Hypergraphs with infinitely many extremal constructions

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Abstract

We give the first exact and stability results for a hypergraph Turán problem with infinitely many extremal constructions that are far from each other in edit-distance. This includes an example of triple systems with Turán density 2/9, thus answering some questions posed by the third and fourth authors and Reiher about the feasible region of hypergraphs.

Our novel approach is to construct certain multilinear polynomials that attain their maximum (in the standard simplex) on a line segment and then to use these polynomials to define an operation on hypergraphs that gives extremal constructions.

1 Introduction

1.1 Turán number and stability

For $r \ge 2$ an r-uniform hypergraph (henceforth r-graph) $\mathcal{H}$ is a collection of r-subsets of some finite set V. Given a family $\mathcal{F}$ of r-graphs we say $\mathcal{H}$ is $\mathcal{F}$-free if it does not contain any member of $\mathcal{F}$ as a subgraph. The *Turán number* $\mathrm{ex}(n,\mathcal{F})$ of $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free r-graph on n vertices. The *Turán density* $\pi(\mathcal{F})$ of $\mathcal{F}$ is defined as $\pi(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n,\mathcal{F})/\binom{n}{r}$. A family $\mathcal{F}$ is called *nondegenerate* if $\pi(\mathcal{F}) > 0$. The study of $\mathrm{ex}(n,\mathcal{F})$ is perhaps the central topic in extremal graph and hypergraph theory.

Much is known about $\mathrm{ex}(n,\mathcal{F})$ when $r = 2$, and one of the most famous results in this regard is Turán's theorem, which states that for every integer $\ell \ge 2$ the Turán number $\mathrm{ex}(n, K_{\ell+1})$ is uniquely achieved by the balanced ℓ-partite graph on n vertices, which is called the Turán graph $T(n,\ell)$.

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For $r \geq 3$ determining $\pi(\mathcal{F})$ for a family $\mathcal{F}$ of r-graphs is known to be notoriously hard in general. Indeed, the problem of determining $\pi(K_\ell^r)$ raised by Turán [29], where K_ℓ^r is the complete r-graph on ℓ vertices, is still wide open for all $\ell > r \geq 3$. Erdős offered \$500 for the determination of any $\pi(K_\ell^r)$ with $\ell > r \geq 3$ and \$1000 for the determination of all $\pi(K_\ell^r)$ with $\ell > r \geq 3$.

Conjecture 1.1 (Turán [29])**.** *For every integer $\ell \geq 3$ we have $\pi(K_{\ell+1}^3) = 1 - 4/\ell^2$.*

The case $\ell = 3$ above, which states that $\pi(K_4^3) = 5/9$ has generated a lot of interest and activity over the years. Many constructions (e.g. see [3, 15, 8]) are known to achieve the value in Conjecture 1.1 for $\ell = 3$. In particular, Kostochka [15] showed that there are at least 2^{n-2} nonisomorphic extremal K_4^3-free constructions on $3n$ vertices (assuming Turán's Tetrahedron conjecture is true). This is perhaps one of the reasons why it is so challenging. On the other hand, successively better upper bounds for $\pi(K_4^3)$ were obtained by de Caen [5], Giraud (see [4]), Chung and Lu [4], and Razborov [26]. The current record is $\pi(K_4^3) \leq 0.561666$, which was obtained by Razborov [26] using the Flag Algebra machinery.

In this work, we continue the research on hypergraph Turán problems with the flavour of trying to show that certain phenomena impossible for graphs are possible for r-graphs when $r \geq 3$. There are several results that belong to this category. For instance, the well known work of Frankl and Rödl on jumps [12], and Pikhurko's theorem that for $r \geq 3$ there exist uncountably many Turán densities of infinite families of r-graphs [25]. More specifically, we will show that there are infinitely many finite families $\mathcal{F}$ with similar properties as K_4^3 (assuming Turán's conjecture) in terms of the number of extremal constructions.

Many families $\mathcal{F}$ have the property that there is a unique $\mathcal{F}$-free hypergraph $\mathcal{G}$ on n vertices achieving $\mathrm{ex}(n, \mathcal{F})$, and moreover, any $\mathcal{F}$-free hypergraph $\mathcal{H}$ of size close to $\mathrm{ex}(n, \mathcal{F})$ can be transformed to $\mathcal{G}$ by deleting and adding very few edges. Such a property is called the stability of $\mathcal{F}$. The first stability theorem, which says that K_ℓ is stable for all integers $\ell \geq 3$, was proved independently by Erdős and Simonovits [28]. Their result motivated the third author [24] to make the following definition.

Definition 1.2 (t-stable)**.** *Let $r \geq 2$ and $t \geq 1$ be integers. A family $\mathcal{F}$ of r-graphs is t-stable if for every $m \in \mathbb{N}$ there exist r-graphs $\mathcal{G}_1(m), \ldots, \mathcal{G}_t(m)$ on m vertices such that the following holds. For every $\delta > 0$ there exist $\epsilon > 0$ and N_0 such that for all $n \geq N_0$ if $\mathcal{H}$ is an $\mathcal{F}$-free r-graph on n vertices with $|\mathcal{H}| > (1 - \epsilon)\mathrm{ex}(n, \mathcal{F})$ then $\mathcal{H}$ can be transformed to some $\mathcal{G}_i(n)$ by adding and removing at most δn^r edges. Say $\mathcal{F}$ is stable if it is 1-stable.*

Denote by $\xi(\mathcal{F})$ the minimum integer t such that $\mathcal{F}$ is t-stable, and set $\xi(\mathcal{F}) = \infty$ if there is no such t. Call $\xi(\mathcal{F})$ *the stability number* of $\mathcal{F}$.

The classical Erdős–Stone–Simonovits theorem [7, 6] and Erdős–Simonovits stability theorem [28] imply that every nondegenerate family of graphs is stable. For hypergraphs there are many families (whose Turán densities are unknown) which are conjecturally not stable. Two famous examples are Turán's conjecture on tetrahedra (i.e. K_4^3) mentioned above (e.g. see [27, 15, 20]), and the Erdős–Sós conjecture on triple systems with bipartite links (e.g. see [9, 20]). In fact, families that are not stable and whose Turán densities can be determined were constructed only very recently. In [20], the third and fourth authors constructed the first finite 2-stable family of 3-graphs. Later in [22], together with Reiher, they further constructed the first finite t-stable family of 3-graphs for every integer $t \geq 3$. In [21], using a similar method that was used for the hypergraph jump problem (see e.g. [11]) their constructions were extended to every integer $r \geq 4$.

Using Kostochka's constructions the authors proved in [20] that $\xi(K_4^3) = \infty$ (assuming Turán's conjecture). However, the methods used in [20] and [22] for constructing the 2-stable family and t-stable families cannot be extended further to construct a finite family whose stability number is infinite, and this was left as an open question in both papers.

Our main result in this work is to give a novel method to construct (finite) families $\mathcal{F}$ with $\xi(\mathcal{F}) = \infty$. Our method is based on a simple property of multilinear polynomials, which will be stated in Section 1.4. We will use this method to give infinitely many (finite) families of 3-graphs whose number of extremal constructions grows with the number of vertices. Further, using the machinery provided in [23] we also prove an Andrásfai–Erdős–Sós type [1] stability theorem for these families.

We identify an r-graph $\mathcal{H}$ with its edge set, use $V(\mathcal{H})$ to denote its vertex set, and denote by $v(\mathcal{H})$ the size of $V(\mathcal{H})$. An r-graph $\mathcal{H}$ is a *blowup* of an r-graph $\mathcal{G}$ if there exists a map $\psi\colon V(\mathcal{H}) \to V(\mathcal{G})$ so that $\psi(E) \in \mathcal{G}$ iff $E \in \mathcal{H}$. We say $\mathcal{H}$ is *$\mathcal{G}$-colorable* if there exists a map $\phi\colon V(\mathcal{H}) \to V(\mathcal{G})$ so that $\phi(E) \in \mathcal{G}$ for all $E \in \mathcal{H}$, and we call such a map ϕ a *homomorphism* from $\mathcal{H}$ to $\mathcal{G}$. In other words, $\mathcal{H}$ is $\mathcal{G}$-colorable if and only if $\mathcal{H}$ occurs as a subgraph in some blowup of $\mathcal{G}$.

Theorem 1.3. *For every integer $t \geq 3$ there exists a finite family $\mathcal{F}_t$ of 3-graphs such that the following statements hold.*

(a) We have $\mathrm{ex}(n, \mathcal{F}_t) \leq \frac{(t-2)(t-1)}{6t^2} n^3$ for all $n \in \mathbb{N}$, and equality holds whenever $t \mid n$.

(b) If $t \mid n$, then the number of nonisomorphic maximum $\mathcal{F}_t$-free 3-graphs on n vertices is at least $n/2t$.

(c) We have $\xi(\mathcal{F}_t) = \infty$.

(d) For every integer $t \geq 4$ there exist constants $\epsilon = \epsilon(t) > 0$ and $N_0 = N_0(t)$ such that the following holds for every integer $n \geq N_0$. Every n-vertex $\mathcal{F}_t$-free 3-graph with minimum degree at least $\left(\frac{(t-2)(t-1)}{2t^2} - \epsilon\right) n^2$ is Γ_t-colorable, where Γ_t is some fixed 3-graph on $t+2$ vertices.

Motivated by the hypergraph jump problem[1], the authors asked in [22] whether there exists a finite family whose Turán density is 2/9 that is not stable? By letting $t = 3$ in Theorem 1.3 we see that the finite family $\mathcal{F}_3$ of 3-graphs has Turán density 2/9 but $\xi(\mathcal{F}_3) = \infty$, thus answering this question.

1.2 Multilinear polynomials

In this short section, we state a simple result about polynomials. Later we will see that our main theorem reduces to this result.

Denote by Δ_{m-1} the standard $(m-1)$-dimensional simplex, i.e.

$$\Delta_{m-1} = \{(x_1, \ldots, x_m) \in [0,1]^m \colon x_1 + \cdots + x_m = 1\}.$$

Given an m-variable continuous function f we define

$$\lambda(f) = \max\{f(x_1, \ldots, x_m) \colon (x_1, \ldots, x_m) \in \Delta_{m-1}\},$$

[1] Our results in this paper are not very related to the Jump problem, so we omit the related definitions here. We refer the interested reader to e.g. [12, 11, 2] for related results.

and

$$Z(f) = \{(x_1, \ldots, x_m) \in \Delta_{m-1} \colon f(x_1, \ldots, x_m) - \lambda(f) = 0\}.$$

Since Δ_{m-1} is compact, a well known theorem of Weierstraß implies that the restriction of f to Δ_{m-1} attains a maximum value. Thus $\lambda(f)$ and $Z(f)$ are well-defined.

Let $p(X_1, \ldots, X_m)$ be a polynomial, where $X_1, \ldots, X_m$ are indeterminates. We say p is *multilinear* if for every $\alpha \in \mathbb{R}$ and for every $i \in [m]$ we have $p(X_1, \ldots, \alpha \cdot X_i, \ldots, X_m) = \alpha \cdot p(X_1, \ldots, X_i, \ldots, X_m)$. Note that this is equivalent to the fact that each term of p is of the form $\prod_{i \in S} x_i$ for some S. We say p is *nonnegative* (or *nonpositive*) if $p(x_1, \ldots, x_m) \geq 0$ (or $p(x_1, \ldots, x_m) \leq 0$) for all $(x_1, \ldots, x_m) \in \Delta_{m-1}$. For a pair $\{i, j\} \subset [m]$ we say p is symmetric with respect to X_i and X_j if

$$p(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_m) = p(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_m).$$

An easy observation is that a multilinear polynomial $p(X_1, \ldots, X_m)$ is symmetric with respect to X_i and X_j iff there exist polynomials p_1, p_2, p_3 without indeterminates X_i and X_j such that $p = p_1 + p_2(X_i + X_j) + p_3 X_i X_j$.

Given two vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$ define the line segment $L(\vec{x}, \vec{y})$ with endpoints $\vec{x}$ and $\vec{y}$ as

$$L(\vec{x}, \vec{y}) = \{\alpha \cdot \vec{x} + (1 - \alpha) \cdot \vec{y} \colon \alpha \in [0, 1]\}.$$

We will later prove the following result about multilinear polynomials.

Proposition 1.4. *Let $p(X_1, \ldots, X_m) = p_1 + p_2(X_i + X_j) + p_3 X_i X_j$ be an m-variable multilinear polynomial that is symmetric with respect to X_i and X_j. Suppose that p_3 is nonnegative, and p_4, p_5 are nonnegative polynomials satisfying $p_4 + p_5 = p_3$. Then the $(m+2)$-variable polynomial*

$$\begin{aligned} &\hat{p}(X_1, X_1', X_2, X_2', X_3, \ldots, X_m) \\ &= p_1 + p_2(X_1 + X_1' + X_2 + X_2') + p_4(X_1 + X_1')(X_2 + X_2') + p_5(X_1 + X_2)(X_1' + X_2') \end{aligned}$$

satisfies $\lambda(\hat{p}) = \lambda(p)$, and moreover, for every $(x_1, \ldots, x_m) \in Z(p)$ we have $L(\vec{y}, \vec{z}) \subset Z(\hat{p})$, where $\vec{y}, \vec{z} \in \Delta_{m+1}$ are defined by

$$\begin{aligned} \vec{y} &= (x_1, \ldots, x_{i-1}, (x_i + x_j)/2, 0, x_{i+1}, \ldots, x_{j-1}, 0, (x_i + x_j)/2, x_{j+1}, \ldots, x_m), \\ \vec{z} &= (x_1, \ldots, x_{i-1}, 0, (x_i + x_j)/2, x_{i+1}, \ldots, x_{j-1}, (x_i + x_j)/2, 0, x_{j+1}, \ldots, x_m). \end{aligned}$$

1.3 Nonminimal hypergraphs

Now we return to hypergraphs. Our constructions are closely related to the so-called nonminimal hypergraphs. To define them properly we need some definitions related to the Lagrangian of a hypergraph, which was introduced by Frankl and Rödl in [12].

For an r-graph $\mathcal{G}$ on m vertices (let us assume for notational transparency that $V(\mathcal{G}) = [m]$) the multilinear polynomial $p_{\mathcal{G}}$ is defined by

$$p_{\mathcal{G}}(X_1, \ldots, X_m) = \sum_{E \in \mathcal{G}} \prod_{i \in E} X_i.$$

The *Lagrangian* of $\mathcal{G}$ is defined by $\lambda(\mathcal{G}) = \lambda(p_{\mathcal{G}})$. Define

$$Z(\mathcal{G}) = Z(p_{\mathcal{G}}) = \{(x_1, \ldots, x_n) \in \Delta_{m-1} \colon p_{\mathcal{G}}(x_1, \ldots, x_m) = \lambda(\mathcal{G})\}.$$

For a vertex set $S \subset V(\mathcal{G})$ we use $\mathcal{G} - S$ to denote the induced subgraph of $\mathcal{G}$ on $V(\mathcal{G}) \setminus S$.

Definition 1.5. *An r-graph $\mathcal{G}$ is minimal if $\lambda(\mathcal{G} - v) < \lambda(\mathcal{G})$ for every vertex $v \in \mathcal{G}$. Otherwise, it is nonminimal.*

A substantial amount of research on hypergraph Turán problems is about minimal hypergraphs. Indeed, for every nondegenerate family $\mathcal{F}$ of hypergraphs that was studied before this work, the extremal $\mathcal{F}$-free constructions are either minimal or blowups of minimal hypergraphs or close to them in edit-distance. Conversely, in [25], Pikhurko proved that for every minimal hypergraph $\mathcal{G}$ there exists a finite family $\mathcal{F}$ whose unique extremal construction is a blowup of $\mathcal{G}$ [2]. In contrast, there was very little literature about nonminimal hypergraphs.

Parallel to Plikurko's result about minimal hypergraphs we have the following result about nonminimal hypergraphs.

Proposition 1.6. *Let $r \geq 3$ be an integer. For every nonminial r-graph $\mathcal{G}$ there exists a finite family $\mathcal{F}$ of r-graphs such that for every integer $n \in \mathbb{N}$ there exists a maximum $\mathcal{F}$-free r-graph on n-vertices which is a blowup of $\mathcal{G}$.*

Next, we introduce an operation motivated by Proposition 1.4. It shows that for a certain class of 3-graphs $\mathcal{G}$ we can produce a nonminimal 3-graph containing $\mathcal{G}$.

For a 3-graph $\mathcal{G}$ and a pair of vertices $\{u, v\} \subset V(\mathcal{G})$ the *neighborhood* $N_{\mathcal{G}}(u, v)$ and *codegree* $d_{\mathcal{G}}(u, v)$ of $\{u, v\}$ are defined as

$$N_{\mathcal{G}}(u, v) = \{w \in \mathcal{G} \colon \{u, v, w\} \in \mathcal{G}\} \quad \text{and} \quad d_{\mathcal{G}}(u, v) = |N_{\mathcal{G}}(u, v)|.$$

We say $\mathcal{G}$ is 2*-covered* if $d_{\mathcal{G}}(u, v) \geq 1$ for every pair $\{u, v\} \subset V(\mathcal{G})$. For a vertex $u \in V(\mathcal{G})$ the *link* of v is defined as

$$L_{\mathcal{G}}(u) = \{\{v, w\} \subset \mathcal{G} \colon \{u, v, w\} \in \mathcal{G}\}.$$

Vertices u and u' in $\mathcal{G}$ are clones if $L_{\mathcal{G}}(u) = L_{\mathcal{G}}(v)$.

Definition 1.7 (Crossed blowup)**.** *Let $\mathcal{G}$ be a 3-graph and $\{v_1, v_2\} \subset \mathcal{G}$ be a pair of vertices with $d(v_1, v_2) = k \geq 2$. Fix an ordering of vertices in $N_{\mathcal{G}}(v_1, v_2)$, say $N_{\mathcal{G}}(v_1, v_2) = \{u_1, \ldots, u_k\}$. The crossed blowup $\mathcal{G} \boxplus \{v_1, v_2\}$ of $\mathcal{G}$ on $\{v_1, v_2\}$ is defined as follows.*

(a) Remove all edges containing the pair $\{v_1, v_2\}$ from $\mathcal{G}$,

(b) add two new vertices v_1' and v_2', make v_1' a clone of v_1 and v_2' a clone of v_2,

(c) for every $i \in [k-1]$ add the edge set $\{u_i v_1 v_2, u_i v_1 v_2', u_i v_1' v_2, u_i v_1' v_2'\}$, and for $i = k$ add the edge set $\{u_k v_1 v_1', u_k v_1 v_2', u_k v_2 v_1', u_k v_2 v_2'\}$.

Remark. If in Step (c) we add the edge set $\{u_k v_1 v_2, u_k v_1 v_2', u_k v_1' v_2, u_k v_1' v_2'\}$ instead of $\{u_k v_1 v_1', u_k v_1 v_2', u_k v_2 v_1', u_k v_2 v_2'\}$, then we obtain an ordinary blowup of $\mathcal{G}$.

Definition 1.8. *Let $\mathcal{G}$ be a 3-graph. A pair $\{v_1, v_2\} \subset V(\mathcal{G})$ is symmetric in $\mathcal{G}$ if*

$$L_{\mathcal{G}}(v_1) - v_2 = L_{\mathcal{G}}(v_2) - v_1.$$

The crossed blowup of a 3-graph has the following properties.

[2] Pikhurko's results are much more general than what we stated here, and we refer the reader to [25] for more details.

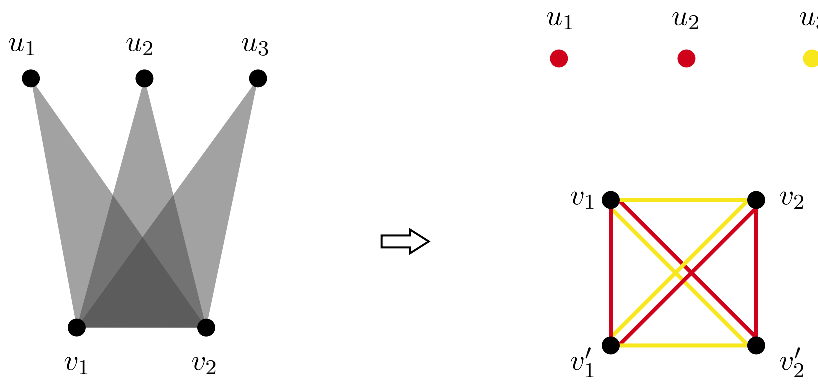

Figure 1: $\{u_1v_1v_2, u_2v_1v_2, u_3v_1v_2\}$ and $\{u_1v_1v_2, u_2v_1v_2, u_3v_1v_2\} \boxplus \{v_1, v_2\}$. The link of red vertices is the red $K_{2,2}$, the link of the yellow vertex is the yellow $K_{2,2}$.

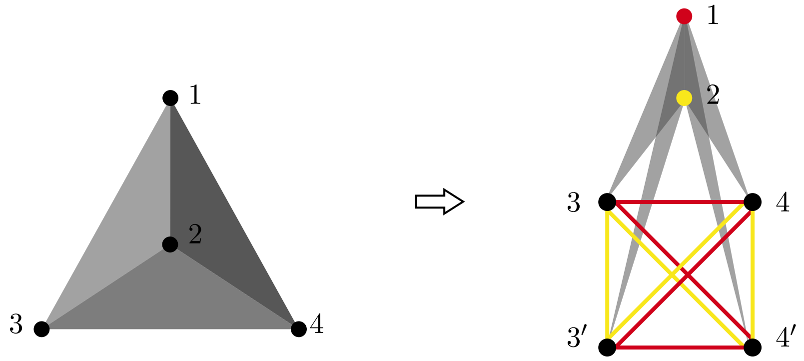

Figure 2: K_4^3 and $K_4^3 \boxplus \{3, 4\}$.

Proposition 1.9. *Suppose that $\mathcal{G}$ is an m-vertex 3-graph and $\{v_1, v_2\} \subset V(\mathcal{G})$ is a pair of vertices with $d(v_1, v_2) \geq 2$. Then the following statements hold.*

(a) The 3-graph $\mathcal{G}$ is contained in $\mathcal{G} \boxplus \{v_1, v_2\}$ as an induced subgraph. In particular, $\lambda(\mathcal{G}) \leq \lambda(\mathcal{G} \boxplus \{v_1, v_2\})$.

(b) The 3-graph $\mathcal{G} \boxplus \{v_1, v_2\}$ is 2-covered iff $\mathcal{G}$ is 2-covered.

(c) If $\{v_1, v_2\}$ is symmetric in $\mathcal{G}$, then $\lambda(\mathcal{G} \boxplus \{v_1, v_2\}) = \lambda(\mathcal{G})$. If, in addition, there exists $(x_1, \ldots, x_m) \in Z(\mathcal{G})$ with $x_1 + x_2 > 0$, then the set $Z(\mathcal{G} \boxplus \{v_1, v_2\})$ contains a one-dimensional simplex (i.e. a nontrivial line segment).

Remarks.

- It is easy to see that Proposition 1.9 (b) does not necessarily hold for the ordinary blowup of $\mathcal{G}$.
- The requirement that $\{v_1, v_2\}$ is symmetric in Proposition 1.9 (c) cannot be removed since some calculations show that $\lambda(C_5^3) = 1/25$ while $\lambda(C_5^3 \boxplus \{4, 5\}) = 4/81$, where $C_5^3 = \{123, 234, 345, 451, 512\}$.

Using Propositions 1.6 and 1.9 and some further argument we obtain the following theorem.

Theorem 1.10. *Let $\mathcal{G}$ be a minimal 3-graph. Suppose that $\{v_1, v_2\} \subset V(\mathcal{G})$ is symmetric in $\mathcal{G}$ and $d(v_1, v_2) \geq 1$. Then there exists a finite family $\mathcal{F}$ of 3-graphs with $\xi(\mathcal{F}) = \infty$*

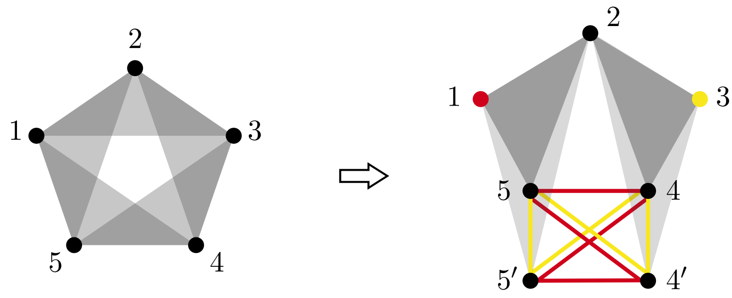

Figure 3: C_5^3 and $C_5^3 \boxplus \{4,5\}$.

such that $\pi(\mathcal{F}) = 6\lambda(\mathcal{G})$, and for every $n \in \mathbb{N}$ there exists a maximum $\mathcal{F}$-free 3-graph on n-vertices which is a blowup of $\mathcal{G}$.

1.4 Feasible region

Theorem 1.3 has some interesting application to the so-called feasible region of hypergraphs.

Given an r-graph $\mathcal{H}$ the *shadow* of $\mathcal{H}$ is defined as

$$\partial\mathcal{H} = \left\{ A \in \binom{V(\mathcal{H})}{r-1} : \text{there is } B \in \mathcal{H} \text{ such that } A \subseteq B \right\}.$$

The *edge density* of $\mathcal{H}$ is defined as $\rho(\mathcal{H}) = |\mathcal{H}|/\binom{v(\mathcal{H})}{r}$, and the *shadow density* of $\mathcal{H}$ is defined as $\rho(\partial\mathcal{H}) = |\partial\mathcal{H}|/\binom{v(\mathcal{H})}{r-1}$.

For a family $\mathcal{F}$ the *feasible region* $\Omega(\mathcal{F})$ of $\mathcal{F}$ is the set of points $(x, y) \in [0,1]^2$ such that there exists a sequence of $\mathcal{F}$-free r-graphs $(\mathcal{H}_k)_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} v(\mathcal{H}_k) = \infty, \quad \lim_{k\to\infty} \rho(\partial\mathcal{H}_k) = x, \quad \text{and} \quad \lim_{k\to\infty} \rho(\mathcal{H}_k) = y.$$

The feasible region unifies and generalizes many classical problems such as the Kruskal–Katona theorem [16, 13] and the Turán problem. It was introduced in [19] to understand the extremal properties of $\mathcal{F}$-free hypergraphs beyond just the determination of $\pi(\mathcal{F})$. The general shape of $\Omega(\mathcal{F})$ was analyzed in [19] as follows: For some constant $c(\mathcal{F}) \in [0,1]$ the projection to the first coordinate,

$$\text{proj}\Omega(\mathcal{F}) = \{x : \text{there is } y \in [0,1] \text{ such that } (x,y) \in \Omega(\mathcal{F})\},$$

is the interval $[0, c(\mathcal{F})]$. Moreover, there is a left-continuous almost everywhere differentiable function $g(\mathcal{F}) : \text{proj}\Omega(\mathcal{F}) \to [0,1]$ such that

$$\Omega(\mathcal{F}) = \left\{(x,y) \in [0, c(\mathcal{F})] \times [0,1] : 0 \le y \le g(\mathcal{F})(x)\right\}.$$

Let us call $g(\mathcal{F})$ the *feasible region function* of $\mathcal{F}$. It was shown in [19] that $g(\mathcal{F})$ is not necessarily continuous, and in [17], it was shown that $g(\mathcal{F})$ can have infinitely many local maxima even for some simple and natural families $\mathcal{F}$.

The stability number of $\mathcal{F}$ can give information about the number of global maxima of $g(\mathcal{F})$ (e.g. see [22]). The family $\mathcal{M}$ of triple systems from [20] for which $\xi(\mathcal{M}) = 2$ has the following additional property: not only are the two near extremal constructions for $\mathcal{M}$ far from each other in edit-distance, but the same is true of their shadows. As a consequence, in addition to $\xi(\mathcal{M}) = 2$, the function $g(\mathcal{M})$ has exactly two global maxima. Similarly, it was proved in [22] that for the t-stable family $\mathcal{M}_t$ the function $g(\mathcal{M}_t)$ has exactly t global maxima for every positive integer t.

It was left as an open question in [22] whether there exists a finite family $\mathcal{F}$ so that the function $g(\mathcal{F})$ has infinitely many global maxima. Here we would like to remind the reader that even though there are infinitely many extremal constructions for Turán's tetrahedron conjecture (if it is true), the shadow of all these constructions is complete. Hence, solving Turán's tetrahedron conjecture will not answer the question asked in [22].

Our next result shows that the same family $\mathcal{F}_t$ as in Theorem 1.3 has the property that $g(\mathcal{F}_t)$ attains its maximum on a nontrivial interval, thus giving a positive answer to the question in [22].

Theorem 1.11. *For every integer $t \geq 3$ we have $\mathrm{proj}\Omega(\mathcal{F}_t) = \left[0, \frac{t+1}{t+2}\right]$, and $g(\mathcal{F}_t, x) \leq (t-2)(t-1)/t^2$ for all $x \in \mathrm{proj}\Omega(\mathcal{F}_t)$. Moreover, if $t \geq 4$, then $g(\mathcal{F}_t, x) = (t-2)(t-1)/t^2$ iff $x \in \left[\frac{t-1}{t}, \frac{t-1}{t} + \frac{1}{t^2}\right]$.*

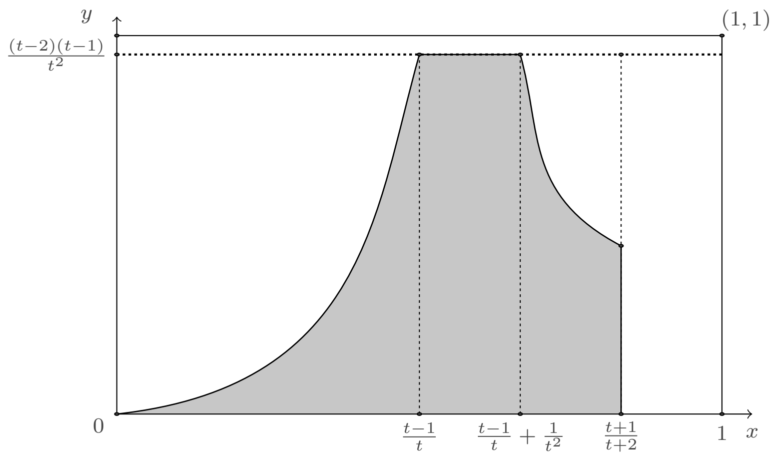

Figure 4: The function $g(\mathcal{F}_t)$ attains its maximum on the interval $\left[\frac{t-1}{t}, \frac{t-1}{t} + \frac{1}{t^2}\right]$.

The remainder of this paper is organized as follows. In Section 2 we introduce some preliminary definitions and results. In Section 3 we prove Proposition 1.4. In Section 4 we prove Propositions 1.6 and 1.9 and Theorem 1.10. In Section 5 we prove Theorem 1.3 (a), (b), and (c). In Section 6 we prove Theorem 1.3 (d). In Section 7 we prove Theorem 1.11. The final section conatins some concluding remarks.

2 Preliminaries

We present some definitions and useful results in this section.

Recall that the shadow of an r-graph $\mathcal{H}$ is

$$\partial\mathcal{H} = \left\{ A \in \binom{V(\mathcal{H})}{r-1} : \text{there is } B \in \mathcal{H} \text{ such that } A \subseteq B \right\}.$$

By setting $\partial_1\mathcal{H} = \partial\mathcal{H}$ for every $i \in [r-1]$ we define $\partial_i\mathcal{H}$ inductively as $\partial_i\mathcal{H} = \partial\partial_{i-1}\mathcal{H}$. In particular, $\partial_{r-2}\mathcal{H}$ is a graph.

Given an r-graph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$ the *neighborhood* of v is defined as

$$N_{\mathcal{H}}(v) = \{u \in V(\mathcal{H}) \setminus \{v\} \colon \exists E \in \mathcal{H} \text{ such that } \{u, v\} \subset E\}.$$

Recall that the *link* of v in $\mathcal{H}$ is $L_{\mathcal{H}}(v) = \{e \in \partial\mathcal{H} \colon e \cup \{v\} \in \mathcal{H}\}$. The *degree* of v is $d_{\mathcal{H}}(v) = |L_{\mathcal{H}}(v)|$. Denote by $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ the minimum degree and maximum degree of $\mathcal{H}$, respectively. Recall that the neighborhood of $\{u, v\} \subset V(\mathcal{H})$ in $\mathcal{H}$ is $N_{\mathcal{H}}(u, v) = \{w \in V(\mathcal{H}) \colon \{u, v, w\} \in \mathcal{H}\}$. The codegree of $\{u, v\}$ is $d_{\mathcal{H}}(u, v) = |N_{\mathcal{H}}(u, v)|$. Denote by $\delta_2(\mathcal{H})$ and $\Delta_2(\mathcal{H})$ the minimum degree and maximum codegree of $\mathcal{H}$, respectively. We will omit the subscript $\mathcal{H}$ from our notations if it is clear from the context.

Let $\mathcal{G}$ be an r-graph with the vertex set $[m]$, and $V_1, \ldots, V_m$ be m disjoint sets (each V_i is allowed to be empty). The blowup $\mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$ is obtained from $\mathcal{G}$ by replacing each vertex i with the set V_i and replacing each edge $i_1 \cdots i_r$ with the complete r-partite r-graph with parts $V_{i_1}, \ldots, V_{i_r}$.

For a hypergraph $\mathcal{G}$ the maximum number of edges in a blowup of $\mathcal{G}$ is related to $\lambda(\mathcal{G})$ (e.g. see Frankl and Füredi [10] or Keevash's survey [14, Section 3]).

Lemma 2.1 ([12, 14])**.** *Suppose that $\mathcal{G}$ is an r-graph and $V_1, \ldots, V_m$ are m disjoint sets with $\sum_{i \in [m]} |V_i| = n$. Then $|\mathcal{G}[V_1, \ldots, V_m]| = p_{\mathcal{G}}(|V_1|/n, \ldots, |V_m|/n)\, n^r$. In particular, $|\mathcal{G}[V_1, \ldots, V_m]| \le \lambda(G) n^r$.*

Given an r-graph F we say $\mathcal{H}$ is *F-hom-free* if there is no homomorphism from F to $\mathcal{H}$. This is equivalent to say that every blowup of $\mathcal{H}$ is F-free. For a family $\mathcal{F}$ of r-graphs we say $\mathcal{H}$ is *$\mathcal{F}$-hom-free* if it is F-hom-free for all $F \in \mathcal{F}$. An easy observation is that if an r-graph F is 2-covered, then $\mathcal{H}$ is F-free iff it is F-hom-free.

Definition 2.2 (Blowup-invariance)**.** *A family $\mathcal{F}$ of r-graphs is blowup-invariant if every $\mathcal{F}$-free r-graph is also $\mathcal{F}$-hom-free.*

Let $\mathcal{H}$ be an r-graph and $\{u, v\} \subset V(\mathcal{H})$ be two non-adjacent vertices (i.e., no edge contains both u and v). We say u and v are *equivalent* if $L_{\mathcal{H}}(u) = L_{\mathcal{H}}(v)$. Otherwise we say they are *non-equivalent*. An equivalence class of $\mathcal{H}$ is a maximal vertex set in which every pair of vertices are equivalent. We say $\mathcal{H}$ is *symmetrized* if it does not contain non-equivalent pairs of vertices. In [23], the authors summarized the well known method of Zykov [30] symmetrization for solving Turán problems into the following statement.

Theorem 2.3 (see e.g. [23])**.** *Suppose that $\mathcal{F}$ is a blowup-invariant family of r-graphs. If $\mathfrak{H}$ denotes the class of all symmetrized $\mathcal{F}$-free r-graphs, then $ex(n, \mathcal{F}) = \mathfrak{h}(n)$ holds for every $n \in \mathbb{N}^+$, where $\mathfrak{h}(n) = \max\{|\mathcal{H}| \colon \mathcal{H} \in \mathfrak{H}$ and $v(\mathcal{H}) = n\}$.*

3 Multilinear polynomials

We prove Proposition 1.4 in this section. First, we need the following simple lemma.

Lemma 3.1. *Suppose that $p(X_1, \ldots, X_m) = p_1 + p_2(X_i + X_j) + p_3 X_i X_j$ is an m-variable multilinear polynomial that is symmetric with respect to X_i and X_j, and p_3 is nonnegative. Then for every $(x_1, \ldots, x_m) \in Z(p)$ we have $(x_1, \ldots, x_{i-1}, (x_i + x_j)/2, x_{i+1}, \ldots, x_{j-1}, (x_i + x_j)/2, x_{j+1}, \ldots, x_m) \in Z(p)$. In particular,*

$$\max\{p(y_1, \ldots, y_m) \colon (y_1, \ldots, y_m) \in \Delta_{m-1} \text{ and } y_i = y_j\} = \lambda(p).$$

Proof. Using the AM-GM inequality we obtain

$$\begin{aligned}
&p(x_1, \ldots, x_{i-1}, \frac{x_i + x_j}{2}, x_{i+1}, \ldots, x_{j-1}, \frac{x_i + x_j}{2}, x_{j+1}, \ldots, x_m) \\
&= p_1 + p_2 \left(\frac{x_i + x_j}{2} + \frac{x_i + x_j}{2} \right) + p_3 \left(\frac{x_i + x_j}{2} \right)^2 \\
&\geq p_1 + p_2(x_i + x_j) + p_3 x_i x_j = \lambda(p),
\end{aligned}$$

where the last equality follows from our assumption that $(x_1, \ldots, x_m) \in Z(p)$. By the definition of $\lambda(p)$, the inequality above is actually an equality, so we have $(x_1, \ldots, x_{i-1}, (x_i + x_j)/2, x_{i+1}, \ldots, x_{j-1}, (x_i + x_j)/2, x_{j+1}, \ldots, x_m) \in Z(p)$. ∎

Now we are ready to prove Proposition 1.4.

Proof of Proposition 1.4. Let $p, p_1, p_2, p_3, p_4, p_5, \hat{p}$ be polynomials that satisfy the assumptions in Proposition 1.4. By symmetry, we may assume that $\{i, j\} = \{1, 2\}$. Let $\widehat{X}_1 = X_1 + X_1'$ and $\widehat{X}_2 = X_2 + X_2'$. It follows from the AM-GM inequality that

$$\begin{aligned}
&\hat{p}(X_1, X_1', X_2, X_2', X_3, \ldots, X_m) \\
&= p_1 + p_2(X_1 + X_1' + X_2 + X_2') + p_4(X_1 + X_1')(X_2 + X_2') + p_5(X_1 + X_2)(X_1' + X_2') \\
&\leq p_1 + p_2(\widehat{X}_1 + \widehat{X}_2) + p_4 \left(\frac{\widehat{X}_1 + \widehat{X}_2}{2} \right)^2 + p_5 \left(\frac{\widehat{X}_1 + \widehat{X}_2}{2} \right)^2 \\
&= p_1 + p_2(\widehat{X}_1 + \widehat{X}_2) + p_3 \left(\frac{\widehat{X}_1 + \widehat{X}_2}{2} \right)^2 \\
&= p((\widehat{X}_1 + \widehat{X}_2)/2, (\widehat{X}_1 + \widehat{X}_2)/2, X_3, \ldots, X_m). \qquad (1)
\end{aligned}$$

This implies that $\lambda(\hat{p}) \leq \lambda(p)$.

Let $\alpha \in [0, 1]$ and

$$\vec{y} = \alpha \cdot (\frac{x_1 + x_2}{2}, 0, 0, \frac{x_1 + x_2}{2}, x_3, \ldots, x_m) + (1 - \alpha) \cdot (0, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, 0, x_3, \ldots, x_m).$$

Then it follows from Equation (1) that

$$\hat{p}(\vec{y}) = p((x_1 + x_2)/2, (x_1 + x_2)/2, x_3, \ldots, x_m).$$

By Lemma 3.1, we have

$$p((x_1 + x_2)/2, (x_1 + x_2)/2, x_3, \ldots, x_m) = p(x_1, x_2, x_3, \ldots, x_m) = \lambda(p).$$

Therefore, $\vec{y} \in Z(\hat{p})$. ∎

4 Nonmininal hypergraphs

In this section, we prove results that were stated in Section 1.3.

4.1 Proof for Proposition 1.6

In this subsection we prove Proposition 1.6.

For every r-graph $\mathcal{G}$ let $\mathcal{F}_{\infty}(\mathcal{G})$ be the (infinite) family of all r-graphs that are not $\mathcal{G}$-colorable, i.e.

$$\mathcal{F}_{\infty}(\mathcal{G}) = \{r\text{-graph } F\colon \text{ and } F \text{ is not } \mathcal{G}\text{-colorable}\}.$$

For every positive integer M define the family $\mathcal{F}_M(\mathcal{G})$ of r-graphs as

$$\mathcal{F}_M(\mathcal{G}) = \{F \in \mathcal{F}_{\infty}(\mathcal{G})\colon v(F) \le M\}.$$

It is clear that $\mathcal{F}_M(\mathcal{G})$ is nonempty if $M \ge v(\mathcal{G}) + 1$ since it contains all r-graphs F with at most M vertices and $K_{v(\mathcal{G})+1} \subset \partial_{r-2}F$. It is also clear from the definition that $\mathcal{F}_M(\mathcal{G}) \subset \mathcal{F}_{M'}(\mathcal{G})$ for all $M' \ge M$.

It follows easily from the definition that every $\mathcal{F}_{\infty}(\mathcal{G})$-free r-graph is $\mathcal{G}$-colorable. In particular, every maximum $\mathcal{F}_{\infty}(\mathcal{G})$-free r-graph on n vertices is $\mathcal{G}$-colorable. One of the main result in [25] implies the following compactness result. For every minimum r-graph $\mathcal{G}$ there exists a constant $M_0 \in \mathbb{N}$ such that every maximum $\mathcal{F}_{\infty}(\mathcal{G})$-free r-graph on n vertices is $\mathcal{G}$-colorable. Proposition 1.6 actually shows that a similar compactness result holds for all r-graphs $\mathcal{G}$.

The following simple lemma is a special case of Lemma 8 in [25].

Lemma 4.1 ([25])**.** *For every r-graph $\mathcal{G}$ and every positive integer M the family $\mathcal{F}_M(\mathcal{G})$ is blowup-invariant.*

The following lemma extends Observation 2.3 (a) in [18].

Lemma 4.2. *Let $\mathcal{G}$ be an r-graph. Let m denote the order of a maximum complete subgraph in $\partial_{r-2}\mathcal{G}$ and let $M = m + 1 + (r-2)\binom{m+1}{2}$. Suppose that an r-graph $\mathcal{H}$ is $\mathcal{F}_M(\mathcal{G})$-free. Then $\partial_{r-2}\mathcal{H}$ is K_{m+1}-free.*

Proof. Suppose to the contrary that there exists a set $C \subset V(\mathcal{H})$ of size $m+1$ such that the induced subgraph of $\partial\mathcal{H}$ on C is complete. For every pair $\{u, v\} \subset C$ let $E_{u,v} \in \mathcal{H}$ be an edge that contains $\{u, v\}$. Let $F = \{E_{u,v}\colon \{u, v\} \subset C\}$. It is clear that $v(F) \le m + 1 + (r-2)\binom{m+1}{2} = M$. So it follows from our assumption that F is $\mathcal{G}$-colorable. In other words, there exists a homomorphism $\phi\colon V(F) \to V(\mathcal{G})$ from F to $\mathcal{G}$. Since every pair $\{u, v\} \subset C$ is contained in an edge of F, we have $\phi(u) \ne \phi(v)$. Therefore, $|\phi(C)| = |C| = m+1$. However, $u \sim v$ in $\partial_{r-2}F$ implies that $\phi(u) \sim \phi(v)$ in $\partial_{r-2}\mathcal{G}$, which means that the induced subgraph of $\partial_{r-2}\mathcal{G}$ on $\phi(C)$ is complete, a contradiction. ∎

We will prove the following statement which implies Proposition 1.6.

Proposition 4.3. *Suppose that $r \ge 3$, $\mathcal{G}$ is an r-graph, and $M = v(\mathcal{G}) + 1 + (r-2)\binom{v(\mathcal{G})+1}{2}$. Then $\mathrm{ex}(n, \mathcal{F}_M(\mathcal{G})) = \max\{|\mathcal{H}|\colon v(\mathcal{H}) = n$ and $\mathcal{H}$ is $\mathcal{G}$-colorable$\}$ for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{G}$ be an r-graph and let $m = v(\mathcal{G})$. Let $\mathfrak{G}$ be the collection of all $\mathcal{G}$-colorable r-graphs. Define $\mathfrak{g}(n) = \max\{|\mathcal{H}|\colon \mathcal{H} \in \mathfrak{H}$ and $v(\mathcal{H}) = n\}$. Let $M = m + 1 + (r-2)\binom{m+1}{2}$ and $\mathcal{F} = \mathcal{F}_M(\mathcal{G})$. It follows from Lemma 4.1 that $\mathcal{F}$ is blowup-invariant. Therefore, by Theorem 2.3, it suffices to prove the following claim.

Claim 4.4. *Every symmetrized $\mathcal{F}$-free r-graph is contained in $\mathfrak{G}$.*

Proof. Let $\mathcal{H}$ be a symmetrized $\mathcal{F}$-free r-graph. Let $C \subset V(\mathcal{H})$ be a set that contains exactly one vertex from each equivalence class of $\mathcal{H}$ and let $\mathcal{T}$ denote the induced subgraph of $\mathcal{H}$ on C. It is clear from the definition of symmetrized hypergraphs that $\mathcal{H}$ is a blowup of $\mathcal{T}$. Since $\partial_{r-2}\mathcal{T} \cong K_{|C|}$, it follows from Lemma 4.2 that $|C| \le m$. Since $\mathcal{T} \notin \mathcal{F}$, we know that $\mathcal{T}$ is $\mathcal{G}$-colorable. This implies that $\mathcal{H}$ is also $\mathcal{G}$-colorable. Therefore, $\mathcal{H} \in \mathfrak{G}$. ∎

This completes the proof of Proposition 4.3. ∎

4.2 Proofs for Proposition 1.9 and Theorem 1.10

In this subsection we prove Proposition 1.9 and Theorem 1.10.

Let us assume that the vertex set of $\mathcal{G}$ is $\{v_1, v_2, \ldots, v_m\}$. Let v_1', v_2' denote the clones of v_1, v_2 in $\mathcal{G} \boxplus \{v_1, v_2\}$, respectively. Let $U_1 = \{v_1, v_2', v_3, \ldots, v_m\}$ and $U_2 = \{v_1', v_2, v_3, \ldots, v_m\}$. It is easy to see from the definition that the induced subgraph of $\mathcal{G} \boxplus \{v_1, v_2\}$ on U_1 and U_2 are both isomorphic to $\mathcal{G}$, this proves (a). Statement (b) follows easily from the definition of crossed blowup. So we may focus on (c).

We will prove the following statement which implies Proposition 1.9 (c).

Proposition 4.5. *Suppose that $\mathcal{G}$ is an m-vertex 3-graph and $\{v_1, v_2\} \subset \mathcal{G}$ is a pair of vertices with $d(v_1, v_2) \ge 2$. Suppose that $\{v_1, v_2\}$ is symmetric in $\mathcal{G}$. Then $\lambda(\mathcal{G} \boxplus \{v_1, v_2\}) = \lambda(\mathcal{G})$. Moreover, for every $(z_1, \ldots, z_m) \in Z(\mathcal{G})$ we have*

$$L\left((\bar{z}, 0, 0, \bar{z}, z_3, \ldots, z_m), (0, \bar{z}, \bar{z}, 0, z_3, \ldots, z_m)\right) \subset Z(\mathcal{G} \boxplus \{v_1, v_2\}),$$

where $\bar{z} = (z_1 + z_2)/2$.

Proof. Let $k = d(v_1, v_2)$ and assume that $N(v_1, v_2) = \{v_{i_1}, \ldots, v_{i_k}\}$. Let $W = \{v_3, \ldots, v_m\}$. Define polynomials $p_1, \ldots, p_5$ as

$$p_1 = \sum_{v_i v_j v_k \in \mathcal{G}[W]} X_i X_j X_k,$$

$$p_2 = \sum_{v_i v_j \in L_{\mathcal{G}}(v_1)[W]} X_i X_j = \sum_{v_i v_j \in L_{\mathcal{G}}(v_2)[W]} X_i X_j,$$

$$p_3 = X_{i_1} + \cdots + X_{i_k}, \quad p_4 = X_{i_1} + \cdots + X_{i_{k-1}}, \quad \text{and} \quad p_5 = X_k.$$

It is easy to see that $p_{\mathcal{G}}(X_1, \ldots, X_m) = p_1 + p_2(X_1 + X_2) + p_3 X_1 X_2$. On the other hand, it follows from the definition of crossed blowup that

$$\begin{aligned} & p_{\mathcal{G} \boxplus \{v_1, v_2\}}(X_1, X_1', X_2, X_2', X_3 \ldots, X_m) \\ & = p_1 + p_2(X_1 + X_2 + X_1' + X_2') + p_4(X_1 + X_1')(X_2 + X_2') + p_5(X_1 + X_2)(X_1' + X_2'). \end{aligned}$$

So by Proposition 1.4 we have

$$L\left((\bar{z}, 0, 0, \bar{z}, z_3, \ldots, z_m), (0, \bar{z}, \bar{z}, 0, z_3, \ldots, z_m)\right) \subset Z(\mathcal{G} \boxplus \{v_1, v_2\}).$$

∎

Now we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. If $d_{\mathcal{G}}(v_1, v_2) \geq 2$, then we let $\mathcal{G}' = \mathcal{G} \boxplus \{v_1, v_2\}$, and it follows from Proposition 1.9 (c) that $\lambda(\mathcal{G}') = \lambda(\mathcal{G})$.

If $d_{\mathcal{G}}(v_1, v_2) = 1$, then we let $\{w\} = N_{\mathcal{G}}(v_1, v_2)$ and let $\mathcal{G}'$ be obtained as follows. First, we take two vertex-disjoint copies of $\mathcal{G}$ and identify them on the set $V(\mathcal{G})\setminus\{w\}$ (i.e. we blowup w into two vertices). Denote by $\mathcal{G}^2$ the resulting 3-graph. For example, if $\mathcal{G} = \{125, 345\}$ and $\{v_1, v_2\} = \{1, 2\}$, then $\mathcal{G}^2 = \{125, 345, 126, 346\}$. Let $\mathcal{G}' = \mathcal{G}^2 \boxplus \{v_1, v_2\}$. Since $\mathcal{G}^2$ is $\mathcal{G}$-colorable and $\mathcal{G} \subset \mathcal{G}^2$, we have $\lambda(\mathcal{G}^2) = \lambda(\mathcal{G})$. Notice that $\{v_1, v_2\}$ is still symmetric in $\mathcal{G}^2$, so it follows from Proposition 1.9 (c) that $\lambda(\mathcal{G}') = \lambda(\mathcal{G}^2) = \lambda(\mathcal{G})$.

Let $m = v(\mathcal{G}')$ and $M = m + 1 + (r-2)\binom{m+1}{2}$. Let $\mathcal{F} = \mathcal{F}_M(\mathcal{G}')$. It follows from Proposition 4.3 that $\mathrm{ex}(n, \mathcal{F}) = \max\{|\mathcal{H}| \colon v(\mathcal{H}) = n \text{ and } \mathcal{H} \text{ is } \mathcal{G}'\text{-colorable}\}$. Then by Lemma 2.1, we have $\pi(\mathcal{F}) \leq 6\lambda(\mathcal{G}') = 6\lambda(\mathcal{G})$. The converse of this inequality follows from the fact that every blowup of $\mathcal{G}'$ is $\mathcal{F}$-free. Therefore, we have $\pi(\mathcal{F}) = 6\lambda(\mathcal{G})$.

Now we prove that $\xi(\mathcal{F}) = \infty$. Suppose to the contrary that $\xi(\mathcal{F}) = t$ for some $t \in \mathbb{N}^+$. For every $n \in \mathbb{N}$ let $\mathcal{S}_1(n), \ldots, \mathcal{S}_t(n)$ be the n-vertex 3-graphs that witness the t-stability of $\mathcal{F}$. For every $i \in [t]$ and $n \in \mathbb{N}$ let $\rho_{i,n} = \rho(\mathcal{G}', \mathcal{S}_i(n)) = N(\mathcal{G}', \mathcal{S}_i(n))/\binom{n}{m}$, where $N(\mathcal{G}', \mathcal{S}_i(n))$ is the number of copies of $\mathcal{G}'$ in $\mathcal{S}_i(n)$.

We will get a contradiction by first showing that for all sufficiently large n there exists a blowup $\widehat{\mathcal{G}}$ of $\mathcal{G}$ on n vertices whose edge density is close to $\pi(\mathcal{F})$ but whose shadow density is far from $\rho_{i,n}$ for every $i \in [t]$. Then we argue that since every shadow edge is contained $\Omega(n)$ edges in $\widehat{\mathcal{G}}$, the edit-distance of $\widehat{\mathcal{G}}$ and $\mathcal{S}_i(n)$ is $\Omega(n^3)$ for all $i \in [t]$. This contradicts the definition of t-stable.

Notice that $v(\mathcal{G}) = m-2$ if $d_{\mathcal{G}}(v_1, v_2) \geq 2$ and $v(\mathcal{G}^2) = m-2$ if $d_{\mathcal{G}}(v_1, v_2) = 1$. Let $(z_1, z_2, z_3, \ldots, z_{m-2}) \in Z(\mathcal{G})$ (or $Z(\mathcal{G}^2)$ if $d_{\mathcal{G}}(v_1, v_2) = 1$) be a vector such that $\zeta := \min\{z_1, z_2, z_3, \ldots, z_{m-2}\}$ is maximized. By Lemma 3.1, we may assume that $z_1 = z_2$ since otherwise we can replace z_1 and z_2 by $(z_1 + z_2)/2$.

Claim 4.6. *We have $\zeta > 0$.*

Proof. If $d(v_1, v_2) \geq 2$, then the minimality of $\mathcal{G}$ implies that every vector in $Z(\mathcal{G})$ has only positive coordinate. Therefore, $\zeta > 0$.

Suppose that $d(v_1, v_2) = 1$. Without loss of generality let us assume that $N_{\mathcal{G}}(v_1, v_2) = \{v_3\}$ and $N_{\mathcal{G}^2}(v_1, v_2) = \{v_3, v_4\}$. Since v_4 is a clone of v_3 in $\mathcal{G}^2$, $(z_1, z_2, z_3, z_4, \ldots, z_{m-2}) \in Z(\mathcal{G}^2)$ implies that $(z_1, z_2, (z_3 + z_4)/2, (z_3 + z_4)/2, \ldots, z_{m-2}) \in Z(\mathcal{G}^2)$. For the same reason, $(z_1, z_2, z_3, z_4, \ldots, z_{m-2}) \in Z(\mathcal{G}^2)$ implies that $(z_1, z_2, z_3 + z_4, z_5, \ldots, z_{m-2}) \in Z(\mathcal{G})$. The minimality of $\mathcal{G}$ implies that each coordinate of $(z_1, z_2, z_3 + z_4, z_5, \ldots, z_{m-2})$ is positive. Therefore, every coordinate in $(z_1, z_2, (z_3 + z_4)/2, (z_3 + z_4)/2, \ldots, z_{m-2})$ is positive, and hence, $\zeta > 0$. ∎

Let $\vec{x}_\alpha = \alpha \cdot (z_1, 0, 0, z_2, z_3, \ldots, z_{m-2}) + (1-\alpha) \cdot (0, z_1, z_2, 0, z_3, \ldots, z_{m-2})$ for $\alpha \in [0, 1]$. Let $\widehat{\mathcal{G}}_\alpha(n) = \mathcal{G}'[V_1, V_1', V_2, V_2', V_3, \cdots, V_{m-2}]$ be a blowup of $\mathcal{G}'$, where $|V_1| = \lfloor \alpha z_1 n \rfloor$, $|V_1'| = \lfloor (1-\alpha) z_1 n \rfloor$, $|V_2| = \lfloor \alpha z_2 n \rfloor$, $|V_2'| = \lfloor (1-\alpha) z_2 n \rfloor$, and $|V_i| = \lfloor z_i n \rfloor$ for $i \in [3, m-2]$. Let $W = \{v_3, \ldots, v_{m-2}\}$. Then

$$|\partial\widehat{\mathcal{G}}_\alpha(n)| = \sum_{v_i v_j \in \partial\mathcal{G} \cap \binom{W}{2}} |V_i||V_j| + \sum_{v_i \in N_{\mathcal{G}}(v_1)} |V_i|\left(|V_1| + |V_1|'\right) + \sum_{v_i \in N_{\mathcal{G}}(v_2)} |V_i|\left(|V_2| + |V_2|'\right)$$
$$+ \left(|V_1| + |V_1'|\right)\left(|V_2| + |V_2'|\right) + |V_1||V_1'| + |V_2||V_2'|.$$

Taking the limit we obtain

$$\lim_{n\to\infty} \rho\left(\partial\widehat{\mathcal{G}}_\alpha(n)\right) = 2\alpha(1-\alpha)(z_1+z_2)+2C,$$

where

$$C = \sum_{v_iv_j\in\partial\mathcal{G}\cap\binom{W}{2}} z_iz_j + \sum_{v_i\in N_{\mathcal{G}}(v_1)} z_iz_1 + \sum_{v_i\in N_{\mathcal{G}}(v_2)} z_iz_2 + z_1z_2$$

is independent of α.

For $i\in[t+1]$ let $\alpha_i = \frac{i}{2(t+1)}$. Since $z_1+z_2>0$ and $\alpha(1-\alpha)$ is increasing in α for $\alpha\in[0,1/2]$, there exists $\epsilon>0$ and N_0 such that

$$\left|\partial\widehat{\mathcal{G}}_{\alpha_j}(n)\right| - \left|\partial\widehat{\mathcal{G}}_{\alpha_i}(n)\right| > \frac{6\epsilon n^2}{\zeta}$$

for all $1\le i<j\le t+1$. Fix such an $\epsilon>0$. By the definition of t-stable there exists $\delta=\delta(\epsilon)$ and $N_1=N_1(\epsilon)$ such that every n-vertex $\mathcal{F}$-free 3-graph $\mathcal{H}$ with $|\mathcal{H}|\ge(1-\delta)\mathrm{ex}(n,\mathcal{F})$, satisfies $|\mathcal{H}\triangle\mathcal{S}_i(n)|\le\epsilon n^3$ for some $i\in[t]$. Since $\lim_{n\to\infty}\rho\left(\widehat{\mathcal{G}}_\alpha(n)\right)=\pi(\mathcal{F})$, there exists N_2 such that $\left|\widehat{\mathcal{G}}_\alpha(n)\right|\ge(1-\delta)\mathrm{ex}(n,\mathcal{F})$ for all $\alpha\in[0,1]$ and $n\ge N_2$. Therefore, for every $j\in[t+1]$ there exists $i\in[t]$ such that $|\widehat{\mathcal{G}}_{\alpha_j}(n)\triangle\mathcal{S}_i(n)|\le\epsilon n^3$ for all $n\ge\max\{N_0,N_1,N_2\}$. By the Pigeonhole principle, there exists a pair $\{j,k\}$ such that $|\widehat{\mathcal{G}}_{\alpha_j}(n)\triangle\mathcal{S}_i(n)|\le\epsilon n^3$ and $|\widehat{\mathcal{G}}_{\alpha_k}(n)\triangle\mathcal{S}_i(n)|\le\epsilon n^3$. By the triangle inequality, we have $|\widehat{\mathcal{G}}_{\alpha_j}(n)\triangle\widehat{\mathcal{G}}_{\alpha_k}(n)|\le 2\epsilon n^3$.

Without loss of generality we may assume that $\alpha_j<\alpha_k$. Observe that every edge $e\in\partial\widehat{\mathcal{G}}_{\alpha_j}(n)\triangle\partial\widehat{\mathcal{G}}_{\alpha_k}(n)$ is contained in the set $V_1\cup V_1'\cup V_2\cup V_2'$. Let $\{\ell_1,\ell_2\}\subset[3,m-2]$ be the pair such that $N_{\mathcal{G}^2}=\{\ell_1,\ell_2\}$. Then we have $N_{\widehat{\mathcal{G}}_\alpha(n)}(e)\ge\min\{|V_{\ell_1}|,|V_{\ell_2}|\}\ge\lfloor\zeta n\rfloor>\zeta n/2$. Therefore, in order to transform $\widehat{\mathcal{G}}_{\alpha_k}(n)$ into $\widehat{\mathcal{G}}_{\alpha_k}(n)$ we need to remove at least

$$\left(|\partial\widehat{\mathcal{G}}_{\alpha_k}(n)|-|\partial\widehat{\mathcal{G}}_{\alpha_j}(n)|\right)\frac{\zeta n}{2} > \frac{6\epsilon n^2}{\zeta}\frac{\zeta n}{2} > 2\epsilon n^3$$

edges, a contradiction. Therefore, $\xi(\mathcal{F})=\infty$. ∎

5 Proof of Theorem 1.3 (a), (b), and (c)

In this section we prove Theorem 1.3 (a), (b), and (c). First let us define Γ_t and $\mathcal{F}_t$ that were mentioned in Theorem 1.3. For convenience, we will use $t+2$ instead t in the rest of this paper.

Definition 5.1. *Let $t\ge1$ be an integer.*

(a) Let

$$\Gamma_{t+2} = \begin{cases}\{134,234\}\boxplus\{3,4\} & \text{if } t=1,\\ K_{t+2}^3\boxplus\{t+1,t+2\} & \text{if } t\ge2.\end{cases}$$

(b) Let $\mathfrak{d}_{t+2}$ be the collection of all Γ_{t+2}-colorable 3-graphs.

(c) Let $\gamma_{t+2}(n)=\max\{|\mathcal{H}|\colon v(\mathcal{H})=n$ and $\mathcal{H}\in\mathfrak{d}_{t+2}\}$.

(d) Let $\mathcal{F}_{t+2} = \{F \colon v(F) \le 4(t+4)^2$ *and* $F \notin \mathfrak{d}_{t+2}\}$.

Remark. In the rest of the paper, we always assume that for $t \ge 2$ the vertex set of Γ_{t+2} is $[t+4]$, and $t+3, t+4$ are clones of $t+1, t+2$, respectively.

It follows from Proposition 1.9 and Lemma 2.1 that $\gamma_{t+2}(n) \sim \lambda(K^3_{t+2})n^3 = \frac{t(t+1)}{6(t+2)^2}n^3$ (for $t = 1$ we also used $\lambda(\{134, 234\}) = \lambda(K^3_3)$).

Theorem 1.3 (a) follows easily from Proposition 1.6. Theorem 1.3 (c) follows easily from Theorem 1.10. So we just need to prove Theorem 1.3 (b).

Proof of Theorem 1.3 (b). Let $t \ge 2$ and n be an integer satisfying $(t+2) \mid n$. It is easy to see that $\mathrm{ex}(n, \mathcal{F}_{t+2}) = \frac{t(t+1)}{6(t+2)^2}n^3$. Let $\mathcal{H} = \Gamma_{t+2}[V_1, \cdots, V_{t+4}]$ be a blowup of Γ_{t+2} with $|V_1| = \cdots = |V_t| = \frac{n}{t+2}$, $|V_{t+1}| = |V_{t+4}| = \frac{\alpha n}{t+2}$, and $|V_{t+2}| = |V_{t+3}| = \frac{(1-\alpha)n}{t+2}$, where $\alpha \in [0, 1/2]$ satisfies that $\frac{\alpha n}{t+2} \in \mathbb{N}$. It is easy to see that $|\mathcal{H}| = \frac{t(t+1)}{6(t+2)^2}n^3$. So it suffices to show that the set $S = \left\{\alpha \in [0, 1/2] \colon \frac{\alpha n}{t+2} \in \mathbb{N}\right\}$ has size at least $\frac{n}{2(t+2)}$. Indeed, suppose that $\alpha = \frac{p}{q} \in [0, 1/2]$, where $p \le q/2$ are integers that are coprime. Notice that $\frac{p}{q}\frac{n}{t+2} \in \mathbb{N}$ if $q \mid \frac{n}{t+2}$. Since p is coprime with q iff $q - p$ is coprime with q, we have $|\{p \colon (p, q) = 1,\ p \le q/2\}| \ge \varphi(q)/2$, where $\varphi(q)$ is the Euler totient function that denotes the number of positive integers $p \le q$ that are coprime with q. Therefore, $|S| \ge \frac{1}{2}\sum_{q \mid \frac{n}{t+2}} \varphi(q)$. It follows from a well known result of Gauss that $\sum_{q \mid \frac{n}{t+2}} \varphi(q) = \frac{n}{t+2}$. Therefore, $|S| \ge \frac{n}{2(t+2)}$. This implies that there are at least $\frac{n}{2(t+2)}$ nonisomorphic extremal $\mathcal{F}_t$-free 3-graphs on n vertices. The proof for the case $t = 1$ is similar, so we omit it here. ∎

6 Stability

6.1 Preparations

In this section we present some useful theorems and lemmas that will be used later in the proof of Theorem 1.3 (d).

Definition 6.1 (Vertex-extendibility)**.** *Let $\mathcal{F}$ be a family of r-graphs and let $\mathfrak{H}$ be a class of $\mathcal{F}$-free r-graphs. We say that $\mathcal{F}$ is* vertex-extendable *with respect to $\mathfrak{H}$ if there exist $\zeta > 0$ and $N_0 \in \mathbb{N}$ such that for every $\mathcal{F}$-free r-graph $\mathcal{H}$ on $n \ge N_0$ vertices satisfying $\delta(\mathcal{H}) \ge (\pi(\mathcal{F})/(r-1)! - \zeta)n^{r-1}$ the following holds: if $\mathcal{H} - v$ is a subgraph of a member of $\mathfrak{H}$ for some vertex $v \in V(\mathcal{H})$, then $\mathcal{H}$ is a subgraph of a member of $\mathfrak{H}$ as well.*

In [23], the authors developed a machinery that reduces the proof of stability of certain families $\mathcal{F}$ to the simpler question of checking that an $\mathcal{F}$-free hypergraph $\mathcal{H}$ with large minimum degree is vertex-extendable.

Theorem 6.2 ([23])**.** *Suppose that $\mathcal{F}$ is a blowup-invariant nondegenerate family of r-graphs and that $\mathfrak{H}$ is a hereditary class of $\mathcal{F}$-free r-graphs. If $\mathfrak{H}$ contains all symmetrized $\mathcal{F}$-free r-graphs and $\mathcal{F}$ is vertex-extendable with respect to $\mathfrak{H}$, then the following statement holds. There exist $\epsilon > 0$ and N_0 such that every $\mathcal{F}$-free r-graph on $n \ge N_0$ vertices with minimum degree at least $(\pi(\mathcal{F})/(r-1)! - \epsilon)\, n^{r-1}$ is contained in $\mathfrak{H}$.*

The following lemma will be used extensively in the proof of Theorem 1.3 (d).

Lemma 6.3 (see e.g. [22])**.** *Fix a real $\eta \in (0,1)$ and integers $m, n \geq 1$. Let $\mathcal{G}$ be a 3-graph with vertex set $[m]$ and let $\mathcal{H}$ be a further 3-graph with $v(\mathcal{H}) = n$. Consider a vertex partition $V(\mathcal{H}) = \bigcup_{i\in[m]} V_i$ and the associated blow-up $\widehat{\mathcal{G}} = \mathcal{G}[V_1, \ldots, V_m]$ of $\mathcal{G}$. If two sets $T \subseteq [m]$ and $S \subseteq \bigcup_{j\notin T} V_j$ have the properties*

(a) $|V_j| \geq (|S|+1)|T|\eta^{1/3}n$ *for all* $j \in T$,

(b) $|\mathcal{H}[V_{j_1}, V_{j_2}, V_{j_3}]| \geq |\widehat{\mathcal{G}}[V_{j_1}, V_{j_2}, V_{j_3}]| - \eta n^3$ *for all* $\{j_1, j_2, j_3\} \in \binom{T}{3}$, *and*

(c) $|L_{\mathcal{H}}(v)[V_{j_1}, V_{j_2}]| \geq |L_{\widehat{\mathcal{G}}}(v)[V_{j_1}, V_{j_2}]| - \eta n^2$ *for all* $v \in S$ *and* $\{j_1, j_2\} \in \binom{T}{2}$,

then there exists a selection of vertices $u_j \in V_j$ for all $j \in [T]$ such that $U = \{u_j : j \in T\}$ satisfies $\widehat{\mathcal{G}}[U] \subseteq \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] \subseteq L_{\mathcal{H}}(v)[U]$ for all $v \in S$. In particular, if $\mathcal{H} \subseteq \widehat{\mathcal{G}}$, then $\widehat{\mathcal{G}}[U] = \mathcal{H}[U]$ and $L_{\widehat{\mathcal{G}}}(v)[U] = L_{\mathcal{H}}(v)[U]$ for all $v \in S$.

Next we prove some basic properties about Γ_{t+2}.

Observation 6.4. *The following statements hold.*

(a) *If a set $U \subset [t+4]$ satisfies that $\Gamma_{t+2}[U] \cong K^3_{t+2}$, then either $U = [t] \cup \{t+1, t+4\}$ or $U = [t] \cup \{t+2, t+3\}$.*

(b) *We have $L_{\Gamma_{t+2}}(1) \cong \cdots \cong L_{\Gamma_{t+2}}(t)$, and $L_{\Gamma_{t+2}}(t+1) \cong \cdots \cong L_{\Gamma_{t+2}}(t+4)$.*

(c) *We have*

$$d_{\Gamma_{t+2}}(i,j) = \begin{cases} t+2 & \text{if } \{i,j\} \in \binom{[t]}{2}, \\ t+1 & \text{if } i \in [t] \text{ and } j \in [t+1, t+4], \\ t & \text{if } \{i,j\} \in \{\{t+1, t+4\}, \{t+2, t+3\}\}, \\ t-1 & \text{if } \{i,j\} \in \{\{t+1, t+3\}, \{t+2, t+4\}\}, \\ 1 & \text{if } \{i,j\} \in \{\{t+1, t+2\}, \{t+3, t+4\}\}. \end{cases}$$

Using Observation 6.4 (a) we prove the following lemma.

Lemma 6.5. *Suppose that $\mathcal{G}$ is a blowup of Γ_{t+2} and $U, U' \subset V(\mathcal{G})$ are two vertex sets of size $t+2$ with $|U \cap U'| = t+1$. If $\mathcal{G}[U] \cong \mathcal{G}[U'] \cong K^3_{t+2}$, then the set $U \triangle U'$ is not contained in any edge of $\mathcal{G}$.*

Proof. Suppose that $U = \{u_1, \ldots, u_{t+1}, u\}$ and $U' = \{u_1, \ldots, u_{t+1}, u'\}$. Let $\phi\colon U \cup U' \to V(\Gamma_{t+2})$ be a map such that $\phi|_U$ and $\phi|_{U'}$ are homomorphisms from $\mathcal{G}[U]$ and $\mathcal{G}[U']$ to Γ_{t+2}, respectively. Since $|U \cap U'| = t+1$, it follows from Observation 6.4 (a) that either $\phi(U) = \phi(U') = [t] \cup \{t+1, t+4\}$ or $\phi(U) = \phi(U') = [t] \cup \{t+2, t+3\}$. In either case, we have $\phi(u) = \phi(u')$, which implies that u and u' lie in the same part of $\mathcal{G}$. So $\{u, u'\}$ is not contained in any edge of $\mathcal{G}$. ∎

The following lemma follows easily from Observation 6.4 (c).

Lemma 6.6. *Suppose that $\phi\colon V(\Gamma_{t+2}) \to V(\Gamma_{t+2})$ is a homomorphism from Γ_{t+2} to Γ_{t+2}. Then ϕ is bijective, $\phi([t]) = [t]$, and*

$$\{\phi(\{t+1, t+4\}), \phi(\{t+2, t+3\})\} = \{\{t+1, t+4\}, \{t+2, t+3\}\}.$$

If $t \geq 3$, then additionally, $\phi([t-1]) = [t-1]$.

Using Observation 6.4 (c) we further prove the following lemma.

Lemma 6.7. *Suppose that $\phi\colon V(\Gamma_{t+2}) \setminus \{t+3\} \to V(\Gamma_{t+2})$ is a homomorphism from $\Gamma_{t+2} \setminus \{t+3\}$ to Γ_{t+2}. Then ϕ is injective, $\phi([t]) = [t]$, $\phi(\{t+1, t+2, t+4\}) \subset [t+1, t+4]$, and $\{\phi(t+1), \phi(t+4)\} \in \{\{t+1, t+4\}, \{t+2, t+3\}\}$. If $t \geq 3$, then additionally, $\phi([t-1]) = [t-1]$.*

Proof. Let $F = \Gamma_{t+2} \setminus \{t+3\}$. Since ϕ is an embedding, we have $d_F(i,j) \leq d_{\Gamma_{t+2}}(\phi(i), \phi(j))$ for all pairs of vertices in F. Observe that

$$d_F(i,j) = \begin{cases} t+1 & \text{if } \{i,j\} \in \binom{[t]}{2} \text{ or } (i,j) \in [t-1] \times \{t+4\} \text{ or } (i,j) = (t, t+1), \\ t & \text{if } (i,j) \in [t-1] \times \{t+1, t+2\} \text{ or } (i,j) \in \{t\} \times \{t+2, t+4\} \\ & \text{or } \{i,j\} = \{t+1, t+4\}, \\ t-1 & \text{if } \{i,j\} = \{t+2, t+4\}, \\ 1 & \text{if } \{i,j\} = \{t+1, t+2\}. \end{cases}$$

Since the induced subgraph of F on $[t] \cup \{t+1, t+4\}$ is isomorphic to K^3_{t+2}, it follows from from Observation 6.4 (a) that $\phi([t] \cup \{t+1, t+4\}) \in \{[t] \cup \{t+1, t+4\}, [t] \cup \{t+2, t+3\}\}$. By symmetry we may assume that $\phi([t] \cup \{t+1, t+4\}) = [t] \cup \{t+1, t+4\}$. Then we have $\phi(t+2) \in [t+1, t+4] \setminus \phi([t] \cup \{t+1, t+4\}) = \{t+2, t+3\}$ By symmetry we may assume that $\phi(t+2) = t+2$. Since $d_{\Gamma_{t+2}}(\phi(t+2), t+1) = 1$ and $d_{\Gamma_{t+2}}(\phi(t+2), t+4) = t-1$, we have $\phi(t+1) = t+1$ and $\phi(t+4) = t+4$ (otherwise we would have $\phi(i) \in \{t+1, t+4\}$ for some $i \in [t]$, but this implies that $t = d_F(i, t+2) \leq d_{\Gamma_{t+2}}(\phi(i), \phi(t+2)) \leq t-1$, a contradiction). Consequently, $\phi([t]) = [t]$.

Suppose that $t \geq 3$. Let $S = \{t+1, t+2, t+4\}$ Observe that $L_F(i)[S] = L_F(j)[S] \neq L_F(t)(S)$ and $L_{\Gamma_{t+2}}(i)[\phi(S)] = L_{\Gamma_{t+2}}(j)[\phi(S)] \neq L_{\Gamma_{t+2}}(t)(\phi(S))$ and for all $i, j \in [t-1]$. So, we should have $\phi([t-1]) = [t-1]$ and $\phi(t) = \phi(t)$. ∎

The following lemma characterizes the vectors $(x_1, \ldots, x_{t+4}) \in \Delta_{t+3}$ for which the value $p_{\Gamma_{t+2}}(x_1, \ldots, x_{t+4})$ is close to the maximum.

Lemma 6.8. *For every integer $t \geq 2$ and every real number $\epsilon > 0$ there exists a constant $\delta > 0$ such that if a vector $\vec{x} \in \Delta_{t+3}$ satisfies $p_{\Gamma_{t+2}}(x_1, \ldots, x_{t+4}) \geq \lambda(K^3_{t+2}) - \delta$, then there exists a real number $\alpha \in [0,1]$ such that*

$$x_i = \begin{cases} \frac{1}{t+2} \pm \epsilon & \text{if } i \in [t], \\ \frac{\alpha}{t+2} \pm \epsilon & \text{if } i \in \{t+1, t+4\}, \\ \frac{1-\alpha}{t+2} \pm \epsilon & \text{if } i \in \{t+2, t+3\}. \end{cases} \tag{2}$$

Proof. We will use

$$xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2, \tag{3}$$

and the following inequalities (see Lemma 2.2 in [23]),

$$p_{K^3_{t+2}}(y_1, \ldots, y_{t+2}) \leq \frac{t(t+1)}{2(t+2)^2} - \frac{t}{6(t+2)} \sum_{i=1}^{t+2} \left(y_i - \frac{1}{t+2}\right)^2 \tag{4}$$

for all $(y_1, \ldots, y_{t+2}) \in \Delta_{t+1}$.

Let $\delta = \left(\frac{\epsilon}{30t}\right)^2$ and note that $\epsilon = 30t\delta^{1/2}$. Let $p = p_{\Gamma_{t+2}}(x_1, \ldots, x_{t+4})$ and $\sigma_i = \sigma_i(x_1, \ldots, x_t) = \sum_{E \in \binom{[t]}{i}} \prod_{j \in E} x_j$ for $i \in \{1, 2, 3\}$. Let $y_{12} = x_{t+1} + x_{t+2}$, $y_{34} = x_{t+3} + x_{t+4}$, $y_{13} = x_{t+1} + x_{t+3}$, $y_{24} = x_{t+2} + x_{t+4}$, and $y_{t+1} = y_{t+2} = (x_{t+1} + x_{t+2} + x_{t+3} + x_{t+4})/2$. Note that $y_{t+1} = y_{t+2} = (y_{12} + y_{34})/2 = (y_{13} + y_{24})/2$. By Equation (3), we have

$$
\begin{aligned}
p &= \sigma_3 + \sigma_2 (1 - \sigma_1) + \sum_{i=1}^{t-1} x_i y_{12} y_{34} + x_t y_{13} y_{24} \\
&\leq \sigma_3 + \sigma_2 (1 - \sigma_1) + \sigma_1 y_{t+1} y_{t+2} - (\sigma_1 - x_t) \left(\frac{y_{12} - y_{34}}{2}\right)^2 - x_t \left(\frac{y_{13} - y_{24}}{2}\right)^2 \\
&= p_{K^3_{t+2}}(x_1, \ldots, x_t, y_{t+1}, y_{t+2}) - (\sigma_1 - x_t) \left(\frac{y_{12} - y_{34}}{2}\right)^2 - x_t \left(\frac{y_{13} - y_{24}}{2}\right)^2.
\end{aligned}
$$

Since $p \geq \frac{t(t+1)}{2(t+2)^2} - \delta$, it follows from Equation (4) and the inequality above that

$$
\begin{aligned}
\delta \geq \frac{t}{6(t+2)} \sum_{i=1}^{t} \left(x_i - \frac{1}{t+2}\right)^2 + \sum_{i=t+1}^{t+2} \left(y_i - \frac{1}{t+2}\right)^2 \\
+ (\sigma_1 - x_t) \left(\frac{y_{12} - y_{34}}{2}\right)^2 + x_t \left(\frac{y_{13} - y_{24}}{2}\right)^2.
\end{aligned}
\tag{5}
$$

First, it follows that

$$
\begin{aligned}
\left| x_i - \frac{1}{t+2} \right| &\leq \sqrt{\frac{6(t+2)}{t}\delta} \leq 6\delta^{1/2} \quad \text{for } i \in [t], \text{ and} \\
\left| y_i - \frac{1}{t+2} \right| &\leq \sqrt{\frac{6(t+2)}{t}\delta} \leq 6\delta^{1/2} \quad \text{for } i \in \{t+1, t+2\}.
\end{aligned}
$$

In particular, the first inequality implies that $x_t \geq 1/(t+2) - 6\delta^{1/2} \geq 1/4t^2$ and $\sigma_1 - x_t = \sum_{i=1}^{t-1} x_i \geq 1/4t^2$. So it follows from Inequality (5) implies that

$$
\max\{|y_{12} - y_{34}|, |y_{13} - y_{24}|\} \leq \max\left\{2\sqrt{\delta/x_t}, 2\sqrt{\delta/(\sigma_1 - x_t)}\right\} \leq 4t\delta^{1/2}.
$$

Since $y_{12} + y_{34} = y_{13} + y_{24} = 2y_{t+1}$, we obtain

$$
\max\left\{\left|y_{12} - \frac{1}{t+2}\right|, \left|y_{23} - \frac{1}{t+2}\right|, \left|y_{13} - \frac{1}{t+2}\right|, \left|y_{24} - \frac{1}{t+2}\right|\right\} \leq 4t\delta^{1/2} + 6\delta^{1/2} \leq 10t\delta^{1/2}.
$$

Consequently,

$$
|x_{t+1} - x_{t+4}| = |y_{12} - y_{24}| \leq 20t\delta^{1/2}, \quad \text{and} \quad |x_{t+2} - x_{t+3}| = |y_{12} - y_{13}| \leq 20t\delta^{1/2}.
$$

Since $x_{t+1} + x_{t+2} = y_{12} = \frac{1}{t+2} \pm 10t\delta^{1/2}$, there exists $\alpha \in [0, 1]$ such that

$$
x_{t+1} = \frac{\alpha}{t+2} \pm 10t\delta^{1/2}, \quad \text{and} \quad x_{t+2} = \frac{1-\alpha}{t+2} \pm 10t\delta^{1/2}.
$$

Consequently,

$$
x_{t+3} = \frac{1-\alpha}{t+2} \pm 30t\delta^{1/2}, \quad \text{and} \quad x_{t+4} = \frac{\alpha}{t+2} \pm 30t\delta^{1/2}.
$$

This proves Lemma 6.8. ∎

6.2 Proof of Theorem 1.3 (d)

We prove Theorem 1.3 (d) in this section. Recall from Lemma 4.1 that $\mathcal{F}_{t+2}$ is blowup-invariant, and from Claim 4.4 that the hereditary family $\mathfrak{d}_{t+2}$ contains all symmetrized $\mathcal{F}_{t+2}$-free 3-graphs. Therefore, by Theorem 6.2, it suffices to show that $\mathcal{F}_{t+2}$ is vertex-extendable with respect to $\mathfrak{d}_{t+2}$.

Proof of Theorem 1.3 (d). Let $\epsilon > 0$ be a sufficiently small constant and n be a sufficiently large integer. Let $\delta > 0$ be the constant guaranteed by Lemma 6.8. We may assume that $\delta \le \epsilon$. Let $\mathcal{H}$ be an $(n+1)$-vertex Γ_{t+2}-colorable 3-graph with minimum degree at least $3(\lambda - \delta/2)(n+1)^2$, where $\lambda = \lambda(\Gamma_{t+2}) = \frac{t(t+1)}{6(t+2)^2}$.

Suppose that there exists a vertex $v \in V(\mathcal{H})$ such that $\mathcal{H} \setminus \{v\} \in \mathfrak{d}_{t+2}$. Let $V = V(\mathcal{H}) \setminus \{v\}$ and $\mathcal{H}' = \mathcal{H} \setminus \{v\}$. Let $V = V_1 \cup \cdots \cup V_{t+4}$ be a partition such that the map $\phi \colon V(\mathcal{H}') \to V(\Gamma_{t+2})$ defined by $\phi(V_i) = i$ for $i \in [t+4]$ induces a homomorphism from $\mathcal{H}'$ to Γ_{t+2}. Let $\mathcal{G} = \Gamma_{t+2}[V_1, \ldots, V_{t+4}]$ be a blowup of Γ_{t+2}. It is possible that some set V_i is empty. So it will be convenient later to define the following graphs.

Let $w_1, \ldots, w_{t+4}$ be $t+4$ new vertices and let $W_i = V_i \cup \{w_i\}$ for $i \in [t+4]$. Let $\widehat{\mathcal{G}} = \Gamma_{t+2}[W_1, \ldots, W_{t+4}]$ be a blowup of Γ_{t+2}. For every $i \in [t+4]$ define $L_{\mathcal{G}}(i) = L_{\widehat{\mathcal{G}}}(w_i)[V]$ and $N_{\mathcal{G}}(i) = V \setminus V_i$. Notice that $L_{\mathcal{G}}(i)$ is a graph on $N_{\mathcal{G}}(i)$.

We will $N(u)$ and $L(u)$ to denote $N_{\mathcal{H}}(u)$ and $L_{\mathcal{H}}(u)$ respectively for every vertex $u \in V(\mathcal{H})$ if it does not cause any confusion.

Claim 6.9. *It suffices to prove that there exists some $i_0 \in [t+4]$ such that $L(v) \subset L_{\mathcal{G}}(i_0)$.*

Proof. Suppose that there exists some $i_0 \in [t+4]$ such that $L(v) \subset L_{\mathcal{G}}(i_0)$. First notice that since $N_{\mathcal{G}}(i_0) \cap V_{i_0} = \emptyset$, we have $N(v) \cap V_{i_0} = \emptyset$ as well. Let $\widehat{V}_{i_0} = V_{i_0} \cup \{v\}$ and $\widehat{V}_i = V_i$ for $i \in [t+4] \setminus \{i_0\}$. It is clear that $\mathcal{H} \subset \Gamma_{t+2}[\widehat{V}_1, \ldots, \widehat{V}_{t+4}]$. Therefore, $\mathcal{F}_{t+2}$ is vertex-extendable with respect to $\mathfrak{d}_{t+2}$. This proves Theorem 1.3 (d). ∎

Let $x_i = |V_i|/n$ for $i \in [t+4]$. By assumption, we have

$$\delta(\mathcal{H}') \ge \delta(\mathcal{H}) - n \ge 3(\lambda - \delta/2)(n+1)^2 - n \ge 3(\lambda - \delta)n^2,$$

and hence $|\mathcal{H}'| \ge \delta(\mathcal{H}')n/3 \ge (\lambda - \delta)n^3$. Since $p_{\Gamma_{t+2}}(x_1, \ldots, x_{t+4}) \ge |\mathcal{H}'|/n^3 \ge \lambda - \delta$, it follows from Lemma 6.8 that there exists a real number $\alpha \in [0,1]$ such that Equation (2) holds.

Claim 6.10. *For every vertex $u \in V$ we have $|L_{\mathcal{G}}(u) \setminus L(u)| \le 4t\epsilon n^2$.*

Proof. Let us assume that $u \in V_i$ where $i \in [t+4]$.

If $i \in \{1, \ldots, t\}$, then it follows from (2) that

$$\begin{aligned} |L_{\mathcal{G}}(u)|/n^2 &\le \binom{t-1}{2}\left(\frac{1}{t+2} + \epsilon\right)^2 + (t-1)\left(\frac{1}{t+2} + \epsilon\right)\left(\frac{2}{t+2} + 4\epsilon\right) \\ &\quad + \left(\frac{1}{t+2} + 2\epsilon\right)\left(\frac{1}{t+2} + 2\epsilon\right) \\ &\le \frac{t(t+1)}{2(t+2)^2} + \frac{t(t+3)}{t+2}\epsilon + \frac{t^2+5t+2}{2}\epsilon^2 \le \frac{t(t+1)}{2(t+2)^2} + 2t\epsilon. \end{aligned}$$

If $i \in \{t+1, \dots, t+4\}$, then it follows from (2) that

$$|L_{\mathcal{G}}(u)|/n^2 \leq \binom{t}{2}\left(\frac{1}{t+2}+\epsilon\right)^2 + t\left(\frac{1}{t+2}+\epsilon\right)\left(\frac{1}{t+2}+2\epsilon\right)$$
$$\leq \frac{t(t+1)}{2(t+2)^2} + t\epsilon + \frac{t(t+3)}{2}\epsilon^2 \leq \frac{t(t+1)}{2(t+2)^2} + 2t\epsilon.$$

Recall that $\lambda = \frac{t(t+1)}{6(t+2)^2}$ and $\delta \leq \epsilon$. Therefore,

$$|L_{\mathcal{G}}(u) \setminus L(u)|/n^2 \leq \frac{t(t+1)}{2(t+2)^2} + 2t\epsilon - 3(\lambda - \delta) \leq \frac{t(t+1)}{2(t+2)^2} + 2t\epsilon - \frac{t(t+1)}{2(t+2)^2} + 3\epsilon \leq 4t\epsilon.$$

∎

Claim 6.11. *For every $i \in [t+4]$ and every $u \in V_i$ we have $|V_j \setminus N(u)| \leq 12t\epsilon n$ for all $j \in [t+4] \setminus \{i\}$.*

Proof. Fix $i \in [t+4]$ and let $u \in V_i$. If $i \in [t]$, then it follows from Lemma 6.8 that

$$\delta(L_{\mathcal{G}}(u))/n \geq \min\left\{(t-2)\left(\frac{1}{t+2}-\epsilon\right) + 2\left(\frac{\alpha}{t+2} - \epsilon + \frac{1-\alpha}{t+2} - \epsilon\right),\right.$$
$$\left.(t-1)\left(\frac{1}{t+2}-\epsilon\right) + \frac{\alpha}{t+2} - \epsilon + \frac{1-\alpha}{t+2} - \epsilon\right\} \geq \frac{t}{t+2} - (t+2)\epsilon \geq \frac{1}{3}.$$

If $i \in [t+1, t+4]$, then it follows from Lemma 6.8 that

$$\delta(L_{\mathcal{G}}(u))/n \geq \min\left\{(t-1)\left(\frac{1}{t+2}-\epsilon\right) + \frac{\alpha}{t+2} - \epsilon + \frac{1-\alpha}{t+2} - \epsilon, t\left(\frac{1}{t+2}-\epsilon\right)\right\}$$
$$\geq \frac{t}{t+2} - (t+2)\epsilon \geq \frac{1}{3}.$$

Since each vertex in $V_j \setminus N(u)$ contributes at least $\delta(L_{\Gamma_{t+2}}(u))$ edges to $L_{\mathcal{G}}(u) \setminus L(u)$, it follows from Claim 6.10 that

$$|V_j \setminus N(u)| \leq \frac{4t\epsilon n^2}{n/3} = 12t\epsilon n.$$

∎

By symmetry, we may assume that $\alpha \geq 1/2$. Since ϵ is sufficiently small, we have $\min\{x_1, \dots, x_t, x_{t+1}, x_{t+4}\} \geq \frac{1}{4(t+2)}$.

Claim 6.12. *We have $L(v) \cap \binom{V_i}{2} = \emptyset$ for all $i \in [t+4]$.*

Proof. Suppose to the contrary that there exists $i \in [t+4]$ such that $L(v) \cap \binom{V_i}{2} \neq \emptyset$. Let us assume that $\{u_i, u_i'\} \in L(v) \cap \binom{V_i}{2}$.

First let us assume that $i \in \{1, \dots, t, t+1, t+4\}$. Applying Lemma 6.3 with $\eta = 4t\epsilon$ and $T = \{1, \dots, t, t+1, t+4\} \setminus \{i\}$ we obtain $u_j \in V_j$ for every $j \in T$ such that sets $U = \{u_j : j \in T\} \cup \{u_i\}$ and $U' = \{u_j : j \in T\} \cup \{u_i'\}$ satisfy

$$\mathcal{H}[U] = \mathcal{G}[U] \cong K_{t+2}^3 \quad \text{and} \quad \mathcal{H}[U'] = \mathcal{G}[U'] \cong K_{t+2}^3.$$

Let $F = \mathcal{H}[U \cup U'] \cup \{\{v, u_i, u_i'\}\}$. It is clear that $v(F) = t + 4 \leq 4(t+4)^2$. Therefore, F is contained as a subgraph in some blowup of Γ_{t+2}. Since $\mathcal{H}[U] \cong K_{t+2}^3$, $\mathcal{H}[U'] \cong K_{t+2}^3$, and $|U \cap U'| = t+1$, it follows from Lemma 6.5 that $\{u_i, u_i'\}$ is not contained in any edge of F, a contradiction.

Now let us assume that $i \in \{t+2, t+3\}$. Applying Lemma 6.3 with $\eta = 4t\epsilon$ and $T = \{1, \ldots, t, t+1, t+4\}$ we obtain $u_j \in V_j$ for every $j \in T$ such that the set $U = \{u_j : j \in T\}$ satisfies

$$\mathcal{H}[U] = \mathcal{G}[U] \cong K_{t+2}^3 \quad \text{and} \quad L(u_i)[U] = L_{\mathcal{G}}(u_i)[U] = L_{\mathcal{G}}(u_i')[U] = L(u_i')[U].$$

Let $F = \mathcal{H}[U \cup \{u_i, u_i'\}] \cup \{\{v, u_i, u_i'\}\}$. It is clear that $v(F) = t + 5 \leq 4(t+4)^2$. So there exists a homomorphism $\phi\colon V(F) \to V(\Gamma_{t+2})$ from F to Γ_{t+2}. Since $F[U] \cong K_{t+2}^3$, it follows from Observation 6.4 (a) that either $\phi(U) = [t] \cup \{t+1, t+4\}$ or $\phi(U) = [t] \cup \{t+2, t+3\}$. By symmetry we may assume that $\phi(U) = [t] \cup \{t+1, t+4\}$. Since $U \cup \{u_i, u_i'\}$ is 2-covered in F, we have $\{\phi(u_i), \phi(u_i')\} = [t+4] \setminus \phi(U) = \{t+2, t+3\}$. However, notice that $|L_F(u_i)[U] \cap L_F(u_i')[U]| = \binom{t}{2} + t$, while $|L_{\Gamma_{t+2}}(t+2)[\phi(U)] \cap L_{\Gamma_{t+2}}(t+3)[\phi(U)]| = \binom{t}{2} < \binom{t}{2} + t$, a contradiction. ∎

Define

$$I_{small} = \left\{ i \in [t+4] \colon |N(v) \cap V_i| \leq \epsilon^{1/4} n \right\}, \quad \text{and} \quad I_{\emptyset} = \{ i \in [t+4] \colon N(v) \cap V_i = \emptyset \}.$$

It follows from the definition that $I_{\emptyset} \subset I_{small}$.

Next we will consider two cases depending on the sizes of V_{t+2} and V_{t+3}: either V_{t+2} and V_{t+3} are both large (i.e. at least $50t^2\epsilon^{1/4}$) or V_{t+2} and V_{t+3} are both small. The first case is quite easy to handle while the second case needs more work.

Case 1: $\alpha \leq 1 - 101t^3\epsilon^{1/4}$.

Note that in this case we have

$$\min\{|V_{t+1}|, |V_{t+2}|, |V_{t+3}|, |V_{t+4}|\} \geq \frac{101t^3\epsilon^{1/4}}{t+2} n \geq 50t^2\epsilon^{1/4} n.$$

Define an auxiliary graph R on $[t+4]$ in which two vertices i and j are adjacent iff there exists $e \in L(v)$ such that $e \cap V_i \neq \emptyset$ and $e \cap V_j \neq \emptyset$. An easy observation is that $L(v) \subset R[V_1, \ldots, V_{t+4}]$.

Claim 6.13. *We have $L(v) \subset L_{\mathcal{G}}(i)$ for some $i \in I_{small}$.*

Proof. For each edge $ij \in R$ let $e_{ij} \in L(v)$ be an edge with one endpoint in V_i and the other endpoint in V_j. Let $U_1 = \bigcup_{ij \in R} e_{ij}$ be a vertex subset of V and note that $|U_1| \leq 2\binom{t+4}{2}$. Define

$$V_i' = \begin{cases} V_i \cap N(v) \cap \left(\bigcap_{u \in U_1 \setminus V_i} N(u) \right) & \text{for } i \in [t+4] \setminus I_{small}, \\ V_i \cap \left(\bigcap_{u \in U_1 \setminus V_i} N(u) \right) & \text{for } i \in I_{small}. \end{cases}$$

It follows from Claim 6.11 and our assumption that

$$|V_i'| \geq \min\left\{ \epsilon^{1/4} n - (|U_1| + 1) \times 12t\epsilon n,\ 50t^2\epsilon^{1/4} n - |U_1| \times 12t\epsilon n \right\} \geq 4(t+4)^3 \epsilon^{1/3} n.$$

Applying Lemma 6.3 with $\eta = 4t\epsilon$ and $T = [t+4]$ we obtain $u_i \in V_i'$ for $i \in T$ such that the set $U_2 = \{u_i : i \in T\}$ satisfies

(a) $\mathcal{H}[U_2] = \mathcal{G}[U_2] \cong \Gamma_{t+2}$, and

(b) $L(u)[U_2] = L_{\mathcal{G}}(u)[U_2]$ for all $u \in U_1$.

Let $F = \mathcal{H}[U_1 \cup U_2 \cup \{v\}]$. Note that $v(F) \le 2\binom{t+4}{2} + t + 5 \le 3(t+4)^2$. By assumption there exists a homomorphism $\phi\colon V(F) \to V(\Gamma_{t+2})$ from F to Γ_{t+2} (otherwise we would have $F \in \mathcal{F}_{t+2}$, a contradiction). Since $F[U_2] \cong \Gamma_{t+2}$, it follows from Lemma 6.6 that $\phi(u_i) \in [t]$ for $i \in [t]$ (and $\phi(u_t) = t$ if $t \ge 3$), and

$$\{\phi(\{u_{t+1}, u_{t+4}\}), \phi(\{u_{t+2}, u_{t+3}\})\} = \{\{t+1, t+4\}, \{t+2, t+3\}\}\,.$$

By symmetry we may assume that $\phi(u_i) = i$ for $i \in [t+4]$. For every $i \in [t+4]$ and $u \in V_i \cap U_1$ since u is adjacent to all vertices in $U_2 \setminus \{u_i\}$ in ∂F, we have $\phi(u) = i$. Finally, since v is adjacent to all vertices in $\{u_i \colon i \in [t+4] \setminus I_{small}\}$, we have $\phi(v) \notin \phi([t+4] \setminus I_{small})$. In other words, $\phi(v) \in \phi(I_{small})$. This means that there exists some $i \in I_{small}$ such that $\phi(R) \subset L_{\Gamma_{t+2}}(\phi(i))$, and hence $L(v) \subset L_{\mathcal{G}}(i)$. ∎

Case 2: $\alpha \ge 1 - 101t^3\epsilon^{1/4}$.

Note that in this case we have

$$\begin{aligned}
\max\{|V_{t+2}|, |V_{t+3}|\} &\le \frac{101t^3\epsilon^{1/4}}{t+2}n + \epsilon n \le 102t^2\epsilon^{1/4}n, \quad \text{and} \\
\min\{|V_{t+1}|, |V_{t+4}|\} &\ge \frac{1 - 100t^3\epsilon^{1/4}}{t+2}n - \epsilon n \ge \frac{n}{t+2} - 50t^2\epsilon^{1/4}n.
\end{aligned}$$

For convenience, let $S = [t] \cup \{t+1, t+4\}$.

Claim 6.14. *We have* $|I_{small} \cap S| \le 1$.

Proof. Suppose to the contrary that $|I_{small} \cap S| \ge 2$. Then it follows from Lemma 6.8 that

$$\begin{aligned}
|L(v)|/n^2 &\le \binom{t}{2}\left(\frac{1}{t+2} + \epsilon\right)^2 + 2 \times 102t^2\epsilon^{1/4} + 2\epsilon^{1/4} \\
&= \frac{t(t+1)}{2(t+2)^2} - \frac{t}{(t+2)^2} + \frac{t(t-1)}{t+2}\epsilon + \frac{t(t-1)}{2}\epsilon^2 + 204t^2\epsilon^{1/4} + 2\epsilon^{1/4} \\
&< 3(\lambda - \delta),
\end{aligned}$$

a contradiction. ∎

Next we will consider two cases depending on the value of $|I_{small} \cap S|$: either $|I_{small} \cap S| = 1$ or $|I_{small} \cap S| = 0$.

Case 2.1: $|I_{small} \cap S| = 1$.

We may assume that $I_{small} = \{1\}$. The proof for other cases follows analogously. Let $W = \bigcup_{i \in S \setminus I_{small}} V_i$. It follows from our assumption that

$$|L(v)[W]|/n^2 \ge \delta(\mathcal{H})/n^2 - 2 \times 102t^2\epsilon^{1/4} - \epsilon^{1/4} \ge \frac{t(t+1)}{2(t+2)^2} - 205t^2\epsilon^{1/4}.$$

Notice that the induced subgraph of $L_{\mathcal{G}}(1)$ on W is the blowup $K_{t+1}[V_2, \ldots, V_t, V_{t+1}, V_{t+4}]$ of K_{t+1}. Using the same argument as in the proof of Claim 6.10, we get

$$|L_{\mathcal{G}}(1) \setminus L(v)[W]|/n^2 \leq \frac{t(t+1)}{2(t+2)^2} + 2t\epsilon - \left(\frac{t(t+1)}{2(t+2)^2} - 205t^2\epsilon^{1/4}\right) \leq 206t^2\epsilon^{1/4}.$$

Also, using the same argument as in the proof of Claim 6.11, we get

$$|N(v) \cap V_i| \geq |V_i| - \frac{206t^2\epsilon^{1/4}n^2}{n/3} \geq \frac{n}{t+2} - 50t^2\epsilon^{1/4}n - 700t^2\epsilon^{1/4}n \geq \frac{n}{2(t+2)}.$$

Claim 6.15. *We have $N(v) \cap V_1 = \emptyset$.*

Proof. Suppose to the contrary that there exists a vertex $u_1 \in N(v) \cap V_1$. Let $V_i' = V_i \cap N(v) \cap N(u_1)$ for $i \in S \setminus \{1\}$. Notice that $|V_i'| \geq \frac{n}{2(t+2)} - 12t\epsilon n > 206(t+4)^4\epsilon^{1/12}n$. Applying Lemma 6.8 with $\eta = 206t^2\epsilon^{1/4}$ and $T = S \setminus \{1\}$, we obtain a vertex $u_i \in V_i'$ for every $i \in T$ such that the set $U = \{u_i \colon i \in T\}$ satisfies

$$\mathcal{H}[U \cup \{u_1\}] = \mathcal{G}[U \cup \{u_1\}] \cong K_{t+2}^3 \quad \text{and} \quad \mathcal{H}[U \cup \{v\}] = \mathcal{G}[U \cup \{v\}] \cong K_{t+2}^3.$$

Let $e \in \mathcal{H}$ be an edge that contains $\{v, u_1\}$ and let $F = \mathcal{H}[U \cup \{v, u_1\}] \cup \{e\}$. By assumption, F should be contained in some blowup of Γ_{t+2}, but this would contradict Lemma 6.5 since $\{v, u_1\}$ is contained in e. ∎

Claim 6.16. *We have $L(v) \subset L_{\mathcal{G}}(1)$.*

Proof. Suppose to the contrary that there exists an edge $e \in L_{\mathcal{G}}(1) \setminus L(v)$. By symmetry we may assume that $e = \{u_{t+1}, u_{t+2}\}$ and $(u_{t+1}, u_{t+2}) \in V_{t+1} \times V_{t+2}$. Let

$$V_i' = \begin{cases} V_i \cap N(u_{t+1}) \cap N(u_{t+2}) & \text{if } i = 1, \\ V_i \cap N(v) \cap N(u_{t+2}) & \text{if } i = t+1, \\ V_i \cap N(v) \cap N(u_{t+1}) \cap N(u_{t+2}) & \text{if } i \in [2, t] \cup \{t+4\}. \end{cases}$$

Notice that $|V_i'| \geq \frac{n}{2(t+2)} - 2 \times 12t\epsilon > 206(t+4)^4\epsilon^{1/12}$. Applying Lemma 6.8 with $\eta = 206t^2\epsilon^{1/4}$ and $T = S$, we obtain a vertex $u_i' \in V_i'$ for every $i \in T$ such that the set $U = \{u_i' \colon i \in T\}$ satisfies

(a) $\mathcal{H}[U] = \mathcal{G}[U] \cong K_{t+2}^3$,

(b) $L(v)[U \setminus \{u_1\}] \cong K_{t+1}$, and

(c) $L(u_i)[U] = L_{\mathcal{G}}(u_i)[U]$ for $i \in \{t+1, t+2\}$.

Let $F = \mathcal{H}[U \cup \{v, u_{t+1}, u_{t+2}\}]$ and $U' = (U \cup \{u_{t+1}\}) \setminus \{u_{t+1}'\}$. Since $v(F) = t + 5 \leq 4(t+4)^2$, by assumption, there is a homomorphism $\phi \colon V(F) \to V(\Gamma_{t+2})$ from F to Γ_{t+2}. Since $\mathcal{H}[U] \cong \mathcal{H}[U'] \cong K_{t+2}^3$, we have $\Gamma_{t+2}[\phi(U)] \cong \Gamma_{t+2}[\phi(U')] \cong K_{t+2}^3$. Since $|\phi(U) \cap \phi(U')| = |U \cap U'| = t+1$, it follows from Lemma 6.5 that $\phi(U) = \phi(U')$ and $\phi(u_{t+1}) = \phi(u_{t+1}')$.

Let $U_1 = U \cup \{u_{t+2}\}$. Since $F[U_1] \cong \Gamma_{t+2} \setminus \{t+3\}$, it follows from Lemma 6.7 that $\phi(\{u_i' \colon i \in [t]\}) = [t]$ (and $\phi(u_t') = t$ if $t \geq 3$), $\phi(\{u_{t+1}', u_{t+2}, u_{t+4}'\}) \subset [t+1, t+4]$, and $\phi(\{u_{t+1}', u_{t+4}'\}) \in \{\{t+1, t+4\}, \{t+2, t+3\}\}$. By symmetry we may assume that $\phi(u'_i) = i$ for $i \in S$ and $\phi(u_{t+2}) = t+2$. Hence $\phi(u'_{t+1}) = \phi(u_{t+1}) = t+1$. Since v is adjacent to all vertices in $U_1 \setminus \{u_1\}$, we have $\phi(v) \in [t+4] \setminus \phi(U_1 \setminus \{u_1\}) = \{1, t+3\}$. If $\phi(v) = t+3$, then $\{v, u_{t+1}, u_{t+2}\} \in F$ implies that $\{t+1, t+2, t+3\} = \{\phi(v), \phi(u_{t+1}), \phi(u_{t+2})\} \in \Gamma_{t+2}$, a contradiction. Therefore, we may assume that $\phi(v) = 1$. Then $\big\{\{u'_1, u'_{t+1}, u'_{t+4}\}, \{u'_1, u_{t+2}, u'_{t+4}\}, \{v, u_{t+1}, u_{t+2}\}\}\big\} \subset F$ implies that

$$\begin{aligned}&\{\{1, t+1, t+4\}, \{1, t+2, t+4\}, \{1, t+1, t+2\}\}\} \\ =&\big\{\{\phi(u'_1), \phi(u'_{t+1}), \phi(u'_{t+4})\}, \{\phi(u'_1), \phi(u_{t+2}), \phi(u'_{t+4})\}, \{\phi(v), \phi(u_{t+1}), \phi(u_{t+2})\}\}\big\} \subset \Gamma_{t+2}.\end{aligned}$$

This means that the triangle $\{t+1, t+2, t+4\}$ is contained in the induced subgraph of $L_{\Gamma_{t+2}}(1)$ on $\{t+1, t+2, t+3, t+4\}$, contradicts the fact that the induced subgraph of $L_{\Gamma_{t+2}}(1)$ on $\{t+1, t+2, t+3, t+4\}$ is a copy of C_4. ∎

Recall that $S = [t] \cup \{t+1, t+2\}$

Case 2.2: $I_{small} \cap S = \emptyset$.

Recall that R is a graph on $[t+4]$ in which two vertices i and j are adjacent iff there exists $e \in L(v)$ such that $e \cap V_i \neq \emptyset$ and $e \cap V_j \neq \emptyset$. Recall the observation that $L(v) \subset R[V_1, \ldots, V_{t+4}]$.

Claim 6.17. *If $V_{t+2} \cup V_{t+3} \neq \emptyset$, then either $L(v) \subset L_{\mathcal{G}}(t+2)$ or $L(v) \subset L_{\mathcal{G}}(t+3)$.*

Proof. For every $ij \in R$ let $e_{ij} \in L(v)$ be an edge such that $e_{ij} \cap V_i \neq \emptyset$ and $e_{ij} \cap V_j \neq \emptyset$. Let $U_1 = \bigcup_{ij \in R_2} e_{ij}$ be a vertex subset of V. Note that $|U_1| \leq 2\binom{t+4}{2}$.

By symmetry we may assume that $V_{t+2} \neq \emptyset$. Fix a vertex $u_{t+2} \in V_{t+2}$ (it is possible that $u_{t+2} \in U_1$ as well). For every $i \in S$ let $V'_i = V_i \cap N(v) \cap N(u_{t+2}) \cap \Big(\bigcap_{u \in U_1 \setminus V_i} N(u)\Big)$. Notice that $|V'_i| \geq \frac{n}{2(t+2)} - (2\binom{t+4}{2} + 1)12t\epsilon > 4(t+4)^3\epsilon^{1/3}n$. Applying Lemma 6.3 with $\eta = 4t\epsilon$ and $T = S$ we obtain $u_i \in V'_i$ for every $i \in T$ such that the set $U = \{u_i \colon i \in T\}$ satisfies

(a) $\mathcal{H}[U] = \mathcal{G}[U] \cong K^3_{t+2}$,

(b) $L(u)[U] = L_{\mathcal{G}}(u)[U]$ for all $u \in U_1$, and

(c) $L(u_{t+2})[U] = L_{\mathcal{G}}(u_{t+2})[U]$.

Let $F = \mathcal{H}[U \cup U_1 \cup \{v, u_{t+2}\}]$. Since $v(F) \leq 2\binom{t+4}{2} + t + 5 \leq 4(t+4)^3$, by assumption there is a homomorphism $\phi \colon V(F) \to V(\Gamma_{t+2})$ from F to Γ_{t+2}. Since $F[U \cup \{u_{t+2}\}] \cong \Gamma_{t+2} \setminus \{t+3\}$, it follows from Lemma 6.7 that $\phi(\{u_i \colon i \in [t]\}) = [t]$ (and $\phi(u_t) = t$ if $t \geq 3$), $\phi(\{u_{t+1}, u_{t+2}, u_{t+4}\}) \subset [t+1, t+4]$, and $\phi(\{u_{t+1}, u_{t+4}\}) \in \{\{t+1, t+4\}, \{t+2, t+3\}\}$. By symmetry, we may assume that $\phi(u_i) = i$ for $i \in [t] \cup \{t+1, t+2, t+4\}$. For every $i \in [t] \cup \{t+1, t+4\}$ and every $u \in U_1 \cap V_i$, since $L(u)[U] = L(u_i)[U] = L_{\mathcal{G}}(u_i)[U]$ and u is adjacent to all vertices in $U \setminus \{u_i\}$, we have $\phi(u) = \phi(u_i) = i$. Finally, since v is adjacent to all vertices in $\{u_i \colon i \in [t] \cup \{t+1, t+3\}\}$, we have $\phi(v) \in [t+4] \setminus \phi(\{u_i \colon i \in [t] \cup \{t+1, t+3\}\}) = \{t+2, t+3\}$. If $\phi(v) = t+2$, then $\phi(R) \subset L_{\Gamma_{t+2}}(\phi(v)) = L_{\Gamma_{t+2}}(t+2)$, which means that $L(v) \subset L_{\mathcal{G}}(t+2)$. If $\phi(v) = t+3$, then $\phi(R) \subset L_{\Gamma_{t+2}}(\phi(v)) = L_{\Gamma_{t+2}}(t+3)$, which means that $L(v) \subset L_{\mathcal{G}}(t+3)$. ∎

Now we may assume that $V_{j+2} \cup V_{j+3} = \emptyset$.

Claim 6.18. *We have either $L(v) \subset L_{\mathcal{G}}(t+2)$ or $L(v) \subset L_{\mathcal{G}}(t+3)$.*

Proof. For each edge $ij \in R$ let $e_{i,j} \in L(v)$ be an edge with one endpoint in V_i and the other endpoint in V_j. Let $U_1 = \bigcup_{ij\in R} e_{i,j}$ be a vertex subset of V and note that $|U_1| \le 2\binom{t+2}{2}$. Define

$$V_i' = V_i \cap N(v) \cap \left(\bigcap_{u \in U_1 \setminus V_i} N(u) \right) \quad \text{for } i \in [t] \cup \{t+1, t+4\}.$$

Notice that $|V_i'| \ge \frac{n}{2(t+2)} - (2\binom{t+4}{2} + 1)12t\epsilon > 4(t+4)^3\epsilon^{1/3}n$. Applying Lemma 6.3 with $\eta = 4t\epsilon$ and $T = [t] \cup \{t+1, t+4\}$ we obtain $u_i \in V_i'$ for $i \in T$ such that the set $U_2 = \{u_i : i \in T\}$ satisfies

(a) $\mathcal{H}[U_2] = \mathcal{G}[U_2] \cong K_{t+2}^3$, and

(b) $L(u)[U_2 \setminus \{u_i\}] = L_{\mathcal{G}}(u)[U_2 \setminus \{u_i\}] \cong K_{t+1}$ for all $u \in U_1 \cap V_i$ and $i \in S$.

Let $F = \mathcal{H}[U_1 \cup U_2 \cup \{v\}]$. Suppose that there exists a homomorphism $\phi \colon V(F) \to V(\Gamma_{t+2})$ from F to Γ_{t+2}. Since the set $F[U_2] \cong K_{t+2}^3$ we have either $\phi(U_2) = [t] \cup \{t+1, t+4\}$ or $\phi(U_2) = [t] \cup \{t+2, t+3\}$. By symmetry we may assume that the former case holds and $\phi(u_i) = i$ for $i \in [t] \cup \{t+1, t+4\}$. Since $L_{\Gamma_{t+2}}(t+2)$ and $L_{\Gamma_{t+2}}(t+3)$ do not contain K_{t+1} as a subgraph, it follows from (*b*) that for every $u \in V_i \cap U_1$ and $i \in [t] \cup \{t+1, t+4\}$ we have $\phi(u) \notin \{t+2, t+3\}$. Moreover, since u is adjacent to all vertices in $U_2 \setminus \{u_i\}$ in ∂F, we have $\phi(u) \notin \phi(U_2 \setminus \{u_i\})$ as well. Therefore, $\phi(u) = \phi(u_i) = i$. Finally, since $U_2 \cup \{v\}$ is 2-covered in F, we have $\phi(v) \in \{t+2, t+3\}$. If $\phi(v) = t+2$, then we have $L(v) \subset L_{\mathcal{G}}(t+2)$. If $\phi(v) = t+3$, then we have $L(v) \subset L_{\mathcal{G}}(t+3)$. ∎

This completes the proof of Theorem 1.3 (d). ∎

7 Feasible region

We prove Theorem 1.11 in this section. First we need the following simple corollary of Theorem 1.3 (d). For convenience, we will keep using $t+2$ instead of t in this section.

Recall that for an n-vertex r-graph $\mathcal{H}$ the edge density of $\mathcal{H}$ is $\rho(H) = |\mathcal{H}|/\binom{n}{r}$, and the shadow density of $\mathcal{H}$ is $|\partial\mathcal{H}|/\binom{n}{r-1}$.

Corollary 7.1. *For every integer $t \ge 2$ there exist constants $\epsilon_0 > 0$ and N_0 such that the following statement holds for all $\epsilon \le \epsilon_0$ and $n \ge N_0$. Suppose that $\mathcal{H}$ is an n-vertex $\mathcal{F}_{t+2}$-free 3-graph with $\rho(\mathcal{H}) \ge \frac{t(t+1)}{2(t+2)^2} - \epsilon$. Then there exists a set $Z_\epsilon \subset V(\mathcal{H})$ of size at most $\epsilon^{1/2}n$ such that $\mathcal{H} \setminus Z_\epsilon$ is Γ_{t+2}-colorable and $\delta(\mathcal{H}) \ge \left(\frac{t(t+1)}{2(t+2)^2} - 3\epsilon^{1/2}\right)n^2$.*

The idea for proving Corollary 7.1 is to show that after removing a few vertices with small degree from $\mathcal{H}$ the remaining 3-graph has large minimum degree, and hence, by Theorem 1.3 (d), it is Γ_{t+2}-colorable. We refer the reader to the proof of Theorem 4.1 in [22] for detailed calculations.

Lemma 7.2. *Suppose that $\mathcal{H}$ is an n-vertex Γ_{t+2}-colorable 3-graph with $|\mathcal{H}| \ge \left(\frac{t(t+1)}{6(t+2)^2} - \epsilon\right)n^3$. Then $\left(\frac{t+1}{2(t+2)} - 100t^4\epsilon^{1/2}\right)n^2 \le |\partial\mathcal{H}| \le \left(\frac{t^2+3t+3}{2(t+2)^2} + 5000t^4\epsilon^{1/2}\right)n^2$.*

Proof. Let $V = V_1 \cup \cdots \cup V_{t+4}$ be a partition such that $\mathcal{H} \subset \Gamma_{t+2}[V_1, \ldots, V_{t+4}]$. Let $\mathcal{G} = \Gamma_{t+2}[V_1, \ldots, V_{t+4}]$. Let $x_i = |V_i|/n$ for $i \in [t+4]$. By Lemma 6.8, there exists some $\alpha \in [0, 1]$ such that

$$x_i = \begin{cases} \frac{1}{t+2} \pm 30t\epsilon^{1/2} & \text{if } i \in [t], \\ \frac{\alpha}{t+2} \pm 30t\epsilon^{1/2} & \text{if } i \in \{t+1, t+4\}, \\ \frac{1-\alpha}{t+2} \pm 30t\epsilon^{1/2} & \text{if } i \in \{t+2, t+3\}. \end{cases}$$

Notice that for every pair $\{u, v\} \in \partial\mathcal{G}$ the number of edges in $\mathcal{G}$ containing $\{u, v\}$ is at least $\left(\frac{1}{t+2} - 30t\epsilon^{1/2}\right) n \geq \frac{n}{2(t+2)}$. Therefore, it follows from a simple double counting that

$$|\partial\mathcal{G} \setminus \partial\mathcal{H}| \leq \frac{3|\mathcal{G} \setminus \mathcal{H}|}{n/2(t+2)} \leq \frac{3\epsilon n^3}{n/2(t+2)} \leq 6\epsilon(t+2)n^2.$$

Therefore,

$$\begin{aligned}
|\partial\mathcal{H}|/n^2 &\geq |\partial\mathcal{G}|/n^2 - 6\epsilon(t+2) = p_{\partial\mathcal{G}}(x_1, \ldots, x_{t+4}) - 6(t+2)\epsilon \\
&\geq \binom{t}{2}\left(\frac{1}{t+2} - 30t\epsilon^{1/2}\right)^2 + t\left(\frac{1}{t+2} - 30t\epsilon^{1/2}\right)\left(\frac{2}{t+2} - 4 \times 30t\epsilon^{1/2}\right) \\
&\quad + \left(\frac{\alpha}{t+2} - 30t\epsilon^{1/2}\right)^2 + \left(\frac{1-\alpha}{t+2} - 30t\epsilon^{1/2}\right)^2 \\
&\quad + 4\left(\frac{\alpha}{t+2} - 30t\epsilon^{1/2}\right)\left(\frac{1-\alpha}{t+2} - 30t\epsilon^{1/2}\right) - 6(t+2)\epsilon \\
&\geq \frac{t+1}{2(t+2)} + \frac{2\alpha(1-\alpha)}{(t+2)^2} - 30t(t+3)\epsilon^{1/2} + 450t^2\left(t^2 + 7t + 12\right)\epsilon - 6(t+2)\epsilon \\
&\geq \frac{t+1}{2(t+2)} - 100t^4\epsilon^{1/2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\partial\mathcal{H}|/n^2 &\leq |\partial\mathcal{G}|/n^2 = p_{\partial\mathcal{G}}(x_1, \ldots, x_{t+4}) \\
&\leq \binom{t}{2}\left(\frac{1}{t+2} + 30t\epsilon^{1/2}\right)^2 + t\left(\frac{1}{t+2} + 30t\epsilon^{1/2}\right)\left(\frac{2}{t+2} + 4 \times 30t\epsilon^{1/2}\right) \\
&\quad + \left(\frac{\alpha}{t+2} + 30t\epsilon^{1/2}\right)^2 + \left(\frac{1-\alpha}{t+2} + 30t\epsilon^{1/2}\right)^2 \\
&\quad + 4\left(\frac{\alpha}{t+2} + 30t\epsilon^{1/2}\right)\left(\frac{1-\alpha}{t+2} + 30t\epsilon^{1/2}\right) \\
&\leq \frac{t+1}{2(t+2)} + \frac{2\alpha(1-\alpha)}{(t+2)^2} + 30t(t+3)\epsilon^{1/2} + 450t^2\left(t^2 + 7t + 12\right)\epsilon \\
&\leq \frac{t+1}{2(t+2)} + \frac{1}{2(t+2)^2} + 5000t^4\epsilon^{1/2} = \frac{t^2 + 3t + 3}{2(t+2)^2} + 5000t^4\epsilon^{1/2}.
\end{aligned}$$

∎

Now we are ready to prove Theorem 1.11.

Proof of Theorem 1.11. First, we prove that $\mathrm{proj}\Omega(\mathcal{F}_{t+2}) = \left[0, \frac{t+3}{t+4}\right]$. By a result in [19], this is equivalent to show that $\frac{t+3}{t+4} \in \mathrm{proj}\Omega(\mathcal{F}_{t+2})$ and $\mathrm{proj}\Omega(\mathcal{F}_{t+2}) \subset \left[0, \frac{t+3}{t+4}\right]$. Let $\mathcal{H}$

be an n-vertex $\mathcal{F}_{t+2}$-free 3-graph. By Lemma 4.2, the graph $\partial\mathcal{H}$ is K_{t+5}-free. Thus, by Turán's theorem, we have $\rho(\partial\mathcal{H}) \le \frac{t+3}{t+4}$. Therefore, we have $\mathrm{proj}\Omega(\mathcal{F}_{t+2}) \subset \left[0, \frac{t+3}{t+4}\right]$. On the other hand, since the balanced blowup of Γ_{t+2} on n vertices has edge density $\frac{t+3}{t+4}$ as $n \to \infty$, we have $\frac{t+3}{t+4} \in \mathrm{proj}\Omega(\mathcal{F}_{t+2})$. Therefore, $\mathrm{proj}\Omega(\mathcal{F}_{t+2}) = \left[0, \frac{t+3}{t+4}\right]$.

Let $I_t = \left[\frac{t+1}{t+4}, \frac{t^2+3t+3}{(t+2)^2}\right]$. Next, we show that $I_t \times \left\{\frac{t(t+1)}{(t+2)^2}\right\} \subset \Omega(\mathcal{F}_{t+2})$. This is done by constructing for every $x \in I_t$ a sequence of $\mathcal{F}_{t+2}$-free 3-graphs whose shadow densities approach x, and whose edge densities approach $\frac{t(t+1)}{(t+2)^2}$. Let $\alpha \in [0, 1/2]$ be a real number. Let $x_1 = \cdots = x_t = \frac{1}{t+2}$, $x_{t+1} = x_{t+4} = \frac{\alpha}{t+2}$, and $x_{t+2} = x_{t+3} = \frac{1-\alpha}{t+2}$. For every $n \in \mathbb{N}$ let $\mathcal{H}_n(\alpha) = \Gamma_{t+2}[V_1, \ldots, V_{t+4}]$ be a blowup of Γ_{t+2} such that $|V_i| = \lfloor x_i n \rfloor$ for $i \in [t+4]$. It is easy to see that

$$\lim_{n\to\infty} \rho\left(\partial\mathcal{H}_n(\alpha)\right) = 2 \cdot p_{K_{t+4}}(x_1, \ldots, x_{t+4}) = \frac{t^2+3t+2+4\alpha(1-\alpha)}{(t+2)^2},$$

and

$$\lim_{n\to\infty} \rho\left(\mathcal{H}_n(\alpha)\right) = 6 \cdot p_{\Gamma_{t+2}}(x_1, \ldots, x_{t+4}) = \frac{t(t+1)}{(t+2)^2}.$$

Letting α vary from 0 to $1/2$, the function $\frac{t^2+3t+2+4\alpha(1-\alpha)}{2(t+2)^2}$ grows from $\frac{t+1}{t+4}$ to $\frac{t^2+3t+3}{(t+2)^2}$. Therefore, $I_t \times \left\{\frac{t(t+1)}{6(t+2)^2}\right\} \subset \Omega(\mathcal{F}_{t+2})$.

Finally, we show that the feasible region function $g(\mathcal{F}_{t+2})$ attains its maximum only on the interval I_t. Suppose that $(\mathcal{H}_n)_{n=1}^{\infty}$ is a sequence of $\mathcal{F}_{t+2}$-free 3-graphs with $\lim_{n\to\infty} \rho(\partial\mathcal{H}_n) = x$ and $\lim_{n\to\infty} \rho(\mathcal{H}_n) = \frac{t(t+1)}{(t+2)^2}$. To keep our notations simple, let us assume that $v(\mathcal{H}_n) = n$ for all $n \ge 1$. Let $\epsilon > 0$ be a sufficiently small constant. Then there exists N_0 such that $\rho(\partial\mathcal{H}_n) = x \pm \epsilon$ and $\rho(\mathcal{H}_n) \ge \frac{t(t+1)}{(t+2)^2} - \epsilon$ for all $n \ge N_0$. It follows from Corollary 7.1 that there exists a set $Z_n \subset V(\mathcal{H}_n)$ of size $\epsilon^{1/2}n$ such that the 3-graph $\mathcal{H}'_n$ obtained from $\mathcal{H}_n$ by removing all edges that have nonempty intersection with Z_n is Γ_{t+2}-colorable with minimum degree at least $\delta(\mathcal{H}) \ge \left(\frac{t(t+1)}{2(t+2)^2} - 3\epsilon^{1/2}\right)n^2$. Fixing a Γ_{t+2}-coloring $V(\mathcal{H}'_n) = V_{n,1} \cup \cdots \cup V_{n,t+4}$ of $\mathcal{H}'_n$ and let $\mathcal{G}_n = \Gamma_{t+2}[V_{n,1} \cup \cdots \cup V_{n,t+4}]$ be the blowup of Γ_{t+2}.

Since $\rho(\mathcal{H}'_n) \ge \rho(\mathcal{H}_n) - |Z_\epsilon|n^2/\binom{n}{3} \ge \rho(\mathcal{H}_n) - 6\epsilon^{1/2} \ge \frac{t(t+1)}{(t+2)^2} - 7\epsilon^{1/2}$, it follows from Lemma 7.2 that

$$\frac{t+1}{t+2} - 2800t^4\epsilon^{1/4} \le \rho(\partial\mathcal{H}'_n) \le \rho(\partial\mathcal{G}_n) \le \frac{t^2+3t+3}{(t+2)^2} + 140000t^4\epsilon^{1/4}.$$

By Claim 6.12, for every $v \in Z_n$ we have $L_{\mathcal{H}_n}(v) \cap V_{n,i} = \emptyset$ [3] for all $i \in [t+4]$. Therefore, $\rho(\partial\mathcal{H}_n) \le \rho(\partial\mathcal{G}_n) + |Z_n|n/\binom{n}{2} \le \rho(\partial\mathcal{G}_n) + 3\epsilon^{1/2}$. So,

$$\frac{t+1}{t+2} - 2800t^4\epsilon^{1/4} \le \rho(\partial\mathcal{H}'_n) \le \rho(\partial\mathcal{H}_n) \le \rho(\partial\mathcal{G}_n) + 3\epsilon^{1/2} \le \frac{t^2+3t+3}{(t+2)^2} + 140001t^4\epsilon^{1/4}.$$

Consequently,

$$\frac{t+1}{t+2} - 2800t^4\epsilon^{1/4} - \epsilon \le x \le \frac{t^2+3t+3}{(t+2)^2} + 140001t^4\epsilon^{1/4} + \epsilon.$$

[3] Note that in the proof of Claim 6.12 we do not require $d_{\mathcal{H}_n}(v)$ to be large.

Letting $\epsilon \to 0$, we have

$$\frac{t+1}{t+2} \le x \le \frac{t^2+3t+3}{(t+2)^2}.$$

This completes the proof of Theorem 1.11. ∎

8 Concluding remarks

• We defined the crossed blowup and used it to construct finite family of triple systems with infinitely many extremal constructions. One could extended this operation in the following way.

Instead of replacing a pair of vertices (i.e. $\{v_1, v_2\}$) by four vertices (i.e. $\{v_1, v_1', v_2, v_2'\}$), one could replace it by 2^k vertices for $k \ge 2$.

Let Q_k denote the vertex set of the k-dimensional hypercube (each vertex is represented by a length-k binary string). For $i \in [k]$ let $Q_k(i,0) \subset Q_k$ and $Q_k(i,1) \subset Q_k$ denote the collection of vertices whose i-th coordinate is 0 and 1, respectively.

Definition 8.1 (k-crossed blowup). *Let $k \ge 2$ be an integer. Let $\mathcal{G}$ be a 3-graph and $\{v_1, v_2\} \subset V(\mathcal{H})$ be a pair with $d_{\mathcal{G}}(v_1, v_2) = d \ge k$. Assume that $N_{\mathcal{G}}(v_1, v_2) = \{u_1, \ldots, u_d\}$. The k-crossed blowup $\mathcal{G} \boxplus^k \{v_1, v_2\}$ of $\mathcal{G}$ on $\{v_1, v_2\}$ is obtained in the following way.*

(a) Remove all edges in $\mathcal{G}$ that contain $\{v_1, v_2\}$,

(b) replace $\{v_1, v_2\}$ by 2^k new vertices Q_k so that every vertex in $Q(1,0)$ is a clone of v_1 and every vertex in $Q(1,1)$ is a colon of v_2,

(c) for $i \in [k-1]$ we add a set $\mathcal{E}_i$ of triples containing u_i so that $L_{\mathcal{E}_i}(u_i)$ is the complete bipartite graph $B[Q(i,0), Q(i,1)]$ with two parts $Q(i,0), Q(i,1)$,

(d) for $i \in [k,d]$ we add a set $\mathcal{E}_i$ of triples containing u_i so that $L_{\mathcal{E}_i}(u_i)$ is the complete bipartite graph $B[Q(k,0), Q(k,1)]$ with two parts $Q(k,0), Q(k,1)$.

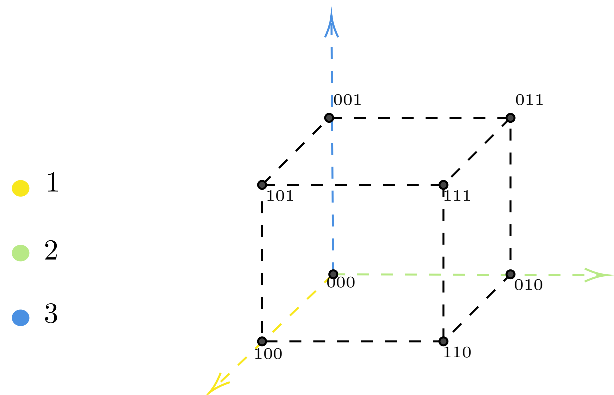

Figure 5: $\{145, 245, 345\} \boxplus^3 \{4,5\}$, in which $L(1)$ is the complete bipartite graph with parts $\{000, 001, 010, 011\}$ and $\{100, 101, 110, 111\}$ (these two squares are perpendicular to the yellow axis). Similarly to $L(2)$ and $L(3)$.

Notice that the crossed blowup we defined in Section 1 is 2-crossed blowup. Using a similar argument, one could extend Proposition 1.9 to k-crossed blowups for all $k \ge 2$.

There is also a natural extension to r-graphs for every $r \geq 4$. We omit the definition here.

• There are two ways to extend Theorem 1.3 to r-graphs for every $r \geq 4$. One is to use Γ_{t+2} to construct an r-graph in the following way: take $r-3$ new vertices $\{u_1, \ldots, u_{r-3}\}$ and let

$$\Gamma_{t+2}^r = \{\{u_1, \ldots, u_{r-3}\} \cup E \colon E \in \Gamma_{t+2}\} .$$

Following the argument as in [21] one could extend Theorem 1.3 to r-graphs.

Another way is to consider the crossed blowup of the r-uniform complete graph K_{t+2}^r. Following the argument as in the present paper one could extend Theorem 1.3 to r-graphs as well.

• For $t = 1$ one could use the fact that the set $Z(\Gamma_1)$ is a two-dimensional simplex

$$\{\alpha \vec{x}_1 + \beta \vec{x}_2 + (1-\alpha-\beta)\vec{x}_3 \colon \alpha \geq 0, \beta \geq 0, \alpha+\beta \leq 1\} ,$$

where $\vec{x}_1 = \left(\frac{1}{3}, 0, \frac{1}{3}, 0, 0, \frac{1}{3}\right)$, $\vec{x}_2 = \left(0, \frac{1}{3}, \frac{1}{3}, 0, 0, \frac{1}{3}\right)$, and $\vec{x}_3 = \left(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}, 0, \right)$, to improve Theorem 1.3 (b) to show that the number of maximum $\mathcal{F}_1$-free 3-graphs on n vertices is $\Theta(n^2)$.

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