A FAMILY OF SWITCH EQUIVALENT GRAPHS

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ABSTRACT. Let $2G$ be the graph consisting of two disjoint copies of G. We prove that every graph of the form $2H$ can be transformed to every other graph of the form $2K$ using the following operations:
(i) adding edge uv if $d(u) = d(v)$ and uv is not present,
(ii) deleting edge uv if $d(u) = d(v)$ and uv is present.

1. INTRODUCTION

A sequence of integers $D = d_1 \leq \cdots \leq d_n$ is *graphic* if it is the degree sequence of a simple graph. A *2-switch* in a simple graph G is the replacement of a pair of edges xy and zw in G by the edges yz and wx, given that yz and wx were not present in G originally. One motivation for this paper is the following theorem sometimes attributed to Berge [1], but essentially proven by Havel [5] and Hakimi [4] independently.

Theorem 1.1. *If G and H are simple graphs with vertex set V, then $d_G(v) = d_H(v)$ for every $v \in V$ if and only if there is a sequence of 2-switches that transforms G into H.*

An application of such a result to analyze the design and dynamic operation of lightwave networks, like traffic patterns can be found in [2].

The *odd set* of a graph G is the set of integers k such that G has an odd number of vertices of degree k. The *switch operation* is the addition or deletion of an edge whose endpoints have the same degree. Graphs H and H' are *switch equivalent* if there is a sequence of switches transforming H to H'. Note that a necessary condition for H and H' to be switch equivalent is that they have the same order and odd sets. Let $G + G = 2G$ denote the graph consisting of two vertex disjoint copies of G.

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Chen et al. [3] proved that two graphs with the same order and odd sets can be obtained from each other by switches and 2-switches. They also proved, by construction, that this is not true if only the switch operation is allowed, thereby answering a conjecture of McCanna [6] negatively. In fact, [3] suggests the question of which pairs of graphs H, H' with the same order and odd sets are switch equivalent, and proves that the graphs H, H' are switch equivalent if each has at least three more isolated vertices than its maximum degree.

In this paper, we prove that G and G' are switch equivalent if they have the same order and odd sets, and $G = 2H$ and $G' = 2K$ for some H, K. Thus in our result both G and G' can be quite dense, but they must have a special structure, namely that each consists of two copies of some other graph.

Theorem 1.2. *Every graph of the form $2H$ can be reduced to the empty graph by a sequence of switches.*

Theorem 1.2 easily implies the following Corollary, which seems an independently interesting fact in graph theory. We omit its easy proof.

Corollary 1.3. *Any graph can be transformed into any other graph using the following operations:*

(i) switches,

(ii) replacing a graph by two identical copies,

(iii) replacing two identical copies of a graph by a single copy, and

(iv) deleting vertices of degree zero.

2. OUTLINE OF THE PROOF

In this section we give the proof of Theorem 1.2 while deferring the proof of a key lemma to Sections 3–5.

Let H and H' be two copies of the same graph and let $G = H + H'$. Suppose for a contradiction G is a counterexample to the theorem. We may choose G to be minimal with respect to the number of edges, and among all such graphs we choose the one that minimizes the number of vertices. Let $d_1, \ldots, d_n$ be the different values for the degrees of the vertices of H. Without loss of generality we can assume $d_1 < \cdots < d_n$. By minimality of the number of vertices we know that H has no isolated vertex so $d_1 > 0$. Partition the vertices of H into sets $V_1, \ldots, V_n$ so that all vertices of V_i have degree d_i. Observe that deleting an edge with both endpoints in the same set V_i corresponds to a switch. It follows by minimality of the number of edges that each set V_i is a stable set. Similarly, we partition the vertices of H' into stable sets $V'_1, \ldots, V'_n$. Throughout this paper when we talk about *minimum* *counterexample* we mean a graph $G = H + H'$ with stable sets $V_1, \ldots, V_n$ and $V_1', \ldots, V_n'$ with the properties described above.

We say that a sequence of switches is *restricted* to a subset of edges if only edges in that subset are added or removed. We say that a sequence of switches is restricted to a subset of vertices if it is restricted to edges with both endpoints in that subset. For a positive integer n, we write $[n]$ for $\{1, \ldots, n\}$. The key to proving Theorem 1.2 is the following lemma.

Key Lemma. *Let G be a minimal counterexample. Then there exists integers k, r, m with $1 \leq k \leq r \leq m < n$ and vertices $u \in V_r, u' \in V_r', v \in V_{m+1}, v' \in V_{m+1}'$ where $uv, u'v'$ are edges of G. Moreover, there exists a sequence of switches restricted to the set $S = \cup_{l=k}^{m}(V_l \cup V_l')$ and to edges distinct from uu' such that in the resulting graph K:*

$$d_K(u) = d_K(u') \quad \text{and} \quad 0 \leq d_K(u) - d_K(v) \leq 1.$$

Observe that since all switches are restricted to S, $d_K(v) = d_K(v') = d_{m+1}$. This lemma implies the main theorem.

Proof of Theorem 1.2. Let G be a minimum counterexample and let K be the graph obtained from the Key Lemma. If $d_K(u) - d_K(v) = 1$ then add edge vv'. Now vertices u, u', v, v' all have the same degree. Remove uv and $u'v'$, and add uu'. Note that these are all switches. Every vertex in the set $S = \cup_{l=k}^{m}(V_l \cup V_l')$ has the same degree now as in K. Since all the switches used to construct K from G were restricted to the set S and to edges distinct from uu' we can repeat each of these switches in the reverse order. At the end remove the edge vv' if it is present. The resulting graph is the graph obtained from G by removing edges uv and $u'v'$, a contradiction to the minimality of the counterexample. $\square$

In the remainder of this section we give the basic idea behind the proof of the Key Lemma. The first observation is the following.

Lemma 2.1. *Let G be a minimum counterexample. Then there exists $j \in [n-1]$ such that*

$$|V_j| \geq d_{j+1} - d_j + 1.$$

For the proof of this result only the cardinality of the stable sets V_i, the number of edges between V_i ($i \in [n-1]$) and V_n, and the degrees $d_1, \ldots, d_n$ are needed. This information is captured by the object we define next. Let J be a star with vertices $x_1, \ldots, x_n$ ($n \geq 2$) and edges $x_i x_n$ for all $i \in [n-1]$. We associate two positive integers s_i, d_i with every vertex x_i. We also associate a positive integer

w_i with every edge x_ix_n. We say that the 4-tuple (J,s,d,w) is an SDW-star if the following relations hold:

$$0 < d_1 < d_2 \cdots < d_n \tag{2.1a}$$

$$d_n = \frac{1}{s_n}\sum_{i=1}^{n-1} w_i \tag{2.1b}$$

$$w_i \le s_i s_n \quad \text{for all} \quad i \in [n-1]. \tag{2.1c}$$

Remark 2.2. Let G be a minimum counterexample and let J be the star with vertices $x_1,\dots,x_n$ edges x_ix_n for all $i \in [n-1]$. Associate with every vertex x_i of J the integers d_i and $s_i = |V_i|$. Associate with every edge x_ix_n the integer w_i which is equal to the number of edges of H with one endpoint in V_i and one endpoint in V_n. Then (J,s,d,w) is an SDW-star.

In the previous remark Relation (2.1a) is trivially satisfied, (2.1b) follows from the fact that V_n is a stable set, and (2.1c) follows from the fact that H is simple. The following result implies Lemma 2.1.

Lemma 2.3. *Let (J,s,d,w) be an SDW-star. Then there exists $j \in [n-1]$ such that*

$$s_j \ge d_{j+1} - d_j + 1. \tag{2.2}$$

Moreover, if j is the smallest such integer, then $\sum_{i=j}^{n-1} w_i > 0$.

Proof. Let j be the largest integer in $[n]$ such that $s_i \le d_{i+1} - d_i$ for all $i \in [j-1]$. Summing all these inequalities we obtain $\sum_{i=1}^{j-1} s_i \le d_j - d_1$. Thus $d_n \ge d_j \ge \sum_{i=1}^{j-1} s_i + d_1$. To complete the proof it suffices to show that $\sum_{i=j}^{n-1} w_i > 0$, as this implies that $j < n$, and hence the relation (2.2).

Suppose for a contradiction that $\sum_{i=j}^{n-1} w_i = 0$. It follows from (2.1b) that $d_n = \frac{1}{s_n}\sum_{i=1}^{j-1} w_i$. Relation (2.1c) states that $\frac{w_i}{s_n} \le s_i$, thus $d_n \le \sum_{i=1}^{j-1} s_i$. But then the lower and upper bound on d_n imply $d_1 = 0$, a contradiction to (2.1a). □

3. Switches

In this section we describe various constructions using switches for the proof of the Key Lemma.

Lemma 3.1. *Let K be a graph, let $T \subseteq S \subseteq V(K)$, and let the subgraph $K[S]$ induced by S be a perfect matching M. Let K' be obtained from K by adding all $T, S-T$-edges. Suppose that*

(i) all vertices of S have the same degree in K,
(ii) $|e \cap T| \equiv 0 (mod 2)$ for every edge $e = \{v, v'\}$ in M.

Then there exists a sequence of switches, restricted to both S and $E(S) - M$, that transforms K to K'.

Proof. See Appendix. □

Applying Lemma 3.1 yields the following two similar results. Before stating them, we need the following setup.

Setup: Consider a graph K with two (possibly empty) disjoint subsets of vertices A and B where $|A|, |B|$ are both even and $A \cup B$ is a stable set of K. Suppose that vertices of A have degree $d - \delta$, vertices of B degree $d - \delta + 1$, and $e \geq d + \gamma$.

Lemma 3.2. *Let $\delta = 1, \gamma = 0$. Then there exists a sequence of switches restricted to $A \cup B$ such that the resulting graph K' has two (possibly empty) subsets of vertices $X, Y \subseteq A \cup B$, where $|X|, |Y|$ are both even with the following properties:*

(i) $X \cup Y$ is a stable set of K',
(ii) vertices of X have degree $e - 1$, vertices of Y have degree e,
(iii) $|X| + |Y| \geq |A| + |B| - e + d - 1$.

Proof. See Appendix. □

Lemma 3.3. *Let $\gamma = 1$ and let u, u' be two vertices of B. Then there exists a sequence of switches restricted to $A \cup B$ and to edges other than uu' such that the resulting graph K' has two (possibly empty) subsets of vertices $X, Y \subseteq A \cup B$ where $|X|, |Y|$ are both even with the following properties:*

(i) $X \cup Y$ is a stable set of K',
(ii) vertices of X have degree $e - \delta'$, vertices of Y degree $e + 1 - \delta'$ where δ' is either 0 or 1,
(iii) $|X| + |Y| \geq |A| + |B| - e + d - 1$,
(iv) if $X \cup Y$ is non-empty, then $u, u' \in Y$.

Proof. See Appendix. □

In the proof of the Key Lemma, constructions (similar to that of Lemma 3.1) are iterated several times. This requires a generalization of Lemma 2.3 which is presented in Section 4. Finally the Key Lemma is proved in Section 5.

4. SDW-STARS

Our objective in this section is to prove the following result which we use later on.

Lemma 4.1. *Let G be a minimum counterexample. Then there exist k, m with $1 \leq k \leq m < n$ such that for all l with $k \leq l \leq m$,*

$$\sum_{i=k}^{l} |V_i| \geq d_{l+1} - d_k + 1.$$

Moreover, $\cup_{i=k}^{m} V_i$ is a stable set and there exists an edge uv with $u \in \cup_{i=k}^{m} V_i$ and $v \in V_{m+1}$.

By restricting the result to the case where $l = 1$ we see that this generalizes Lemma 2.1.

Proposition 4.2. *Let (J, s, d, w) be an SDW-star. Then there exists $k \in [n-1]$ such that for all l with $k \leq l < n$,*

$$\sum_{i=k}^{l} s_i \geq d_{l+1} - d_k + 1. \tag{4.3}$$

Moreover, $\sum_{i=k}^{n-1} w_i > 0$.

We first show that the above proposition implies Lemma 4.1, and then we prove the proposition.

Proof of Lemma 4.1. Let (J, s, d, w) be the SDW-star defined as in Remark 2.2. Let k be the integer from Proposition 4.2. Since $\sum_{i=k}^{n-1} w_i > 0$ there is an edge between $\cup_{i=k}^{n-1} V_i$ and V_n. Thus the following statement is true (choose $m = n - 1$ for instance): for some integer m with $k \leq m < n$ there is an edge between $\cup_{i=k}^{m} V_i$ and V_{m+1}. If m is the smallest such integer, then $\cup_{i=k}^{m} V_i$ is a stable set. □

Proof of Proposition 4.2. Let us proceed by induction on the number of vertices n of J. Consider first the base case $n = 2$, i.e. the star has only two vertices. By Lemma 2.3 we have, $s_1 \geq d_2 - d_1 + 1$ and $w_1 > 0$, as required.

Thus we may assume $n \geq 3$. Consider (J, s, d, w) and let j be smallest integer for which (2.2) holds. If $j = n-1$ then (2.2) is the same as (4.3) (with $k = j = n-1$ and thus $l = k$). Moreover, also by Lemma 2.3 we have $w_{n-1} > 0$, which completes the proof in this case. Thus we will assume $j \leq n-2$. Let J' be the star with vertices $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$ and edges $x_i x_n$ for each $i \in [n-1] - \{j\}$. Define s'_i to be equal to s_i if $i \notin \{j, j+1\}$ and equal to $s_j + s_{j+1}$ if $i = j + 1$. Define w'_i to be equal to w_i if $i \notin \{j, j+1\}$ and equal to $w_j + w_{j+1}$ if $i = j + 1$. Define d'_i to be equal to d_i if $i \notin \{j, j+1\}$ and equal to d_j if $i = j + 1$ (note that s'_j, w'_j, d'_j are not defined).

Claim: (J', v', d', w') is an SDW-star.

Proof of Claim: Clearly, (2.1a) holds, as (J, s, d, w) is an SDW-star. Since $j \leq n-2$, $d'_n = d_n, s'_n = s_n$ and $d_n = \frac{1}{s_n}\sum_{i=1}^{n-1} w_i = \frac{1}{s'_n}\sum_{i=1, i\neq j}^{n} w'_i$, relation (2.1b) is satisfied. Note that $w'_{j+1} = w_j + w_{j+1} \leq s_j s_n + s_{j+1}s_n = s'_{j+1}s'_n$, thus relation (2.1c) is also satisfied.

End of Claim

Let σ be the function defined as follows: $\sigma(l) = l+1$ if $l \neq j-1$ and $\sigma(j-1) = j+1$. It follows from the claim and by induction that there exists $k \in [n-1] - \{j\}$ such that for all $l \neq j$ with $k \leq l < n$,

$$(*) \qquad \sum_{\substack{i=k\\ i\neq j}}^{l} s'_i \geq d'_{\sigma(l)} - d'_k + 1.$$

In particular, when $l = k$ we obtain $s'_k \geq d'_{\sigma(k)} - d'_k + 1$. If $k \leq j-2$, then this last relation can be rewritten as $s_k \geq d_{k+1} - d_k + 1$, a contradiction to the choice of j. If $k = j-1$, then the relation can be rewritten as $s_{j-1} \geq d'_{j+1} - d'_{j-1} + 1 = d_j - d_{j-1} + 1$, again a contradiction to the choice of j. Thus $k \geq \sigma(j-1) = j+1$.

If $k \geq j+2$, then (*) can be rewritten as $\sum_{i=k}^{l} s_i \geq d_{l+1} - d_l + 1$ for all l with $k \leq l < n$. Thus (4.3) is satisfied, and by induction $0 < \sum_{i=k}^{n-1} w'_i = \sum_{i=k}^{n-1} w_i$. Thus we may assume $k = j+1$, and (*) can be rewritten as $s'_{j+1} + \sum_{i=j+2}^{l} s_i \geq d_{l+1} - d'_{j+1} + 1$. Because $s'_{j+1} = s_j + s_{j+1}$ and $d'_{j+1} = d_j$, this can again be rewritten as $\sum_{i=j}^{l} s_i \geq d_{l+1} - d_j + 1$ where $l \geq j+1$. Thus (4.3) is satisfied with $k = j$ for all $l \geq j+1$. If $l = j$, then (4.3) becomes (2.2) which holds by the choice of j. Finally, by induction $0 < \sum_{i=j+1}^{n-1} w'_i = w'_{j+1} + \sum_{i=j+2}^{n-1} w_i = \sum_{i=j}^{n-1} w_i$, where the last equality follows from the fact that $w'_{j+1} = w_j + w_{j+1}$. $\square$

5. Proof of the Key Lemma

Proof of the Key Lemma. Let $G = H + H'$ be a minimum counterexample. Let us write $\bar{V}_i$ for $V_i \cup V'_i$. Let us apply Lemma 4.1 to both H and H' and add both corresponding inequalities. We obtain that there exist integers k, m with $1 \leq k \leq m < n$ such that for all integers l with $k \leq l \leq m$,

$$(*) \qquad \sum_{i=k}^{l} |\bar{V}_i| \geq 2(d_{l+1} - d_k + 1) \geq (l+1-k) + d_{l+1} - d_k + 2.$$

Define $S_l = \cup_{i=k}^{l} \bar{V}_i$. By Lemma 4.1 we also know that S_l is a stable set, and that there are edges $uv \in E(H)$ and $u'v' \in E(H')$ where $u, u' \in \bar{V}_r$ for some r with $k \leq r \leq m$ and $v, v' \in \bar{V}_{m+1}$. We need the following claim, which we prove subsequently, to complete the proof of the key lemma.

Claim: For every l with $k-1 \leq l \leq m$ there exists a sequence of switches, restricted to S_l and restricted to edges distinct from uu', such that the resulting graph G^l has two subsets of vertices $X, Y \subseteq S_l$ where $|X|, |Y|$ are both even with the following properties:

(a) $X \cup Y$ is a stable set,
(b) $|X| + |Y| \geq \sum_{i=k}^{l} |\bar{V}_i| - d_{l+1} + d_k - (l-k) - 1$,
(c) if $r > l$, then vertices of X have degree $d_{l+1} - 1$ and vertices of Y degree d_{l+1},
(d) if $r \leq l$, then $u, u' \in Y$, vertices of X have degree $d_{l+1} - \delta_l$ and vertices of Y degree $d_{l+1} - \delta_l + 1$, where δ_l is either 0 or 1.

Proof of the Key Lemma continued: Since $r \leq m$ we obtain from (d) that there is a sequence of switches restricted to $S = S_l$ and edges distinct from uu', such that in the resulting graph vertices $u, u' \in Y$ have degree $d_{m+1} + 1 - \delta_m$ where δ_m is either 0 or 1. This completes the proof. □

Finally, we conclude with the proof of the claim used in proving the Key Lemma.
Proof of Claim: Let us proceed by induction on l. The base case is when $l = k-1$. Choose $X = Y = \emptyset, G = G^l$ and do no switches. Note that (a),(c) trivially hold. Since $|X| + |Y| = 0 \geq 0 = \sum_{i=k}^{k-1} |\bar{V}_i| - d_{(k-1)+1} + d_k - (k-1-k) - 1$, (b) is satisfied. As $r \geq k > l-1$ we do not need to check (d). This completes the base case. Assume now the claim holds for some l with $k-1 \leq l < m$, and let X', Y' be the corresponding sets of G^l. We will find sets X, Y in a graph G^{l+1} which satisfy properties (a)-(d) for $l+1$. *We will denote these properties by (a')-(d') to distinguish them from the corresponding statement for* l.

Sub claim: If $|X| + |Y| \geq |X'| + |Y'| + |\bar{V}_{l+1}| - d_{l+2} + d_{l+1} - 1$ then (b') is satisfied and $X \cup Y \neq \emptyset$.
Proof of Sub claim: Since by induction (b) holds for X', Y',

$$
\begin{aligned}
&(|X'| + |Y'|) + |\bar{V}_{l+1}| - d_{l+2} + d_{l+1} - 1 \\
&\quad \geq \left(\sum_{i=k}^{l} |\bar{V}_i| + |\bar{V}_{l+1}| - d_{l+1} + d_k - (l-k) - 1 \right) - d_{l+2} + d_{l+1} - 1 \\
&\quad = \sum_{i=k}^{l+1} |\bar{V}_i| - d_{l+2} + d_k - (l+1-k) - 1.
\end{aligned}
$$

Moreover, because of (*) with $l+1$ this last expression is at least

$$l + 2 - k + d_{l+2} - d_k + 2 - d_{l+2} + d_k - l - 1 + k - 1 = 2.$$

End of Sub claim; proof of Claim continued:

Case 1 $r > l$, i.e. $u, u' \notin S_l$.

Define $A = X'$ and $B = Y' \cup \bar{V}_{l+1}$. Note that vertices of A have degree $d_{l+1} - 1$ and vertices of B have degree d_{l+1} (both in G^l). Define $d = d_{l+1}, e = d_{l+2}$. Suppose first that $r > l + 1$. Consider the new graph G^{l+1} and the sets X, Y obtained in Lemma 3.2. Now (i) restates (a') and (ii) restates (c'). Since $r > l + 1$ we do not need to check (d'). We know $|A| + |B| = |X'| + |Y'| + |\bar{V}_{l+1}|$. It follows from (iii) that the hypothesis of the sub claim is satisfied, so (b') holds. Suppose now that $r = l + 1$. Then $u, u' \in B$. Consider the new graph G^{l+1} and the sets X, Y obtained in Lemma 3.3 with $\delta = 1$. Again (i) proves (a') and (iii) with the sub-claim implies that (b') is satisfied and $X \cup Y \neq \emptyset$. This implies using (iv) and (ii) that (d') also holds. Finally as $r = l + 1$ we do not need to check (c').

Case 2 $r \leq l$.

By induction $u, u' \in Y$. If $\delta_l = 1$, then define $A = X', B = Y' \cup \bar{V}_{l+1}$. If $\delta_l = 0$, then define $A = X' \cup \bar{V}_{l+1}, B = Y'$. In either case $u, u' \in Y' \subseteq B$. Note that vertices of A have degree $d_{l+1} - \delta_l$ and vertices of B have degree $d_{l+1} - \delta_l + 1$. Define $d = d_{l+1}, e = d_{l+2}$. Consider the new graph G^{l+1} and the sets X, Y obtained in Lemma 3.3. Again (i) proves (a') and (iii) with the sub-claim implies that (b') is satisfied and $X \cup Y \neq \emptyset$. This implies using (iv) and (ii) that (d') also holds. Finally as $r \leq l$ we do not need to check (c').

End of Claim

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6. APPENDIX

Proof of Lemma 3.1: Since $|S|$ is even we can partition the complete graph with vertex set S into $|S| - 1$ perfect matchings $M_1, \ldots, M_{|S|-1}$, where $M_1 = M$. Sequentially for each $i = 2, \ldots, |S| - 1$, add all edges of M_i (one after the other) to K. Observe that these are all valid switches since all vertices of S have the same degree in K.

By hypothesis, M_1 restricted to T is a perfect matching which we denote by N_1. We can partition the complete graph with vertex set T into perfect matchings, $N_1, \ldots, N_{|T|-1}$. Sequentially for each $i = 2, \ldots, |T| - 1$, remove edges of N_i.

Proceeding similarly, we can also remove all edges with both endpoints in $S - T$. This yields the desired graph. $\square$

The following two proofs use the Setup in Section 3.

Proof of Lemma 3.2: Since A (resp. B) has even cardinality, we can pair every vertex v of A (resp. B) with a unique vertex v' of A (resp. B). If both A, B are non-empty, add all edges between paired vertices of A – these are switches as all vertices of A have the same degree. In the resulting graph all vertices of $A \cup B$ have the same degree, say δ, which is either $d-1$ or d. Define α to be equal to $e-\delta$ if $e-\delta$ is even, and $e-\delta-1$ otherwise. Let $S = A \cup B$. Observe that if $\alpha \geq |S|$, then $|A| + |B| \leq \alpha \leq e-\delta \leq e-d+1$, thus $|A| + |B| - e + d - 1 \leq 0$ and choosing $X = Y = \emptyset$ trivially satisfies (i)-(iii). Hence we can assume $\alpha < |S|$.

Choose a subset T of S of cardinality α which keeps paired vertices together – this is possible since α is even. From Lemma 3.1 we know that there is a sequence of switches (restricted to $A \cup B$) which adds all edges with one endpoint in $S - T$ and one endpoint in T. Denote the resulting graph by K^*. Let $U = (S - T) \cap A$ and $V = (S - T) \cap B$. Vertices of $U \cup V$ have degree $\delta + |T| = \delta + \alpha$ in K^*, and

(a) $$e = \delta + (e - \delta) \geq \delta + \alpha \geq \delta + (e - \delta - 1) = e - 1.$$

Also,

(b) $$|U| + |V| = |A| + |B| - \alpha \geq |A| + |B| - (e - \delta) \geq |A| + |B| - e + d - 1.$$

Consider first the case where either A or B is empty. Define $K' = G^*$. Vertices of $U \cup V$ have degree either $e-1$ or e in K'. In the former case set $X = \emptyset, Y = U \cup V$; in the latter one set $X = U \cup V, Y = \emptyset$. Thus (ii) is satisfied. By construction $U \cup V$ is a stable set so (i) is satisfied. Finally, (iii) follows from (b).

Consider now the case where both A, B are non-empty, in this case $\delta = d$. Remove edges in K^* between paired vertices of U and let K' denote the resulting graph. If vertices of U have degree $e-1$ in K', then we can choose $X = U, Y = V$ and we are done as before. Otherwise vertices of $U \cup V$ must have had degree $e-1$ in K^*. This implies that $e - 1 = \delta + \alpha = d + \alpha$. Consequently,

(c) $$|U| + |V| = |A| + |B| - \alpha = |A| + |B| - e + d + 1.$$

If $|U| \leq 2$, then choose $X = V, Y = \emptyset$. Clearly (i),(ii) are satisfied, and (iii) holds because of (c). If $|U| \geq 2$, then pick any paired vertices v, v' of U. From Lemma 3.1 we know that there is a sequence of switches restricted to U which adds all edges with one endpoint in $U - \{v, v'\}$ and one endpoint in v or v'. Define, $X = V, Y = U - \{v, v'\}$. Again (i),(ii) are clearly satisfied, and (iii) holds because of (c). $\square$

Proof of Lemma 3.3: We can pair every vertex v of A with a unique vertex v' of A. Do the same for B and pair u with u'. If A is non-empty, then add all edges between paired vertices of A. In the resulting graph all vertices of $A \cup B$ have the same degree which is $d-\delta+1$. Define α to be equal to $e-d-1+\delta$ if it is even and $e-d+\delta$ otherwise. Let $S = A \cup B$. As in the proof of Lemma 3.2 if $\alpha \geq |S|$, then we may choose $X = Y = \emptyset$. Otherwise choose a subset T of S of cardinality α which keeps paired vertices together and $u, u' \in S-T$ if $S-T$ non-empty. From Lemma 3.1 we know that there is a sequence of switches (restricted to $A \cup B$ and to edges other than uu') which adds all edges with one endpoint in $S-T$ and one endpoint in T. Denote the resulting graph by K^*. Let $X = (S-T) \cap A$ and $Y = (S-T) \cap B$. Vertices of $X \cup Y$ have degree $d - \delta + 1 + \alpha \geq d + 1 - \delta + e - d - 1 + \delta = e$ in K^* and $d + 1 - \delta + \alpha \geq e$. Remove edges between paired vertices of X and call the resulting graph K'. Clearly (ii) is satisfied. By construction $X \cup Y$ is a stable set in $'K$ proving (i). Now $|X| + |Y| = |A| + |B| - \alpha \geq |A| + |B| - e + d - \delta$ which proves (iii). Moreover, if $|X| + |Y| \geq 2$ then we could have chosen T such that $u, u' \in S - T$ which proves (iv). $\square$

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