Perfect Packings in Quasirandom Hypergraphs I

John Lenz *
University of Illinois at Chicago
lenz@math.uic.edu

Dhruv Mubayi †
University of Illinois at Chicago
mubayi@math.uic.edu

February 12, 2016

Abstract

Let $k \geq 2$ and F be a linear k-uniform hypergraph with v vertices. We prove that if n is sufficiently large and $v|n$, then every quasirandom k-uniform hypergraph on n vertices with constant edge density and minimum degree $\Omega(n^{k-1})$ admits a perfect F-packing. The case $k = 2$ follows immediately from the blowup lemma of Komlós, Sárközy, and Szemerédi. We also prove positive results for some nonlinear F but at the same time give counterexamples for rather simple F that are close to being linear. Finally, we address the case when the density tends to zero, and prove (in analogy with the graph case) that sparse quasirandom 3-uniform hypergraphs admit a perfect matching as long as their second largest eigenvalue is sufficiently smaller than the largest eigenvalue.

1 Introduction

A k-uniform hypergraph H (k-graph for short) is a collection of k-element subsets (edges) of a vertex set $V(H)$. For a k-graph H and a subset S of vertices of size at most $k-1$, let $d(S) = d_H(S)$ be the number of subsets of size $k - |S|$ that when added to S form a edge of H. The *minimum degree* of H, written $\delta(H)$, is the minimum of $d(\{s\})$ over all vertices s. The *minimum ℓ-degree* of H, written $\delta_\ell(H)$, is the minimum of $d(S)$ taken over all ℓ-sets of vertices. The *minimum codegree* of H is the minimum $(k-1)$-degree. Let K_t^k be the complete k-graph on t vertices.

Let G and F be k-graphs. We say that G has a *perfect F-packing* if there exists a collection of vertex-disjoint copies of F such that all vertices of G are covered. An important result of Hajnal and Szemerédi [9] states that if r divides n and the minimum degree of an n-vertex graph G is at least $(1-1/r)n$, then G has a perfect K_r-packing. Later Alon and Yuster [2] conjectured that a similar result holds for any graph F instead of just cliques,

*Research partly supported by NSA Grant H98230-13-1-0224.
†Research supported in part by NSF Grants 0969092 and 1300138.

with the minimum degree of G depending on the chromatic number of F. This was proved by Komlós-Sárközy-Szemerédi [19] by using the Regularity Lemma and Blow-up Lemma. Later, Kühn and Osthus [21] found the minimum degree threshold for perfect F-packings up to a constant; the threshold either comes from the chromatic number of F or the so-called critical chromatic number of F.

In the past decade there has been substantial interest in extending this result to k-graphs. Nevertheless, the simplest case of determining the minimum codegree threshold that guarantees a perfect matching was settled only recently by Rödl-Ruciński-Szemerédi [31]. Since then, there are a few results for codegree thresholds for packing other small 3-graphs [5, 14, 20, 27, 28, 30, 34, 35]. For ℓ-degrees with $\ell < k/2$ (in particular the minimum degree), much less is known. After work by many researchers [10, 11, 15, 16, 23, 22, 29], still only the degree threshold for K_3^3-packings, C_4^3-packings, and K_4^4-packings are known ($\frac{5}{9}$, $\frac{7}{16}$ and $\frac{37}{64}$ respectively). For $m \geq 5$ and $k \geq 4$ the packing degree threshold for K_m^k is open ([22] contains the current best bounds).

A key ingredient in the proofs of most of the above results are specially designed random-like properties of k-graphs that imply the existence of perfect F-packings. There is a rather well-defined notion of quasirandomness for graphs that originated in early work of Thomason [32, 33] and Chung-Graham-Wilson [3] which naturally generalizes to k-graphs. Our main focus in this paper is on understanding when perfect F-packings exist in quasirandom hypergraphs. The basic property that defines quasirandomness is uniform edge-distribution, and this extends naturally to hypergraphs. Let $v(H) = |V(H)|$.

Definition. *Let $k \geq 2$, let $0 < \mu, p < 1$, and let H be a k-graph. We say that H is (p,μ)-dense if for all $X_1, \ldots, X_k \subseteq H$,*

$$e(X_1, \ldots, X_k) \geq p|X_1| \cdots |X_k| - \mu n^k,$$

where $e(X_1, \ldots, X_k)$ is the number of $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that $\{x_1, \ldots, x_k\} \in H$ (note that if the X_is overlap an edge might be counted more than once). Say that H is an (n,p,μ) k-graph if H has n vertices and is (p,μ)-dense. Finally, if $0 < \alpha < 1$, then an (n,p,μ) k-graph is an (n,p,μ,α) k-graph if its minimum degree is at least $\alpha\binom{n}{k-1}$.

The F-packing problem for quasirandom graphs with constant density has been solved implicitly by Komlós-Sárközy-Szemerédi [18] in the course of developing the Blow-up Lemma.

Theorem 1. (Komlós-Sárközy-Szemerédi [18]) *Let $0 < \alpha, p < 1$ be fixed and let F be any graph. There exists an n_0 and $\mu > 0$ such that if H is any (n,p,μ,α) 2-graph where $n \geq n_0$, $v(F)|n$ then H has a perfect F-packing.*

Note that the condition on minimum degree is required, since if the condition "$\delta(H) \geq \alpha n$" in Theorem 1 is replaced by "$\delta(H) \geq f(n)$" for any choice of $f(n)$ with $f(n) = o(n)$, then there exists the following counterexample. Take the disjoint union of the random graph $G(n,p)$ and a clique of size either $\lceil f(n) \rceil + 1$ or $\lceil f(n) \rceil + 2$ depending on which is odd. The minimum degree is at least $f(n)$, there is no perfect matching, and the graph is still (p,μ)-dense. Because of the use of the regularity lemma, the constant n_0 in Theorem 1 is an exponential tower in μ^{-1}. We extend Theorem 1 to a variety of k-graphs. In the process, we also reduce the size of n_0 for all 2-graphs. A basic problem in this area that naturally emerges is the following.

Problem 2. *For which k-graphs F does the following hold: for all $0 < p, \alpha < 1$, there is some n_0 and μ so that if H is an (n, p, μ, α) k-graph with $n \geq n_0$ and $v(F)|n$, then H has a perfect F-packing.*

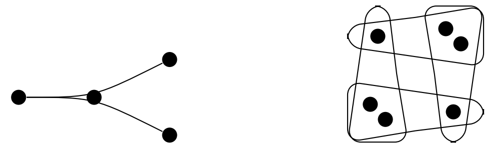
Unlike the graph case, most F will not satisfy Problem 2. Indeed, Rödl observed that for all $\mu > 0$ and there is an n_0 such that for $n \geq n_0$, an old construction of Erdős and Hajnal [6] produces an n-vertex 3-graph which is $(\frac{1}{4}, \mu)$-dense and has no copy of K_4^3. In a forthcoming paper we will show that a stronger notion of quasirandomness suffices to perfectly pack all F.

A hypergraph is *linear* if every two edges share at most one vertex. For a k-graph H, Kohayakawa-Nagle-Rödl-Schacht [17] recently proved an equivalence between $(|H|/\binom{n}{k}, \mu)$-dense and the fact that for each linear k-graph F, the number of labeled copies of F in H is the same as in the random graph with the same density. This leads naturally to the question of whether Problem 2 has a positive answer for linear k-graphs, and our first result shows that this is the case.

Theorem 3. *Let $k \geq 2$, $0 < \alpha, p < 1$, and let F be a linear k-graph. There exists an n_0 and $\mu > 0$ such that if H is an (n, p, μ, α) k-graph where $n \geq n_0$ and $v(F)|n$, then H has a perfect F-packing.*

We restrict our attention only to 3-graphs now although the concepts extend naturally to larger k. Define a 3-graph to be $(2+1)$-*linear* if its edges can be ordered as $e_1, \ldots, e_q$ such that each e_i has a partition $s_i \cup t_i$ with $|s_i| = 2, |t_i| = 1$ and for every $j < i$ we have $e_j \cap e_i \subseteq s_i$ or $e_j \cap e_i \subseteq t_i$. In words, every edge before e_i intersects e_i in a subset of s_i or of t_i. Clearly every linear 3-graph is $(2+1)$-linear, but the converse is false. Keevash's [12] recent proof of the existence of designs and our recent work on quasirandom properties of hypergraphs [25, 24, 26] use a quasirandom property distinct from (p, μ)-dense that Keevash calls *typical* and we call $(2+1)$-*quasirandom* (although the properties are essentially equivalent). These properties imply that the count of all $(2+1)$-linear 3-graphs in a typical 3-graph is the same as in the random 3-graph (see [25, 24]).

Thus a natural direction in which to extend Theorem 3 is to the family of $(2+1)$-linear 3-graphs and we begin this investigation with some of the smallest such 3-graphs. A *cherry* is the 3-graph comprising two edges that share precisely two vertices - this is the "simplest" non-linear hypergraph. A more complicated $(2+1)$-linear 3-graph is $C_4(2+1)$ which has vertex set $\{1, 2, 3, 4, a, b\}$ and edge set $\{12a, 12b, 34a, 34b\}$. The importance of $C_4(2+1)$ lies in the fact that $C_4(2+1)$ is forcing for the class of all $(2+1)$-linear 3-graphs. This means that if F is a $(2+1)$-linear 3-graph and $p, \epsilon > 0$ are fixed, there is n_0 and $\delta > 0$ so that if $n \geq n_0$ and H is an n-vertex 3-graph with $p\binom{n}{3}$ edges and $(1 \pm \delta)p^4 n^6$ labeled copies of $C_4(2+1)$, then the number of labeled copies of F in H is $(1 \pm \epsilon)p^{|F|}n^{v(F)}$ (see [25, 24]).



Cherry $C_4(2+1)$

Figure 1: Two 3-graphs

Theorem 4. *Let* $0 < \alpha, p < 1$*. There exists an* n_0 *and* $\mu > 0$ *such that if* H *is an* (n, p, μ, α) 3*-graph where* $n \geq n_0$*, then* H *has a perfect cherry-packing if* $4|n$ *and a perfect* $C_4(2+1)$ *packing if* $6|n$*.*

One might speculate that Theorem 4 can be extended to the collection of all $(2+1)$-linear F or to the collection of all 3-partite F. However, our next result shows that this is not the case and that solving Problem 2 will be a difficult project. If x is a vertex in a 3-graph H, the *link* of x is the graph with vertex set $V(H) \setminus \{x\}$ and edges those pairs who form an edge with x.

Theorem 5. *Let* F *be any* 3*-graph with an even number of vertices such that there exists a partition of the vertices of* F *into pairs such that each pair has a common edge in their links. Then for any* $\mu > 0$*, there exists an* n_0 *such that for all* $n \geq n_0$*, there exists a* 3*-graph* H *such that*

- $|H| = \frac{1}{8}\binom{n}{3} \pm \mu n^3$,
- H *is* $(\frac{1}{8}, \mu)$*-dense,*
- $\delta(H) \geq (\frac{1}{8} - \mu)\binom{n}{2}$,
- H *has no perfect* F*-packing.*

Two examples of 3-graphs F that satisfy the conditions of Theorem 5 are the complete 3-partite 3-graph $K_{2,2,2}$ with parts of size two and the following $(2+1)$-linear hypergraph. A *cherry 4-cycle* is the $(2+1)$-linear 3-graph with edge set $\{123, 124, 345, 346, 567, 568, 781, 782\}$.

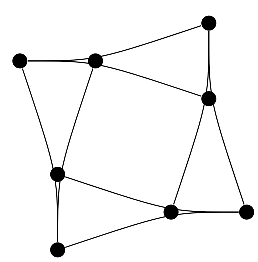


Figure 2: cherry 4-cycle

It is straightforward to see that Theorem 5 applies to the cherry 4-cycle. Therefore one cannot hope that Theorem 3 holds for all $(2+1)$-linear or 3-partite F.

Our final result considers the situation when the density is not fixed and goes to zero. Here the notion of quasirandom is measured by spectral gap. It is a folklore result that large spectral gap guarantees a perfect matching in graphs. For hypergraphs, there are several definitions of eigenvalues. We will use the definitions that originated in the work of Friedman and Wigderson [7, 8] for regular hypergraphs. The definition for all hypergraphs can be found in [25, Section 3] where we specialize to $\pi = 1 + \cdots + 1$. That is, let $\lambda_1(H) = \lambda_{1,1+\cdots+1}(H)$ and let $\lambda_2(H) = \lambda_{2,1+\cdots+1}(H)$, where both $\lambda_{1,1+\cdots+1}(H)$ and $\lambda_{2,1+\cdots+1}(H)$ are as defined in Section 3 of [25]. The only result about eigenvalues that we will require is Proposition 24, which is usually called the Expander Mixing Lemma [25, Theorem 4] (see also [7, 8]).

Theorem 6. *For every $\alpha > 0$, there exists n_0 and $\gamma > 0$ depending only on α such that the following holds. Let H be an n-vertex 3-graph where $3|n$ and $n \geq n_0$. Let $p = 6|H|/n^3$ and assume that $\delta_2(H) \geq \alpha pn$ and*

$$\lambda_2(H) \leq \gamma p^{16} n^{3/2}.$$

Then H contains a perfect matching.

Let $\Delta_2(H)$ be the *maximum codegree of a 3-graph H*, i.e. the maximum of $d(S)$ over all 2-sets $S \subseteq V(H)$. If $\Delta_2(H) \leq cpn$ then $\lambda_1(H) \leq c' pn^{3/2}$ where c' is a constant depending only on c. This implies the following corollary.

Corollary 7. *For every $\alpha > 0$, there exists n_0 and $\gamma > 0$ depending only on α such that the following holds. Let H be an n-vertex 3-graph where $3|n$ and $n \geq n_0$. Let $p = |H|/\binom{n}{3}$ and assume that $\delta_2(H) \geq \alpha pn$, $\Delta_2(H) \leq \frac{1}{\alpha} pn$, and*

$$\lambda_2(H) \leq \gamma p^{15} \lambda_1(H).$$

Then H contains a perfect matching.

The third largest eigenvalue of a graph is closely related to its matching number (see e.g. [4]), but currently we do not know the "correct" definition of λ_3 for hypergraphs. It would be interesting to discover a definition of λ_3 for k-graphs which extends the graph definition and for which a bound on λ_3 forces a perfect matching.

The remainder of this paper is organized as follows. In Section 2 we will develop the tools neccisary for our proofs, including extensions of the absorbing technique and various embedding lemmas. Then in Section 3 we will use these to prove Theorem 3 (Section 3.3) and Theorem 4 (Sections 3.1 and 3.2). Section 4 contains the construction proving Theorem 5 and Section 5 has the proof of the sparse case, Theorem 6.

2 Tools

In this section, we state and prove several lemmas and propositions that we will need; our main tool is the absorbing technique of Rödl-Ruciński-Szemerédi [31].

Definition. *Let F and H be k-graphs and let $A, B \subseteq V(H)$. We say that A* F-absorbs B *or that A is an* F-absorbing set *for B if both $H[A]$ and $H[A \cup B]$ have perfect F-packings. When F is a single edge, we say that A* edge-absorbs B.

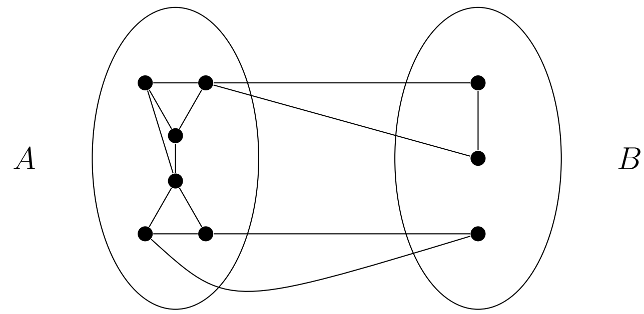


Figure 3: A K_3-absorbs B

Definition. *Let F and H be k-graphs, $\epsilon > 0$, a and b be multiples of $v(F)$, $\mathcal{A} \subseteq \binom{V(H)}{a}$, and $\mathcal{B} \subseteq \binom{V(H)}{b}$. We say that H is* $(\mathcal{A}, \mathcal{B}, \epsilon, F)$-rich *if for all $B \in \mathcal{B}$ there are at least ϵn^a sets in $\mathcal{A}$ which F-absorb B. If $\mathcal{A} = \binom{V(H)}{a}$, we abbreviate this to $(a, \mathcal{B}, \epsilon, F)$-rich and if both $\mathcal{A} = \binom{V(H)}{a}$ and $\mathcal{B} = \binom{V(H)}{b}$, we abbreviate this to (a, b, ϵ, F)-rich.*

The following proposition is one of the main results of this section; the proof appears in Section 2.3.

Proposition 8. *Fix $0 < p < 1$, let F be a k-graph such that F is either linear or k-partite, and let a and b be multiples of $v(F)$. For any $\epsilon > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. If H is an (a, b, ϵ, F)-rich, (n, p, μ) k-graph where $v(F)|n$, then H has a perfect F-packing.*

The proof of Proposition 8 appears in Section 2.3. For Theorem 4, we will need a slight extension of Proposition 8.

Definition. *Let $\zeta > 0$, t be any integer, H be a 3-graph, and $B \subseteq V(H)$ with $|B| = 2t$. We say that B is* ζ-separable *if there exists a partition of B into $B_1, \ldots, B_t$ such that for all i $|B_i| = 2$ and $d_H(B_i) \geq \zeta n$. Set*

$$\mathcal{B}_{\zeta,b}(H) := \left\{ B \in \binom{V(H)}{b} : B \text{ is } \zeta\text{-separable} \right\}.$$

If H is obvious from context, we will denote this by $\mathcal{B}_{\zeta,b}$.

The main result required for the proof of Theorem 4 is that the property (a, b, ϵ, F)-rich can be replaced by $(a, \mathcal{B}_{\zeta,b}, \epsilon, F)$-rich in Proposition 8.

Proposition 9. *Fix $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. Let F be a 3-graph such that F is either linear or k-partite, let $v(F)|a$, and let $v(F)|b$ where in addition b is even. For any $\epsilon > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. If H is an $(a, \mathcal{B}_{\zeta,b}, \epsilon, F)$-rich (n, p, μ, α) 3-graph where $v(F)|n$, then H has a perfect F-packing.*

The proof of Proposition 9 is in Section 2.4. Note that if b is even, H is a 3-graph, and $\delta(H) \geq \alpha\binom{n}{2}$, then Proposition 9 implies Proposition 8. The proofs of Propositions 8 and 9 use the absorbing technique of Rödl-Ruciński-Szemerédi [31]. The two key ingredients are the Absorbing Lemma (Lemma 10) and the Embedding Lemmas (Lemma 11 for linear and Lemma 13 for k-partite). The remainder of this section contains the statements and proofs of these lemmas plus the proofs of both propositions.

2.1 Absorbing Sets

Rödl-Ruciński-Szemerédi [31, Fact 2.3] have a slightly different definition of edge-absorbing where B has size $k + 1$ and one vertex of A is left out of the perfect matching, but the main idea transfers to our setting in a straightforward way as follows. If H is a k-graph, $A \subseteq V(H)$, and $\mathcal{A} \subseteq 2^{V(H)}$, then we say that A *partitions into sets from* $\mathcal{A}$ if there exists a partition $A = A_1 \dot\cup \cdots \dot\cup A_t$ such that $A_i \in \mathcal{A}$ for all i.

Lemma 10. *(Absorbing Lemma) Let F be a k-graph, $\epsilon > 0$, and a and b be multiples of $v(F)$. There exists an n_0 and $\omega > 0$ such that for all n-vertex k-graphs H with $n \geq n_0$, the following holds. If H is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$-rich for some $\mathcal{A} \subseteq \binom{V(H)}{a}$ and $\mathcal{B} \subseteq \binom{V(H)}{b}$, then there exists an $A \subseteq V(H)$ such that A partitions into sets from $\mathcal{A}$ and A F-absorbs all sets C satisfying the following conditions: $C \subseteq V(H) \setminus A$, $|C| \leq \omega n$, and C partitions into sets from $\mathcal{B}$.*

Using the idea of Rödl-Ruciński-Szemerédi [31], Treglown and Zhao [34, Lemma 5.2] proved the above lemma for F a single edge, $a = 2k$, $b = k$, $\mathcal{A} = \binom{V(H)}{a}$ and $\mathcal{B} = \binom{V(H)}{b}$. For the sparse case (Theorem 6) we require a stronger version of Lemma 10 and so a proof of Lemma 10 appears in Section 5 (as a corollary of Lemma 23).

2.2 Embedding Lemmas and Almost Perfect Packings

This section contains embedding lemmas for linear and k-partite k-graphs and a simple corollary of these lemmas which produces a perfect F-packing covering almost all of the vertices.

Definition. *Let F and H be k-graphs with $V(F) = \{w_1, \ldots, w_f\}$. A* labeled copy *of F in H is an edge-preserving injection from $V(F)$ to $V(H)$. A* degenerate labeled copy *of F in H is an edge-preserving map from $V(F)$ to $V(H)$ that is not an injection. Let $1 \leq m \leq f$ and let $Z_1, \ldots, Z_m \subseteq V(H)$. Set inj$[F \to H; w_1 \to Z_1, \ldots, w_m \to Z_m]$ to be the number of edge-preserving injections $\psi : V(F) \to V(H)$ such that $\psi(w_i) \in Z_i$ for all $1 \leq i \leq m$. In other words, inj$[F \to H; w_1 \to Z_1, \ldots, w_m \to Z_m]$ is the number of labeled copies of F in*

H where w_i is mapped into Z_i for all $1 \leq i \leq m$. If $Z_i = \{z_i\}$, we abbreviate $w_i \to \{z_i\}$ as $w_i \to z_i$.

Lemma 11. *Let $0 < p, \alpha < 1$ and let F be a linear k-graph where $0 \leq m \leq v(F)$ and $V(F) = \{s_1, \dots, s_m, t_{m+1}, \dots, t_f\}$ such that there does not exist $E \in F$ with $|E \cap \{s_1, \dots, s_m\}| > 1$ and there do not exist $E_1, E_2 \in F$ with $|E_1 \cap \{s_1, \dots, s_m\}| = 1$, $|E_2 \cap \{s_1, \dots, s_m\}| = 1$, and $E_1 \cap E_2 \cap \{t_{m+1}, \dots, t_f\} \neq \emptyset$.*

For every $\gamma > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. Let H be an n-vertex k-graph with $n \geq n_0$ and let $y_1, \dots, y_m \in V(H), Z_{m+1} \subseteq V(H), \dots, Z_f \subseteq V(H)$. Assume that for every $\{s_i, t_{j_2}, \dots, t_{j_k}\} \in F$

$$\left| \left\{ (z_{j_2}, \dots, z_{j_k}) \in Z_{j_2} \times \cdots \times Z_{j_k} : \{y_i, z_{j_2}, \dots, z_{j_k}\} \in H \right\} \right| \geq \alpha |Z_{j_2}| \cdots |Z_{j_k}| \tag{1}$$

and for every $\{t_{i_1}, \dots, t_{i_k}\} \in F$ and every $Z'_{i_1} \subseteq Z_{i_1}, \dots, Z'_{i_k} \subseteq Z_{i_k}$,

$$e(Z'_{i_1}, \dots, Z'_{i_k}) \geq p|Z_{i_1}| \cdots |Z_{i_k}| - \mu n^k. \tag{2}$$

Then

$$\begin{aligned} inj[F \to H; s_1 \to y_1, \dots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \dots, t_f \to Z_f] \\ \geq \alpha^{d_F(s_1)} \cdots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)} |Z_{m+1}| \cdots |Z_f| - \gamma n^{f-m}. \end{aligned}$$

Proof. Kohayakawa, Nagle, Rödl, and Schacht [17] proved this when $Z_i = V(H)$ for all i, without the distinguished vertices $s_1, \dots, s_m$, and under a stronger condition on H, but it is straightforward to extend their proof to our setup as follows. The lemma is proved by induction on number of edges of F which do not contain any vertex from among $s_1, \dots, s_m$. Let $\mu = (1-p)\gamma$.

First, if every edge of F contains some s_i then F is a vertex disjoint union of stars with centers $s_1, \dots, s_m$ plus some isolated vertices. Therefore, we can form a copy of F of the type we are trying to count by picking an edge of H containing y_i (of the right type) for each edge of F. More precisely, using (1), the fact that all edges of F which use some $s_1, \dots, s_m$ (so all edges of F) do not share any vertices from among $t_{m+1}, \dots, t_f$, and the fact that F is linear, the number of labeled copies of F with $s_i \to y_i$ and $t_j \to Z_j$ is at least

$$\alpha^{|F|} |Z_{m+1}| \cdots |Z_f| = \alpha^{\sum d_F(s_i)} p^0 |Z_{m+1}| \cdots |Z_f|.$$

The proof of the base case is complete.

Now assume F has at least one edge E which does not contain any s_i, with vertices labeled so that $E = \{t_{m+1}, \dots, t_{m+k}\}$. Let F_* be the hypergraph formed by deleting all vertices of E from F and notice that $s_i \in V(F_*)$ for all i. Let F_- be the hypergraph formed by removing the edge E from F but keeping the same vertex set. Let Q_* be an injective edge-preserving map $Q_* : V(F_*) \to V(H)$ where $Q_*(s_i) = y_i$ for $1 \leq i \leq m$ and $Q_*(t_j) \in Z_j$ for $m+1 \leq j \leq f$. For $m+1 \leq j \leq m+k$, define $S_j(Q_*) \subseteq Z_j$ as follows. For each $z \in Z_j$, add z to $S_j(Q_*)$ if $z \notin Im(Q_*)$ and there exists an edge-preserving injection $V(F_*) \cup \{t_j\} \to Im(Q_*) \cup \{z\}$

which when restricted to $V(F_*)$ matches the map Q_*. More informally, $S_j(Q_*)$ consists of all vertices which can be used to extend Q_* to embed a labeled copy of $F_* \cup \{t_j\}$.

By definition, every edge counted by $e(S_{m+1}(Q_*), \ldots, S_{m+k}(Q_*))$ creates a labeled copy of F. Also, every ordered tuple from $S_{m+1}(Q_*) \times \cdots \times S_{m+k}(Q_*)$ creates a labeled copy of F_-. More precisely,

$$\begin{aligned}
\operatorname{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, &t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f] \\
&= \sum_{Q_*} e(S_{m+1}(Q_*), \ldots, S_{m+k}(Q_*)) \\
\operatorname{inj}[F_- \to H; s_1 \to y_1, \ldots, s_m \to y_m, &t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f] \\
&= \sum_{Q_*} |S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)|. \qquad (3)
\end{aligned}$$

For each j, $S_j(Q_*) \subseteq Z_j$ so that (2) implies that

$$\begin{aligned}
\operatorname{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, &t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f] \\
&\geq \sum_{Q_*} \left(p|S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)| - \mu n^k\right) \\
&\geq p \sum_{Q_*} |S_{m+1}(Q_*)| \cdots |S_{m+k}(Q_*)| - \mu n^{f-m}, \qquad (4)
\end{aligned}$$

where the last inequality is because there are at most n^{f-m-k} maps Q_*, since F_* has $f-k$ vertices and $s_i \in V(F_*)$ must map to y_i. Combining (3) and (4) and then applying induction,

$$\begin{aligned}
\operatorname{inj}[F \to H; &s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f] \\
&\geq p \operatorname{inj}[F_- \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f] - \mu n^{f-m} \\
&\geq p \left(\alpha^{\sum d(s_i)} p^{|F|-1-\sum d(s_i)} |Z_{m+1}| \cdots |Z_f| - \gamma n^{f-m}\right) - \mu n^{f-m}.
\end{aligned}$$

Since $\mu = (1-p)\gamma$, the proof is complete. □

Corollary 12. *Let $0 < p < 1$ and let F be a linear k-graph with $V(F) = \{t_1, \ldots, t_f\}$. For every $\gamma > 0$, there exists an n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ) k-graph and let $Z_1 \ldots, Z_f \subseteq V(H)$. Then*

$$\mathit{inj}[F \to H; t_1 \to Z_1, \ldots, t_f \to Z_f] \geq p^{|F|} |Z_1| \cdots |Z_f| - \gamma n^f.$$

Proof. Apply Lemma 11 with $m = 0$. Since H is (p, μ)-dense, (2) holds. Also, (1) is vacuous since $m = 0$. □

Lemma 13. *Let $0 < p < 1$ and let $K_{t_1,\ldots,t_k}$ be the complete k-partite, k-graph with part sizes $t_1, \ldots, t_k$ and parts labeled by $T_1, \ldots, T_k$. For every $0 < \mu < \frac{p}{2}$, there exists n_0 and $0 < \xi < 1$ such that the following holds. Let H be an (n, p, μ) k-graph with $n \geq n_0$. Then for any $X_1, \ldots, X_k \subseteq V(H)$ with $|X_j| \geq (2\mu/p)^{1/k} n$ for all j, the number of labeled copies of $K_{t_1,\ldots,t_k}$ in H with $T_i \subseteq X_i$ for all i is at least $\xi \prod |X_i|^{t_i}$.*

Proof. Let H' be the k-graph on $\sum |X_i|$ vertices with vertex set $Y_1 \dot\cup \cdots \dot\cup Y_t$ where the sets Y_i are disjoint and $Y_i \cong X_i$ for all i. Note that because the sets X_i might overlap, a vertex of H might appear more than once in H'. Make $y_1 \in Y_1, \ldots, y_k \in Y_k$ a hyperedge of H' if $y_1, \ldots, y_k$ are distinct vertices of H and $\{y_1, \ldots, y_k\} \in H$. Let $t = \sum t_i$. Since H is (p, μ)-dense,

$$e(H') = e_H(X_1, \ldots, X_k) \geq p \prod_i |X_i| - \mu n^k \geq p \left(\frac{2\mu}{p}\right) n^k - \mu n^k = \mu n^k \geq \frac{\mu}{k^k} v(H')^k.$$

Therefore, by supersaturation (see [13, Theorems 2.1 and 2.2]), there exists an n_0' and $\xi' > 0$ such that if $v(H') \geq n_0'$ then H' contains at least $\xi' v(H')^t$ labeled copies of $K_{t_1,\ldots,t_k}$. Each of these labeled copies of $K_{t_1,\ldots,k_t}$ in H' produces a possibly degenerate labeled copy of $K_{t_1,\ldots,t_k}$ in H where $T_i \subseteq X_i$ for all i. Pick $\xi = \frac{1}{2}\xi'$, $n_0 \geq n_0'(p/2\mu)^{1/k}$, and $n_0 \geq \frac{1}{\xi}(p/2\mu)^{t/k}$.

Now assume that $n \geq n_0$. This implies that $v(H') \geq |X_1| \geq (2\mu/p)^{1/k} n \geq n_0'$ so that there are at least $\xi' v(H')^t$ labeled copies of $K_{t_1,\ldots,t_k}$ in H'. Therefore, the number of possibly degenerate labeled copies of $K_{t_1,\ldots,t_k}$ in H with $T_i \subseteq X_i$ for all i is at least

$$\xi' v(H')^t = \xi' \prod_i v(H')^{t_i} \geq \xi' \prod_i |X_i|^{t_i} = 2\xi \prod_i |X_i|^{t_i}. \tag{5}$$

Since there are at most n^{t-1} degenerate labeled copies, by the choice of n_0 and since $|X_i| \geq (2\mu/p)^{1/k} n$ for all i, the number of degenerate labeled copies is at most

$$n^{t-1} = \frac{1}{n}\left(\frac{p}{2\mu}\right)^{t/k} \prod_i \left[\left(\frac{2\mu}{p}\right)^{1/k} n\right]^{t_i} \leq \frac{1}{n}\left(\frac{p}{2\mu}\right)^{t/k} \prod_i |X_i|^{t_i} \leq \xi \prod_i |X_i|^{t_i}. \tag{6}$$

Combining (5) with (6) shows that there are at least $\xi \prod_i |X_i|^{t_i}$ labeled copies of $K_{t_1,\ldots,t_k}$ with $T_i \subseteq X_i$ for all i, completing the proof. □

With these lemmas in hand, we can prove that if H is (p, μ)-dense and F is linear or k-partite, then H has an F-packing covering almost all the vertices of H.

Lemma 14. *(Almost Perfect Packing Lemma) Fix $0 < p < 1$ and a k-graph F with f vertices such that F is either linear or k-partite. Let $v(F)|b$. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ) k-graph with $n \geq n_0$ and $f|n$. Then there exists $C \subseteq V(H)$ such that $|C| \leq \omega n$, $b||C|$, and $H[\bar{C}]$ has a perfect F-packing.*

Proof. First, select n_0 large enough and μ small enough so that any vertex set C of size $\lceil \frac{\omega}{2} \rceil$ contains a copy of F. To see that this is possible, there are two cases to consider.

If F is linear, let $\gamma = \frac{1}{2}p^{|F|}(\frac{\omega}{2})^f$ and select n_0 and $\mu > 0$ according to Corollary 12. Now if $C \subseteq V(H)$ with $|C| \geq \frac{\omega}{2}n$, then Corollary 12 implies there are at least $p^{|F|}|C|^f - \gamma n^f \geq p^{|F|}\left(\frac{\omega}{2}\right)^f n^f - \gamma n^f = \gamma n^f > 0$ copies of F inside C.

If F is k-partite, then Lemma 13 is used in a similar way as follows. Let $\mu = \frac{p}{2}\left(\frac{\omega}{2}\right)^k$ and select n_0 and ξ according to Lemma 13. Now by the choice of μ, if $|C| \geq \frac{\omega}{2}$ then $|C| \geq (2\mu/p)^{1/k}n$ so that by Lemma 13, C contains at least $\xi(\frac{\omega}{2})^f n^f > 0$ copies of F.

Now let $F_1, \ldots, F_t$ be a greedily constructed F-packing. That is, $F_1, \ldots, F_t$ are disjoint copies of F and $C := V(H) \setminus V(F_1) \setminus \cdots \setminus V(F_t)$ has no copy of F. By the previous two paragraphs, $|C| \leq \frac{\omega}{2}n$. Since $f|n$ and $H[\bar{C}]$ has a perfect F-packing, $f||C|$. Thus we can let $y \equiv -\frac{|C|}{f} \pmod{b}$ with $0 \leq y < b$ and take y of the copies of F in the F-packing of $H[\bar{C}]$ and add their vertices into C so that $b||C|$. □

2.3 Proof of Proposition 8

Proof of Proposition 8. First, select $\omega > 0$ according to Lemma 10 and $\mu_1 > 0$ accoding to Lemma 14. Also, make n_0 large enough so that both Lemma 10 and 14 can be applied. Let $\mu = \mu_1\omega^k$. All the parameters have now been chosen.

By Lemma 10, there exists a set $A \subseteq V(H)$ such that A F-absorbs C for all $C \subseteq V(H)\setminus A$ with $|C| \leq \omega n$ and $b \mid |C|$. If $|A| \geq (1-\omega)n$, then A F-absorbs $V(H) \setminus A$ so that H has a perfect F-packing. Thus $|A| \leq (1-\omega)n$. Next, let $H' := H[\bar{A}]$ and notice that H' is (p, μ_1)-dense since $v(H') \geq \omega n$ and

$$\mu n^k \leq \frac{\mu}{\omega^k} v(H')^k = \mu_1 v(H')^k.$$

Therefore, by Lemma 14, there exists a vertex set $C \subseteq V(H') = V(H)\setminus A$ such that $|C| \leq \omega n$, $|C|$ is a multiple of b, and $H'[\bar{C}]$ has a perfect F-packing. Now Lemma 10 implies that A F-absorbs C. The perfect F-packing of $A \cup C$ and the perfect F-packing of $H'[\bar{C}]$ produces a perfect F-packing of H. □

2.4 Proof of Proposition 9

This section contains the proof of Proposition 9, but first we need an extension of Lemma 14 that produces a perfect F-packing covering almost all the vertices where in addition the unsaturated vertices are ζ-separable. To do so, we need a well-known probability lemma.

Lemma 15. ***(Chernoff Bound)*** *Let $0 < p < 1$, let $X_1, \ldots, X_n$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all i, and let $X = \sum X_i$. Then for all $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| > a] \leq 2e^{-a^2/2n}.$$

Lemma 16. *Fix $p, \alpha \in (0,1)$, $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$, and a 3-graph F such that either F is linear or F is 3-partite. Let $v(F)|b$ where in addition b is even. For any $0 < \omega < 1$, there exists n_0 and $\mu > 0$ such that the following holds. Let H be an (n, p, μ, α) 3-graph with $n \geq n_0$ and $v(F)|n$. Then there exists a set $C \subseteq V(H)$ such that $|C| \leq \omega n$, C partitions into sets of $\mathcal{B}_{\zeta,b}$, and $H[\bar{C}]$ has a perfect F-packing.*

Proof. Use Lemma 14 to select n_0 and $\mu_1 > 0$ to produce an F-packing $F_1, \dots, F_t$ where $W := V(H) \setminus V(F_1) \setminus \cdots \setminus V(F_t)$ is such that $|W| \leq \frac{\omega\alpha}{4}n$. Let $f = v(F)$ and let

$$\phi = \min\left\{\frac{\omega}{8}, \frac{\alpha}{4}, \frac{p}{4}\right\},$$
$$\mu = \min\left\{\frac{p}{2}\left(\frac{\alpha\phi}{16f}\right)^2, \mu_1\right\}.$$

First, form a vertex set C' by starting with W and for each $1 \leq i \leq t$, add $V(F_i)$ to C' with probability ϕ independently. After this, take $\frac{b}{f} - \frac{|C'|}{f} \pmod{\frac{b}{f}}$ of the unselected copies of F and add their vertices into C' to form the vertex set C.

By construction, $H[\bar{C}]$ has a perfect F-packing (the copies of F which were not selected) and $b||C|$. Since b is even, $|C|$ is also even. So to complete the proof, we just need to show that with positive probability, C is ζ-separable and $|C| \leq \omega n$. (Note that if C is ζ-separable then it can be partitioned into sets from $\mathcal{B}_{\zeta,b}$.)

Let G be the graph where $V(G) = V(H)$ and for every $Z \in \binom{V(G)}{2}$, Z is an edge of G if $d_H(Z) \geq \zeta n$, i.e. the codegree of Z in H is at least ζn. We will now prove that with positive probability, the following two events occur:

- $|C| \leq \frac{1}{2}\omega n$,
- $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$.

First, the expected number of vertices added to W to form C is $\phi f t \leq \frac{\omega}{8}n$ plus potentially a few copies of F to make $b||C|$. By the second moment method, with probability at least $\frac{1}{4}$, at most $\frac{\omega}{4}n$ vertices are added to W so that $|C| \leq \frac{1}{2}\omega n$. Secondly, since $\delta(H) \geq \alpha n^2$ it is the case that $\delta(G) \geq \frac{\alpha}{2}n$. Indeed, if there was some vertex x with $d_G(x) < \frac{\alpha}{2}n$, then $d_H(x) \leq |N_G(x)| \cdot n + n \cdot \zeta n < (\frac{\alpha}{2} + \zeta)n^2$, a contradiction to the fact that $\delta(H) \geq \alpha n^2$ and $\zeta \leq \frac{\alpha}{2}$. Since $|W| \leq \frac{\omega\alpha}{4}n$, we have that any vertex x has at least $\frac{\alpha}{2}n - \frac{\omega\alpha}{4}n > \frac{\alpha}{4}n$ neighbors in G outside W. Since each F_i has size f, the vertex x therefore has a neighbor in G inside at least $\frac{\alpha}{4f}n$ of the copies of F. Therefore, the expected size of $\{y \in C : xy \in E(G)\}$ is at least $\frac{\alpha\phi}{4f}n$ and by Chernoff's Inequality (Lemma 15),

$$\mathbb{P}\left[|\{y \in C : xy \in E(G)\}| < \frac{\alpha\phi}{8f}n\right] \leq e^{-cn}$$

for some constant c. Thus n_0 can be selected large enough so that with probability at most $\frac{1}{4}$, there is some $x \in V(G)$ such that $|\{y \in C : xy \in E(G)\}| < \frac{\alpha\phi}{8f}n$. This implies that with probability at least $\frac{1}{2}$, $|C| \leq \omega n$ and $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$.

To complete the proof, we will show that $\delta(G[C]) \geq \frac{\alpha\phi}{8f}n$ implies that $G[C]$ has a perfect matching (which is equivalent to C being ζ-separable). Divide C into two equal sized parts C_1 and C_2 (recall that $|C|$ is even since b is even and $b||C|$. Assume towards a contradiction that Hall's Condition fails in $G[C_1, C_2]$, i.e. there exists a set $T \subseteq C_1$ such that $|N_G(T) \cap C_2| < |T|$.

In a slight abuse of notation, let $\bar{T} = C_1 \setminus T$. Now $|T| \geq \frac{\alpha\phi}{16f}n$ since $\delta(G[C_1, C_2]) \geq \frac{\alpha\phi}{16f}n$. Similarly, $|\bar{T}| \geq \frac{\alpha\phi}{16f}n$ since if $z \in C_2 \setminus N_G(T)$ then $N_G(z) \cap C_1 \subseteq \bar{T}$. This implies that

$$|C_2 \setminus N_G(T)| = |C_2| - |N_G(T)| = |C_1| - |N_G(T)| > |C_1| - |T| = |\bar{T}| \geq \frac{\alpha\phi n}{16f}.$$

Since there are no edges of G between T and $C_2 \setminus N_G(T)$,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \leq |T| \cdot |C_2 \setminus N_G(T)| \cdot \zeta n = \zeta|T||C_2 \setminus N_G(T)|n. \tag{7}$$

On the other hand, since H is (p, μ)-dense,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq p|T||C_2 \setminus N_G(T)|n - \mu n^3.$$

Since $|T|$ and $|C_2 \setminus N_G(T)|$ are both larger than $\frac{\alpha\phi}{16f}n$,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \left(p - \mu\left(\frac{16f}{\alpha\phi}\right)^2\right)|T||C_2 \setminus N_G(T)|n$$

Since $\mu \leq \frac{p}{2}(\frac{\alpha\phi}{16f})^2$, we have

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \frac{p}{2}|T||C_2 \setminus N_G(T)|n \tag{8}$$

Since $\zeta < \frac{p}{2}$, (8) contradicts (7). Therefore, $G[C_1, C_2]$ satisfies Hall's condition so that $G[C]$ has a perfect matching, i.e. C is ζ-separable. □

Proof of Proposition 9. The proof is similar to the proof of Proposition 8 except Lemma 16 is used instead of Lemma 14. □

3 Rich hypergraphs

This section contains the proofs of Theorems 3 and 4. By the previous section, these proofs come down to showing that (p, μ)-dense and large minimum degree imply either (a, b, ϵ, F)-rich or $(a, \mathcal{B}_{\zeta,b}, \epsilon, F)$-rich, where we get to select a, b, and ϵ but $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. As a warm-up before Theorem 3 (see Section 3.3), we start with the cherry.

3.1 Packing Cherries

Let $K_{1,1,2}$ be the cherry.

Lemma 17. *Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. There exists an n_0, $\epsilon > 0$, and $\mu > 0$ such that if H is an (n, p, μ, α) 3-graph with $n \geq n_0$, then H is $(4, \mathcal{B}_{\zeta,4}, \epsilon, K_{1,1,2})$-rich.*

Proof. Our main task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \mathcal{B}_{\zeta,4}$, there are at least ϵn^4 vertex sets of size four which $K_{1,1,2}$-absorb B; we will define ϵ and μ later.

Fix $B = \{b_1, b_2, b_3, b_4\} \in \mathcal{B}_{\zeta,4}$, labeled so that $d_H(b_1, b_2) \geq \zeta n$ and $d_H(b_3, b_4) \geq \zeta n$. Let $X_1 = N(b_1, b_2) = \{x : xb_1b_2 \in E(H)\} \subseteq V(H)$ and $X_2 = N(b_3, b_4)$ and notice that $|X_1|, |X_2| \geq \zeta n$. Arbitrarily divide X_1 in half and call the two parts Y_1 and Y_2. Let $\mu = \frac{p}{2}(\frac{\zeta}{2})^3$. Since $|Y_1|, |Y_2|, |X_2| \geq \frac{\zeta}{2}n = (2\mu/p)^{1/3}n$, by Lemma 13 there exists a $\xi > 0$ and n_0 such that $H[Y_1, Y_2, X_2]$ contains at least $\xi(\frac{\zeta n}{2})^4$ copies of $K_{1,1,2}$ with one degree two vertex in each of Y_1 and Y_2 and the degree one vertices in X_2. The proof is now complete, since each of these cherries absorbs B. Indeed, let $\epsilon = \xi(\frac{\zeta}{2})^4$ and let $y_1 \in Y_1$, $y_2 \in Y_2$, and $x_1, x_2 \in X_2$ be such that $y_1y_2x_1, y_1y_2x_2 \in E(H)$. Then $A = \{y_1, y_2, x_1, x_2\}$ $K_{1,1,2}$-absorbs B because $b_1b_2y_1, b_1b_2y_2 \in E(H)$ (recall that $Y_1, Y_2 \subseteq N(b_1, b_2)$) and similarly $b_3b_4x_1, b_3b_4x_2 \in E(H)$. Since there are at least ϵn^4 choices for y_1, y_2, x_1, x_2, the proof is complete. $\square$

3.2 Packing Cycles

Throughout this section, let C_4 denote the hypergraph $C_4(2+1)$. This section completes the proof of Theorem 4.

Lemma 18. *Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{4}\}$. There exists an n_0, $\epsilon > 0$, and $\mu > 0$ such that if H is a (n, p, μ, α) 3-graph with $n \geq n_0$, then H is $(18, \mathcal{B}_{\zeta,6}, \epsilon, C_4)$-rich.*

Proof. Similar to the proof of Lemma 17, our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \mathcal{B}_{\zeta,6}$, there are at least ϵn^{18} vertex sets of size eighteen which C_4-absorb B; we will define ϵ and μ later.

Fix $B = \{b_1, b_1', b_2, b_2', b_3, b_3'\} \in \mathcal{B}_{\zeta,6}$ labeled so that $d_H(b_i, b_i') \geq \zeta n$ for all i. For $1 \leq i \leq 3$, let $X_i = N(b_i, b_i')$ and note that $|X_i| \geq \zeta n$. Now for each $1 \leq i \leq 3$, define

$$R_i = \left\{ \{r_1, r_2\} \in \binom{V(H)}{2} : |N(r_1, r_2) \cap X_i| \geq \frac{1}{10} p \zeta n \right\}.$$

In other words, R_i is the set of pairs with neighborhood in X_i at least one-tenth the "expected" size. If $|R_1| \leq \frac{1}{10} p\zeta n^2$, then

$$e(X_1, V(H), V(H)) \leq |R_1| n + \left(\binom{n}{2} - |R_1| \right) \frac{1}{10} p \zeta n \leq \frac{1}{5} p \zeta n^3. \tag{9}$$

On the other hand, since H is (p, μ)-dense,

$$e(X_1, V(H), V(H)) \geq p |X_1| n^2 - \mu n^3 \geq (p\zeta - \mu) n^3.$$

Let $\mu = \frac{p}{2}(\frac{1}{10}p\zeta)^3 < \frac{4}{5}p\zeta$ so that this contradicts (9). Thus $|R_1| \geq \frac{1}{10}p\zeta n^2$ and similarly for $1 \leq i \leq 3$, $|R_i| \geq \frac{1}{10}p\zeta n^2$.

Now fix $r_1r_1' \in R_1$, $r_2r_2' \in R_2$, and $r_3r_3' \in R_3$. There are at least $(\frac{1}{10}p\zeta)^3 n^6$ such choices. For $1 \leq i \leq 3$ let $Y_i = N(r_i, r_i') \cap X_i$ so $|Y_i| \geq \frac{1}{10}p\zeta n = (\frac{2\mu}{p})^{1/3} n$. By Lemma 13, there exists a

$\xi > 0$ such that there are at least $\xi(\frac{1}{10}p\zeta)^{12}n^{12}$ copies of $K_{4,4,4}$ across Y_1, Y_2, Y_3. Let T_1, T_2, T_3 be the three parts of $K_{4,4,4}$ with $T_i \subseteq Y_i$ and let $T_i = \{y_1^i, y_2^i, y_3^i, y_4^i\}$.

Let $\epsilon = \xi(\frac{1}{10}p\zeta)^{15}$; we claim that there are at least ϵn^{18} vertex sets of size 18 which C_4-absorb B. Indeed, $A := \{r_i, r_i', y_j^i : 1 \leq i \leq 3, 1 \leq j \leq 4\}$ forms a C_4-absorbing 18-set for B as follows. First, A has a perfect C_4-packing: one C_4 uses vertices $r_1, r_1', y_1^1, y_2^1, y_3^2, y_3^3$, another uses vertices $r_2, r_2', y_1^2, y_2^2, y_4^3, y_3^1$, and the last uses $r_3, r_3', y_1^3, y_2^3, y_4^1, y_4^2$. Secondly, $A \cup B$ has a perfect C_4-packing: one C_4 using $b_1, b_1', r_1, r_1', y_1^1, y_2^1$, one using $b_2, b_2', r_2, r_2', y_1^2, y_2^2$, one using $b_3, b_3', r_3, r_3', y_1^3, y_2^3$, and one using $y_3^1, y_4^1, y_3^2, y_4^2, y_3^3, y_4^3$. Since there are $(\frac{1}{10}p\zeta)^3 n^6$ choices for $r_1, r_1', r_2, r_2', r_3, r_3'$ and then $\xi(\frac{1}{10}p\zeta)^{12}n^{12}$ choices for y_j^i, there are a total of at least ϵn^{18} choices for A. □

Proof of Theorem 4. Apply Lemmas 17 and 18 and then Proposition 9. □

3.3 Packing Linear Hypergraphs

In this section, we prove Theorem 3.

Lemma 19. *Let $0 < p, \alpha < 1$ and let F be a linear k-graph on f vertices. There exists an n_0, $\epsilon > 0$, and $\mu > 0$ such that if H is a (n, p, μ, α) k-graph with $n \geq n_0$, then H is $(f^2 - f, f, \epsilon, F)$-rich.*

Proof. Let $a = f(f-1)$ and $b = f$. Similar to the proofs in the previous two sections, our task is to come up with an $\epsilon > 0$ such that for large n and all $B \in \binom{V(H)}{b}$, there are at least ϵn^a vertex sets of size a which F-absorb B; we will define ϵ and μ later. Let $V(F) = \{w_0, \ldots, w_{f-1}\}$ and form the following k-graph F'. Let

$$V(F') = \{x_{i,j} : 0 \leq i, j \leq f-1\}.$$

(We think of the vertices of F' as arranged in a grid with i as the row and j as the column.) Form the edges of F' as follows: for each fixed $1 \leq i \leq f-1$, let $\{x_{i,0}, \ldots, x_{i,f-1}\}$ induce a copy of F where $x_{i,j}$ is mapped to $w_{i+j \pmod f}$. More precisely, if $\{w_{\ell_1}, \ldots, w_{\ell_k}\} \in F$, then $\{x_{i,\ell_1 - i \pmod f}, \ldots, x_{i,\ell_k - i \pmod f}\} \in F'$. Similarly, for each fixed $0 \leq j \leq f-1$, let $\{x_{0,j}, \ldots, x_{f-1,j}\}$ induce a copy of F where $x_{i,j}$ is mapped to $w_{i+j \pmod f}$. Note that we therefore have a copy of F in each column and a copy of F in each row besides the zeroth row.

Now fix $B = \{b_0, \ldots, b_{f-1}\} \subseteq V(H)$; we want to show that B is F-absorbed by many a-sets. Note that any labeled copy of F' in H which maps $x_{0,0} \to b_0, \ldots, x_{0,f-1} \to b_{f-1}$ produces an F-absorbing set for B as follows. Let $Q : V(F') \to V(H)$ be an edge-preserving injection where $Q(b_j) = x_{0,j}$ (so Q is a labeled copy of F' in H where the set B is the zeroth row of F'). Let $A = \{Q(x_{i,j}) : 1 \leq i \leq f-1, 0 \leq j \leq f-1\}$ consist of all vertices in rows 1 through $f-1$. Then A has a perfect F-packing consisting of the copies of F on the rows, and $A \cup B$ has a perfect F-packing consisting of the copies of F on the columns. Therefore, A F-absorbs B.

To complete the proof, we therefore just need to use Lemma 11 to show there are many copies of F' with B as the zeroth row. Apply Lemma 11 to F' where $m = f$, $s_1 =$

$x_{0,0}, \dots, s_f = x_{0,f-1}$ and $Z_{m+1} = \cdots = Z_{f^2} = V(H)$. Since $\delta(H) \geq \alpha\binom{n}{k-1}$, (1) holds (with α replaced by $\frac{\alpha}{k^k}$) and since H is (p,μ)-dense (2) holds. Let $\gamma = \frac{1}{2}(\frac{\alpha}{k^k})^{\sum d(x_{0,j})} p^{|F| - \sum d(x_{0,j})}$ and ensure that n_0 is large enough and μ is small enough apply Lemma 11 to show that

$$\operatorname{inj}[F' \to H; x_{0,0} \to b_0, \dots, x_{0,f-1} \to b_{f-1}] \geq \gamma n^{f^2 - f} = \gamma n^a.$$

Each labeled copy of F' produces a labeled F-absorbing set for B, so there are at least $\frac{\gamma}{a!} n^a$ F-absorbing sets for B. The proof is complete by letting $\epsilon = \frac{\gamma}{a!}$. □

Proof of Theorem 3. Apply Lemma 19 and then Proposition 8. □

4 Avoiding perfect F-packings

In this section we prove Theorem 5 using the following construction.

Construction. For $n \in \mathbb{N}$, define a probability distribution $H(n)$ on 3-uniform, n-vertex hypergraphs as follows. Let $G = G^{(2)}(n, \frac{1}{2})$ be the random graph on n vertices. Let X and Y be a partition of $V(G)$ where

- if $n \equiv 0 \pmod 4$, then $|X| = \frac{n}{2} - 1$ and $|Y| = \frac{n}{2} + 1$,
- if $n \equiv 1 \pmod 4$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$,
- if $n \equiv 2 \pmod 4$, then $|X| = |Y| = \frac{n}{2}$,
- if $n \equiv 3 \pmod 4$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$.

Let the vertex set of $H(n)$ be $V(G)$ and make a set $E \in \binom{V(G)}{3}$ into a hyperedge of $H(n)$ as follows. If $|E \cap X|$ is even, then make E into a hyperedge of $H(n)$ if $G[E]$ is a clique. If $|E \cap X|$ is odd, then make E into a hyperedge of $H(n)$ if E is an independent set in G.

Lemma 20. *For every $\epsilon > 0$, with probability going to 1 as n goes to infinity,*

$$\left| |E(H(n))| - \frac{1}{8}\binom{n}{3} \right| \leq \epsilon n^3.$$

Proof. For each $E \in \binom{V(H(n))}{3}$, E is a clique or independent set in $G(n, \frac{1}{2})$ with probability $\frac{1}{8}$. Thus the expected number of edges in $H(n)$ is $\frac{1}{8}\binom{n}{3}$ so the second moment method shows that with probability going to one as n goes to infinity, $|E(H(n)) - \frac{1}{8}\binom{n}{3}| \leq \epsilon n^3$. See [1] or the proof of Lemma 15 in [26] for details about the second moment method. □

Lemma 21. *For every $\epsilon > 0$, with probability going to 1 as n goes to infinity the following holds. Let $X_1, X_2, X_3 \subseteq V(H(n))$. Then*

$$\left| e(X_1, X_2, X_3) - \frac{1}{8}|X_1||X_2||X_3| \right| < \epsilon n^3.$$

Proof. Let $S_1, \ldots, S_n$ be Steiner triple systems that partition $\binom{V(H(n))}{3}$. That is, view $V(H(n)) \cong \mathbb{Z}_{n-1}$ and for $1 \leq i \leq n$ let S_{i+1} consist of the triples $\{a, b, c\}$ such that $a+b+c = i \pmod{n}$. Each triple in $\binom{V(H(n))}{3}$ appears in exactly one S_i and two triples from the same S_i share at most one vertex.

Let $1 \leq i \leq n$ and let $X_1, X_2, X_3 \subseteq V(H(n))$. Let $e_H(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in E(H(n)) \cap S_i$. Let $e_{K_n}(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in S_i$.

The expected value of $e_H(X_1, X_2, X_e; S_i)$ is clearly $\frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i)$. If $E_1, E_2 \in S_i$ then since E_1 and E_2 share at most one vertex the events $E_1 \in E(H(n))$ and $E_2 \in E(H(n))$ are independent. By Chernoff's Bound (Lemma 15),

$$\mathbb{P}\left[\left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i) \right| > \epsilon |S_i| \right] < e^{-cn^2}.$$

for some constant c since $|S_i| = \frac{1}{n}\binom{n}{3}$ and the number of events is $e_{K_n}(X_1, X_2, X_3; S_i) < |S_i|$. By the union bound,

$$\mathbb{P}\left[\exists i, \exists X_1, X_2, X_3, \left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i) \right| > \epsilon |S_i| \right] < e^{-\frac{c}{2}n^2}.$$

Therefore, with high probability, for all i and all X_1, X_2, X_3,

$$\left| e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i) \right| < \epsilon |S_i|. \tag{10}$$

Summing (10) over i completes the proof. □

Lemma 22. *Let F be a 3-graph with an even number of vertices such that there exists a partition of the vertices of F into pairs such that every pair has a common pair in their links. Then $H(n)$ does not have a perfect F-packing for any n.*

Proof. If $n \nmid v(F)$, then obviously $H(n)$ does not have a perfect F-packing. Therefore assume that $n | v(F)$ so that n is even. Let X and Y be the partition of $V(H(n))$ in the definition of $H(n)$. Since n is even, by definition both $|X|$ and $|Y|$ are odd. Let $\{w_1, z_1\}, \{w_2, z_2\}, \ldots, \{w_{v(F)/2}, z_{v(F)/2}\}$ be the partition of $V(F)$ into pairs so that w_i and z_i have a common pair in their link for all i. By construction, if $x \in X$ and $y \in Y$ then there is no pair of vertices $u, v \in V(H(n))$ such that $xuv, yuv \in E(H(n))$ since the parities of $\{x, u, v\} \cap X$ and $\{y, u, v\} \cap X$ are different. This implies that for each i, w_i and z_i must either both appear in X or both appear in Y so that any copy of F in $H(n)$ uses an even number of vertices in X and an even number of vertices in Y. Since $|X|$ is odd, $H(n)$ does not have a perfect F-packing. □

Proof of Theorem 5. By Lemmas 20, 21, and 22, with high probability $H(n)$ has the required properties. □

5 Perfect Matchings in Sparse Hypergraphs

In this section, we prove Theorem 6. We follow the same outline as Section 3.

Lemma 23. *Let $k \geq 2$, $c > 0$, and a, b be multiples of k. There exists an n_0 depending only on k, a, b, and c such that the following holds for all $n \geq n_0$. Let H be an n-vertex k-graph, let $\mathcal{A} \subseteq \binom{V(H)}{a}$, and let $\mathcal{B} \subseteq \binom{V(H)}{b}$. Suppose that $\ell \geq cn^{a-1/2} \log n$ is an integer such that for every $B \in \mathcal{B}$ there are at least ℓ sets in $\mathcal{A}$ which edge-absorb B. Then there exists set $A \subseteq V(H)$ such that A partitions into sets from $\mathcal{A}$ and A edge-absorbs any set C satisfying the following conditions: $C \subseteq V(H) \setminus A$, $|C| \leq \frac{1}{64}\ell^2 n^{-2a+1}$, and C partitions into sets from $\mathcal{B}$.*

Proof. The proof is similar to Treglown-Zhao [34, Lemma 5.2] which in turn is similar to Rödl-Ruciński-Szemerédi [31, Fact 2.3]. Let $q = \frac{1}{8}\ell n^{-2a+1}$ and let $\mathfrak{A} \subseteq \mathcal{A}$ be the family obtained by selecting each element of $\mathcal{A}$ with probability q independently. The expected number of intersecting pairs of elements from $\mathfrak{A}$ is at most $q^2\binom{n}{a}a\binom{n}{a-1} \leq \frac{1}{16}q\ell$. By Markov's inequality, with probability at least $\frac{1}{2}$ there are at most $\frac{1}{8}q\ell$ intersecting pairs of elements from $\mathfrak{A}$.

Now fix $B \in \mathcal{B}$ and let $\Gamma_B \subseteq \{A \in \mathcal{A} : A \text{ edge-absorbs } B\}$ be such that $|\Gamma_B| = \ell$. For each $A \in \Gamma_B$, let X_A be the event that $A \in \mathfrak{A}$. By Chernoff's Bound (Lemma 15),

$$\mathbb{P}\left[\Big||\Gamma_B \cap \mathfrak{A}| - q\ell\Big| > \frac{1}{2}q\ell\right] \leq 2e^{-q\ell/6}.$$

Using that $\ell \geq cn^{a-1/2}\log n$, we have that $q\ell = \frac{1}{8}\ell^2 n^{-2a+1} \geq \frac{c^2}{8}\log^2 n$. By the union bound,

$$\mathbb{P}\left[\exists B, |\Gamma_B \cap \mathfrak{A}| < \frac{1}{2}q\ell\right] \leq \binom{n}{b} 2e^{-q\ell/6} \leq 2e^{b\log n - c^2\log^2 n/48} < \frac{1}{2}$$

for large n. Thus with probability at least $\frac{1}{2}$, $\mathfrak{A}$ is such that for all $B \in \mathcal{B}$, there exist at least $\frac{1}{2}q\ell$ a-sets in $\mathfrak{A}$ which edge-absorb B. Also, with probability at least $\frac{1}{2}$ there are at most $\frac{1}{8}q\ell$ intersecting pairs of elements from $\mathfrak{A}$.

Let $\mathfrak{A}'$ be the subfamily of $\mathfrak{A}$ consisting only of those a-sets A where A is not in any intersecting pair and also there is at least one $B \subseteq V(H)$ (of any size) such that A edge-absorbs B. Thus by the union bound, with positive probability $\mathfrak{A}'$ is such that for all $B \in \mathcal{B}$, there exist at least $\frac{1}{4}q\ell$ a-sets in $\mathfrak{A}'$ which edge-absorb B. Let $\mathfrak{A}'$ be such a family of a-sets and let $A' = \cup\mathfrak{A}'$. First, $H[A']$ has a perfect matching. Indeed, each $A \in \mathfrak{A}'$ edge-absorbs some set so $H[A]$ has a perfect matching, and the sets in $\mathfrak{A}'$ are disjoint so that these perfect matchings combine to form a perfect matching of $H[A']$. Second, A' partitions into sets from $\mathcal{A}$ since the sets in $\mathfrak{A}' \subseteq \mathcal{A}$ are disjoint. Now let $C \subseteq V(H) \setminus A'$ with $|C| \leq \frac{1}{64}\ell^2 n^{-2a+1} = \frac{1}{8}q\ell$ and $C = B_1 \dot\cup \cdots \dot\cup B_t$ with $B_i \in \mathcal{B}$. Using the bound on the size of C, we have that $t < \frac{1}{8}q\ell$. Since each B_i is edge-absorbed by at least $\frac{1}{4}q\ell$ sets in $\mathfrak{A}'$, each B_i can be edge-absorbed by a different a-set in $\mathfrak{A}'$. Therefore, $H[A' \cup B']$ has a perfect matching so the proof is complete. □

Proof of Lemma 10. Let H' be the $v(F)$-uniform hypergraph on the same vertex set as H, where $X \in \binom{V(H)}{v(F)}$ is a hyperedge of H' if $H[X]$ is a copy of F. Let $\ell = \lceil \epsilon n^a \rceil$ and notice since H is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$-rich, for every $B \in \mathcal{B}$ there are at least ℓ sets in $\mathcal{A}$ which edge-absorb B in H'. Also, since a, b are multiples of $v(F)$ they are multiples of the uniformity of H'. Lastly, for large n we have that $\ell \geq n^{a-1/2} \log n$. Therefore, applying Lemma 23 (with $c = 1$) to H' shows that there exists a set $A \subseteq V(H') = V(H)$ such that A partitions into sets from $\mathcal{A}$ and for any $C \subseteq V(H') \setminus A = V(H) \setminus A$ with $|C| \leq \frac{\epsilon^2}{64} n$ and C partitions into sets from $\mathcal{B}$, A edge-absorbs C in H'. Because each edge of H' is a copy of F, this implies that A F-absorbs C in H so the proof is complete by setting $\omega = \frac{\epsilon^2}{64}$. □

Next, similar to the proofs in Sections 3.1, 3.2, and 3.3, we show that a bound on $\lambda_2(H)$ implies that each 3-set is edge-absorbed by many 6-sets. To do so, we need the hypergraph expander mixing lemma, first proved by Friedman and Wigderson [7, 8] (using a slightly different definition of $\lambda_2(H)$) and then extended to our definition of $\lambda_2(H)$ in [25].

Proposition 24. *(Hypergraph Expander Mixing Lemma [25, Theorem 4]). Let H be an n-vertex k-graph and let $S_1, \ldots, S_k \subseteq V(H)$. Then*

$$\left| e(S_1, \ldots, S_k) - \frac{k!|E(H)|}{n^k} \prod_{i=1}^{k} |S_i| \right| \leq \lambda_2(H) \sqrt{|S_1| \cdots |S_k|}.$$

Lemma 25. *Let $\alpha > 0$. Let H be a 3-graph and let $p = 6|E(H)|/n^3$. Assume $\delta_2(H) \geq \alpha p n$ and $\lambda_2(H) \leq \frac{1}{2}\alpha^2 p^{5/2} n^{3/2}$. Then for every $B \subseteq V(H)$ with $|B| = 3$, there are at least $\frac{1}{16}\alpha^4 p^5 n^6$ sets $A \subseteq V(H)$ with $|A| = 6$ such that A edge-absorbs B.*

Proof. Let $B = \{b_1, b_2, b_3\} \subseteq V(H)$. First, there are at least $\frac{1}{8}\alpha p n^3$ edges disjoint from B; let $\{x_1, x_2, x_3\}$ be such an edge. For $1 \leq i \leq 3$, let $Y_i \subseteq N(b_i, x_i) = \{y : y x_i b_i \in H\}$ with $|Y_i| = \alpha p n$. Such a Y_i exists since the minimum codegree is at least $\alpha p n$. By the expander mixing lemma (Proposition 24),

$$e(Y_1, Y_2, Y_3) \geq p|Y_1||Y_2||Y_3| - \lambda_2(H)\sqrt{|Y_1||Y_2||Y_3|} \geq \alpha^3 p^4 n^3 - \lambda_2(H)\alpha^{3/2} p^{3/2} n^{3/2}.$$

Since $\lambda_2(H) \leq \frac{1}{2}\alpha^2 p^{5/2} n^{3/2}$,

$$e(Y_1, Y_2, Y_3) \geq \alpha^3 p^4 n^3 - \frac{1}{2}\alpha^{7/2} p^4 n^3 \geq \frac{1}{2}\alpha^3 p^4 n^3.$$

Let $\{y_1, y_2, y_3\}$ be an edge with $y_1 \in Y_1$, $y_2 \in Y_2$, and $y_3 \in Y_3$. Then $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a six-set that edge-absorbs B and there are at least $\frac{1}{8}\alpha p n^3 \cdot \frac{1}{2}\alpha^3 p^4 n^3 = \frac{1}{16}\alpha^4 p^5 n^6$ such sets. □

Proof of Theorem 6. We are given $\alpha > 0$ such that $\delta_2(H) \geq \alpha p n$. Let $\gamma = 2^{-22}\alpha^{12}$.

First, we can assume that $p \geq \gamma n^{-1/10} \log^{1/5} n$. Indeed, by averaging there exists vertices s_1, s_2 such that the codegree of s_1 and s_2 is at most $2pn$. Then taking $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and S_3 as the non-coneighbors of s_1 and s_2, Proposition 24 shows that

$$\lambda_2(H) \geq p\sqrt{|S_3|} \geq p\sqrt{(1-2p)n}.$$

But by assumption, $\lambda_2(H) \le \gamma p^{16} n^{3/2}$. Therefore,

$$p\sqrt{(1-2p)n} \le \gamma p^{16} n^{3/2}$$

which implies that $p \ge \gamma n^{-1/10} \log^{1/5} n$ (by a large margin).

By Lemma 25, for every $B \subseteq V(H)$ with $|B| = 3$ there are at least $\frac{1}{16}\alpha^4 p^5 n^6$ 6-sets $A \subseteq V(H)$ which edge-absorb B. Let $\ell = \frac{1}{16}\alpha^4 p^5 n^6$. If n is sufficiently large, then since $p \ge \gamma n^{-1/10} \log^{1/5} n$, we have that $\ell = \frac{1}{16}\alpha^4 p^5 n^6 \ge \frac{1}{16}\alpha^4\gamma^5 n^{5.5} \log n$. Let $c = \frac{1}{16}\alpha^4\gamma^5$ so that $\ell \ge cn^{5.5}\log n$. Now by Lemma 23, if n is sufficiently large there exists $A \subseteq V(H)$ such that A edge-absorbs all sets of size a multiple of three and at most

$$\frac{1}{64}\ell^2 n^{-11} = \frac{1}{2^{14}}\alpha^8 p^{10} n. \tag{11}$$

We now show how to construct a perfect matching in H. First, greedily construct a matching in $H[V(H)\backslash A]$. Say the greedy procedure halts with $B \subseteq V(H)\backslash A$ as the unmatched vertices. Since $3|v(H)$ and $3||A|$ (since A is an edge-absorbing set), $3||B|$. By Proposition 24 (recall that $\gamma = 2^{-22}\alpha^{12}$),

$$e(B,B,B) = p|B|^3 \pm \lambda_2(H)|B|^{3/2} \ge p|B|^3 - \frac{1}{2^{22}}\alpha^{12} p^{16} n^{3/2} |B|^{3/2}. \tag{12}$$

If $|B| \ge 2^{-14}\alpha^8 p^{10} n$, then

$$p|B|^3 \ge p|B|^{3/2}\left(\frac{1}{2^{14}}\alpha^8 p^{10} n\right)^{3/2} = \frac{1}{2^{21}}\alpha^{12} p^{16} n^{3/2} |B|^{3/2}.$$

Combining this with (12) shows that $e(B,B,B) > 0$. This contradicts that the greedy procedure halted with B as the unmatched vertices. Thus $|B| \le 2^{-14}\alpha^8 p^{10} n$ and then (11) shows that A edge-absorbs B, producing a perfect matching of H. □

Acknowledgments: We would like to thank Alan Frieze for helpful discussions at the early stages of this project. Thanks also to Daniela Kühn and Sebastian Cioabă for useful comments, and the referees for suggestions that improved the presentation.

References

[1] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[2] N. Alon and R. Yuster. H-factors in dense graphs. *J. Combin. Theory Ser. B*, 66(2):269–282, 1996.

[3] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[4] S. M. Cioabă, D. A. Gregory, and W. H. Haemers. Matchings in regular graphs from eigenvalues. *J. Combin. Theory Ser. B*, 99(2):287–297, 2009.

[5] A. Czygrinow, L. DeBiasio, and B. Nagle. Tiling 3-uniform hypergraphs with $K_4^3 - 2e$. *J. Graph Theory*, 75(2):124–136, 2014.

[6] P. Erdős and A. Hajnal. On Ramsey like theorems. Problems and results. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 123–140. Inst. Math. Appl., Southend, 1972.

[7] J. Friedman. Some graphs with small second eigenvalue. *Combinatorica*, 15(1):31–42, 1995.

[8] J. Friedman and A. Wigderson. On the second eigenvalue of hypergraphs. *Combinatorica*, 15(1):43–65, 1995.

[9] A. Hajnal and E. Szemerédi. Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969)*, pages 601–623. North-Holland, Amsterdam, 1970.

[10] H. Hàn, Y. Person, and M. Schacht. On perfect matchings in uniform hypergraphs with large minimum vertex degree. *SIAM J. Discrete Math.*, 23(2):732–748, 2009.

[11] J. Han and Y. Zhao. Minimum vertex degree threshold for loose Hamilton cycles in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 114:70–96, 2015.

[12] P. Keevash. The existence of designs. http://arxiv.org/abs/1401.3665.

[13] P. Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.

[14] P. Keevash and R. Mycroft. A geometric theory for hypergraph matching. *Mem. Amer. Math. Soc.*, 233(1098):vi+95, 2015.

[15] I. Khan. Perfect matchings in 4-uniform hypergraphs. arXiv:1101.5675.

[16] I. Khan. Perfect matchings in 3-uniform hypergraphs with large vertex degree. *SIAM J. Discrete Math.*, 27(2):1021–1039, 2013.

[17] Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht. Weak hypergraph regularity and linear hypergraphs. *J. Combin. Theory Ser. B*, 100(2):151–160, 2010.

[18] J. Komlós, G. N. Sárközy, and E. Szemerédi. Blow-up lemma. *Combinatorica*, 17(1):109–123, 1997.

[19] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Alon-Yuster conjecture. *Discrete Math.*, 235(1-3):255–269, 2001. Combinatorics (Prague, 1998).

[20] D. Kühn and D. Osthus. Loose Hamilton cycles in 3-uniform hypergraphs of high minimum degree. *J. Combin. Theory Ser. B*, 96(6):767–821, 2006.

[21] D. Kühn and D. Osthus. The minimum degree threshold for perfect graph packings. *Combinatorica*, 29(1):65–107, 2009.

[22] D. Kühn, D. Osthus, and T. Townsend. Fractional and integer matchings in uniform hypergraphs. *European J. Combin.*, 38:83–96, 2014.

[23] D. Kühn, D. Osthus, and A. Treglown. Matchings in 3-uniform hypergraphs. *J. Combin. Theory Ser. B*, 103(2):291–305, 2013.

[24] J. Lenz and D. Mubayi. Eigenvalues of non-regular linear quasirandom hypergraphs. online at `http://arxiv.org/abs/1309.3584`.

[25] J. Lenz and D. Mubayi. Eigenvalues and linear quasirandom hypergraphs. *Forum Math. Sigma*, 3:e2, 26, 2015.

[26] J. Lenz and D. Mubayi. The poset of hypergraph quasirandomness. *Random Structures Algorithms*, 46(4):762–800, 2015.

[27] A. Lo and K. Markström. Minimum codegree threshold for $(K_4^3 - e)$-factors. *J. Combin. Theory Ser. A*, 120(3):708–721, 2013.

[28] A. Lo and K. Markström. F-factors in hypergraphs via absorption. *Graphs Combin.*, 31(3):679–712, 2015.

[29] K. Markström and A. Ruciński. Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees. *European J. Combin.*, 32(5):677–687, 2011.

[30] O. Pikhurko. Perfect matchings and K_4^3-tilings in hypergraphs of large codegree. *Graphs Combin.*, 24(4):391–404, 2008.

[31] V. Rödl, A. Ruciński, and E. Szemerédi. Perfect matchings in large uniform hypergraphs with large minimum collective degree. *J. Combin. Theory Ser. A*, 116(3):613–636, 2009.

[32] A. Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987.

[33] A. Thomason. Random graphs, strongly regular graphs and pseudorandom graphs. In *Surveys in combinatorics 1987 (New Cross, 1987)*, volume 123 of *London Math. Soc. Lecture Note Ser.*, pages 173–195. Cambridge Univ. Press, Cambridge, 1987.

[34] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs. *J. Combin. Theory Ser. A*, 119(7):1500–1522, 2012.

[35] A. Treglown and Y. Zhao. Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II. *J. Combin. Theory Ser. A*, 120(7):1463–1482, 2013.