k(n-k) - 4 + \binom{n-k}{2} \\
&\geq \binom{n}{2} - 190,
\end{aligned} \tag{7}$$

where the last inequality follows from $\min_{k \geq 0} k + k(k-1)(k-2)/105 - \binom{k}{2} = -186$ (achieved by $k = 35$). \square

Remark: Actually, we almost determine $m(n, 4, 6)$ exactly in terms of $\hat{m}(k, 4, 2)$ (off only by 4). To see this, replace $k(k-1)(k-2)/105$ by $\hat{m}(k, 4, 2)$ in (7) and get an upper bound

following the general construction:

$$k + \hat{m}(k, 4, 2) + k(n - k) + \binom{n - k}{2} - 4 \leq |\mathcal{F}| \leq k + \hat{m}(k, 4, 2) + k(n - k) + \binom{n - k}{2},$$

for some $k \geq 0$. The knowledge of $\hat{m}(k, 4, 2)$ for small values of k may lead to the final settlement of $m(n, 4, 6)$.

Proof of Claim 16.

Part 1: Suppose that there are two vertices $a \in A$ and $b \in B$ such that $\{a, b\} \notin \mathcal{F}$. For a 3-set $T \supset \{a, b\}$, if $|\mathcal{F}(T)| = 5$, then $T \in \mathcal{F}$. We may therefore apply Lemma 9 to conclude that $T \notin \mathcal{F}$ for every 3-set $T \supset \{a, b\}$.

Part 2: We first claim that (3) holds in \mathcal{F} . In fact, when $P \in \binom{B}{2}$, we have $\{T \in \binom{[n]}{3} : P \subset T, |\mathcal{F}(T)| = 5\} = \emptyset$. Then we can apply Lemma 9 to obtain (3).

Next, we show that if there exists a set $T \in (F(B))_3$, then we obtain a contradiction. The proof is similar to that of Claim 14 Part 2. First, we claim that $\mathcal{D} = \{S \in \binom{[n]}{4} : T \subset S, |\mathcal{F}(S)| = 6\}$ has exactly one member. Suppose instead, for example, that \mathcal{D} contains $S_1 = \{a\} \cup T$ and $S_2 = \{b\} \cup T$ for some $a \in A$ and $b \in B$ (other cases are similar). It means that there are exactly two sets from $\mathcal{F}(S_1), \mathcal{F}(S_2)$ which contain a, b , respectively. From Part 1, we know that these two sets in S_1 must be $\{a\}$ and $\{a, b_1\}$ and the two sets in S_2 must be $\{b, b_2\}$ and $\{b, b_3\}$, where $b_1, b_2, b_3 \in T$. Assume that, for example, b_1, b_2, b_3 are all distinct. Consider $S_3 = \{a, b, b_1, b_2\}$. It is easy to see that $|\mathcal{F}(S_3)| \leq 5$, a contradiction to Property **D**. Now assume that $\{a\} \cup T$ is the unique member of \mathcal{D} , for example, for some $a \in A$. Let $\mathcal{F}' = \mathcal{F} - T + \{a, b_1, b_2\}$. It is easy to see that \mathcal{F}' is an optimal $(4, 6)$ -system. However, since $\{a, b_1, b_2\} \in \mathcal{F}'$ and $\{a, b_2\} \notin \mathcal{F}'$, this contradicts Part 1.

Since $(F(B))_3 = \emptyset$, $\binom{B}{2} \subset \mathcal{F}$ follows from Property **D**. Consequently $\mathcal{F}(B) = \binom{B}{2}$.

Part 3: We first show that for every $a \in A$, there is at most one vertex $b \in B$ such that $\{a, b\} \notin \mathcal{F}$. Suppose instead, that $\{a, b_1\}, \{a, b_2\} \notin \mathcal{F}$ for some $a \in A$. Consider $S = \{a, b_1, b_2, c\}$ for any vertex $c \in B \setminus \{b_1, b_2\}$. From Part 1 and 2 we know that $(\mathcal{F}(S))_3 = \emptyset$. Consequently $|\mathcal{F}(S)| \leq 5$, a contradiction to Property **D**. Second, for each $b \in B$, there are at most two vertices $a_1, a_2 \in A$ such that $\{a_i, b\} \notin \mathcal{F}$ for $i = 1, 2$. Suppose instead, that there are three vertices $a_1, a_2, a_3 \in A$ such that $\{a_i, b\} \notin \mathcal{F}$. Since $\mathcal{F}(\{a_1, a_2, a_3\}, \{b\}) = \emptyset$, we have $\mathcal{F}(\{a_1, a_2, a_3\}) = \mathcal{F}(\{b, a_1, a_2, a_3\})$, which either violates Property **D** or Property **S**.

Now consider the bipartite graph G whose edge set is $(A \times B) - (\mathcal{F}(A, B))_2$. By the argument

in the previous paragraph, G consists of vertex-disjoint edges or 2-paths whose centers are in B . On the other hand, two independent edges $\{a_i, b_i\}$, $i = 1, 2$ in G (where $a_i \in A$) imply that $\{a_1, a_2\} \in \mathcal{F}$ (by Property **D**). If G contain 3 independent edges $\{a_i, b_i\}$, then $|\mathcal{F}(\{a_1, a_2, a_3\})| = 6$, a contradiction to Property **S**. Therefore G has at most 4 edges which are from two disjoint 2-paths.

Part 4. Clearly $\mathcal{F}_{\geq 2}(A)$ is a $(\widehat{4, 2})$ -system of $[k]$. Let us count the number of triples in a $(\widehat{4, 2})$ -system \mathcal{G} of $[k]$. The following lemma implies that every 7-set of $[k]$ contains at least two triples from \mathcal{G} . We omit its proof because it is an easy case analysis.

Lemma 17. *Every $(\widehat{4, 2})$ -system of $[7]$ contains at least two triples.* □

Using Lemma 17 and the averaging argument, we have $\hat{m}(k, 4, 2) \geq 2 \binom{k}{7} / \binom{k-3}{4}$. □

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