Ordered and colored subgraph density problems

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Abstract

We consider three extremal problems about the number of copies of a fixed graph in another larger graph. First, we correct an error in a result of Reiher and Wagner [19] and prove that the number of k-edge stars in a graph with density $x \in [0,1]$ is asymptotically maximized by a clique and isolated vertices or its complement. Next, among ordered n-vertex graphs with m edges, we determine the maximum and minimum number of copies of a k-edge star whose nonleaf vertex is minimum among all vertices of the star. Finally, for $s \geq 2$, we define a particular 3-edge-colored complete graph F on $2s$ vertices with colors blue, green and red, and determine, for each (x_b, x_g) with $x_b + x_g \leq 1$ and $x_b, x_g \geq 0$, the maximum density of F in a large graph whose blue, green and red edge sets have densities x_b, x_g and $1 - x_b - x_g$, respectively. These are the first nontrivial examples of colored graphs for which such complete results are proved.

1 Introduction

The *density* of a graph G with n vertices and m edges is $\varrho(G) := m/\binom{n}{2}$. For a graph F with $k \leq n$ vertices, let $N(F,G)$ be the number of subgraphs of G that are isomorphic to F. We are interested in the minimum and maximum values of $N(F,G)$ over graphs G with a given value of $\varrho(G)$ as n grows. We note that $N(F,G) \leq (n)_k/|\mathrm{Aut}(F)|$, where $\mathrm{Aut}(F)$ is the automorphism group of F and $(n)_k$ is the falling factorial $n(n-1)\cdots(n-k+1)$. Define the *(labeled) density of F in G* to be

$$\varrho(F,G) := \frac{N(F,G) \cdot |\mathrm{Aut}(F)|}{(n)_k} \in [0,1].$$

We note that the notation $t_{\mathrm{inj}}(F,G)$ is often used in the literature for $\varrho(F,G)$, though it is more convenient to use $\varrho(F,G)$ in this paper.

The classical Kruskal–Katona theorem [9, 10] implies that the maximum density of K_s in a graph of density x is achieved asymptotically by graphs consisting of a clique and isolated vertices. The minimum density of K_s in a graph with density x was determined by Razborov [17] for $s = 3$, by Nikiforov [15] for $s = 4$, and by Reiher [18] for all $s \geq 4$; this is achieved by complete multipartite graphs. Let S_k denote the k-edge star. Ahlswede and Katona [1] determined the maximum number of S_2's in a graph with density x. Reiher and Wagner [19] claim to prove that the asymptotic maximum value of $N(S_k, G)$ when G has density x, is achieved by a clique and isolated vertices or

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its complement. We correct an error in their proof. We also consider this problem in the setting of ordered graphs and prove the first nontrivial results on S_k whose vertices have a particular order.

Finally, we consider similar questions in the induced setting. For G with n vertices and F with k vertices, let $N_{\text{ind}}(F, G)$ be the number of induced subgraphs of G that are isomorphic to F. In other words, $N_{\text{ind}}(F, G)$ is the number of $S \subset V(G)$ such that $G[S] \cong F$. Let $\varrho_{\text{ind}}(F, G) = N_{\text{ind}}(F, G)/\binom{n}{k}$ be the *induced density of F in G*. In [11], the authors consider the region of possible asymptotic values of $\varrho_{\text{ind}}(F, G)$ for graphs G of fixed density. By coloring edges in G and F red, and coloring edges in their complements blue, it is obvious that counting induced subgraphs is the same as counting copies of a two-edge-colored clique in a (larger) two-edge-colored clique. This leads us to consider the maximum asymptotic density of q-edge-colored cliques in (larger) q-edge-colored cliques with given color densities. For $q = 3$, we prove the first nontrivial result in this setting.

In Section 2, we state our results. They are proved in Sections 3, 4, and 5.

2 Statements of results

2.1 Stars

Clearly $N(S_k, G) = \sum_{v \in V(G)} \binom{d(v)}{k}$ and $\varrho(S_k, G) = N(S_k, G) \cdot k!/(n)_{k+1}$. For $x \in [0, 1]$, let $I(S_k, x)$ be the supremum of $\lim_{n\to\infty} \varrho(S_k, G_n)$ over all sequences of graphs $(G_n)_{n=1}^{\infty}$ with $|V(G_n)| \to \infty$, $\varrho(G_n) \to x$ and for which $\lim_{n\to\infty} \varrho(S_k, G_n)$ exists. For each $\gamma \in [0, 1]$, let $\eta = 1 - \sqrt{1-\gamma}$. Then $\gamma^{(k+1)/2}$ and $\eta + (1-\eta)\eta^k$ are the asymptotic S_k-densities in a clique with isolated vertices and the complement of a clique with isolated vertices, both with density γ. Consequently,

$$I(S_k, \gamma) \geq \max\{\gamma^{(k+1)/2},\ \eta + (1-\eta)\eta^k\}.$$

Reiher and Wagner [19] proved matching upper bounds on $I(S_k, \gamma)$. Their results are stated (and proved) using the language of graphons, which are limit object of graphs. Formally, a *graphon* W is a symmetric, measurable function $W : [0, 1]^2 \to [0, 1]$ (see Lovász [12] for background on graphons). For a graphon W, let $d_W(x) = \int_0^1 W(x, y)\ dy$ be the *degree* of x in W, let $t(\,|\,, W) = \int_{[0,1]^2} W(x, y)\ dxdy$ be the *density* of W, and let

$$t(S_k, W) = \int_0^1 d_W^k(x)\ dx$$

be the *homomorphism density of S_k in W*.

Theorem 2.1 (Reiher–Wagner [19]). *Let W be a graphon and let k be a positive integer. Set $\gamma = t(\,|\,, W)$ and $\eta = 1 - \sqrt{1-\gamma}$. Then*

$$t(S_k, W) \leq \max\{\gamma^{(k+1)/2},\ \eta + (1-\eta)\eta^k\}.$$

By the general theory of graphons, Theorem 2.1 gives the same upper bound for $I(S_k, \gamma)$. There appears to be an error in the proof of [19, Proposition 3.7], which is necessary for the proof of Theorem 2.1. We correct this error and prove Theorem 2.1 in Section 3.

2.2 Ordered stars

An *ordered graph* $G = (V, E)$ is a graph with a total order on V. We usually let $V = [n] := \{1, \ldots, n\}$ with the natural ordering. When we refer to an edge ij in E, it is implied that $\{i, j\} \in E$ and $i < j$. Let F be an ordered graph on $[s]$. Let $N_{\text{ord}}(F, G)$ be the number of $\{v_1, \ldots, v_s\} \subseteq [n]$ with $v_1 < v_2 < \cdots < v_s$ such that $v_iv_j \in E(G[v_1, \ldots, v_s])$ whenever $ij \in E(F)$. We consider $N_{\text{ord}}(F, G)$ in the case that F is an appropriate ordered S_k. The following constructions provide lower bounds.

Construction 2.2. For positive integers n and $m \le \binom{n}{2}$, let a be the largest integer such that $f(n,a) := \binom{a}{2} + a(n-a) \le m$. As $f(n,n) = \binom{n}{2}$, we have $0 \le a \le n$. Set $b = m - f(n,a)$. Since $f(n,a+1) - f(n,a) = n-a-1$, we conclude that $0 \le b < n-a-1$. Let $S_L(n,m)$ be the ordered graph with vertex set $[n]$ and edge set

$$\{vw : v \in [a], w \in [n]\} \cup \{\{a+1, j\} : a+2 \le j \le a+b+1\}.$$

In words, $S_L(n,m)$ comprises a complete graph on $[a]$, and in addition has all edges between $[a]$ and $[n] \setminus [a]$ and b edges between $a+1$ and the b smallest vertices in $[n] \setminus [a+1]$ (see Figure 1). Let $S_R(n,m)$ be defined as $S_L(n,m)$, but where the total order on the vertices is reversed.

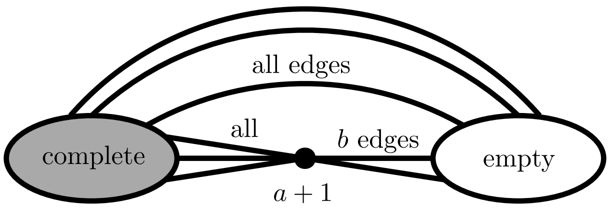

Figure 1: $S_L(n,m)$.

Let $S_L(k) := S_L(k+1,k)$ be the ordered *left star* and $S_R(k) := S_R(k+1,k)$ the ordered *right star*. Note that $S_L(k)$ has $a=1$ and $b=0$ (see Figure 2).

Figure 2: $S_L(k)$ and $S_R(k)$.

Our main result for ordered graphs is the following theorem.

Theorem 2.3. *Let G be an ordered graph with vertex set $[n]$ and m edges. Then*

$$N_{\mathrm{ord}}(S_L(k), S_R(n,m)) \le N_{\mathrm{ord}}(S_L(k), G) \le N_{\mathrm{ord}}(S_L(k), S_L(n,m)).$$

Theorem 2.3 implies similar results for $S_R(k)$. There are, up to obvious symmetries, $\lceil (k+1)/2 \rceil$ different ordered stars with k edges and, apart from $S_L(k)$, it remains open to prove analogous results to Theorem 2.3 for them. Indeed, obtaining sharp bounds for these stars seems nontrivial. We address the first open case when $k=2$.

Let M be the ordered graph with vertex set $[3]$ and edge set $\{12, 23\}$. It seems very difficult to obtain exact results for $N_{\mathrm{ord}}(M,G)$ so we consider asymptotic growth rates. Let

$$\varrho_{\mathrm{ord}}(F,G) := \frac{N_{\mathrm{ord}}(F,G)}{\binom{n}{s}}$$

be the *density of F in G*. Since the vertices are ordered, each s-tuple of vertices can contribute at most one copy of F so $\varrho_{\rm ord}(F,G)\in[0,1]$. Let $(G_n):=(G_n)_{n=1}^{\infty}$ be a sequence of ordered graphs with $\lim_{n\to\infty}|V(G_n)|=\infty$. The sequence (G_n) is *F-good* if both $\lim_{n\to\infty}\varrho(G_n)$ and $\lim_{n\to\infty}\varrho_{\rm ord}(F,G_n)$ exist. In this case, we set $x:=\lim_{n\to\infty}\varrho(G_n)$ and $y:=\lim_{n\to\infty}\varrho_{\rm ord}(F,G_n)$ and say that (G_n) *realizes* (x,y). Define

$$I_{\rm ord}(F,x):=\sup\{y:(x,y)\in[0,1] \text{ is realized by some } F\text{-good } (G_n)\},$$
$$i_{\rm ord}(F,x):=\inf\{y:(x,y)\in[0,1] \text{ is realized by some } F\text{-good } (G_n)\}.$$

We first give a construction that achieves $i_{\rm ord}(,x)$. As in [11], for any integers $n\geq k\geq 2$ and real $x\in(\frac{k-2}{k-1},\frac{k-1}{k}]$, let $H^*(n,x)$ be the complete k-partite graph on n vertices with parts $V_1,\ldots,V_k$ of sizes $|V_1|=\cdots=|V_{k-1}|=\lfloor\alpha_k n\rfloor$ and $|V_k|=n-(k-1)\lfloor\alpha_k n\rfloor$, where

$$\alpha_k=\frac{1}{k}\left(1+\sqrt{1-\frac{k}{k-1}x}\right).$$

It is simple to check that $\lim_{n\to\infty}\varrho(H^*(n,x))=x$. We define

$$g_3(x):=\lim_{n\to\infty}\frac{N(K_3,H^*(n,x))}{\binom{n}{3}}.$$

Let $H'(n,x)$ be an ordered graph obtained from $H^*(n,x)$ with any vertex ordering for which $u<v$ whenever $u\in V_i$, $v\in V_j$ and $i<j$. Then $N_{\rm ord}(M,H'(n,x))=N_{\rm ord}(K_3,H^*(n,x))$ and this implies that $i_{\rm ord}(M,x)\leq g_3(x)$. We further set $\mathcal{G}_x=\{(G_n):\lim_{n\to\infty}\varrho(G_n)=x\}$ and define

$$i(K_3,x):=\min_{(G_n)\in\mathcal{G}_x}\liminf_{n\to\infty}\frac{N(K_3,G_n)}{\binom{n}{3}}.$$

Lovász and Simonovits [13] conjectured that $i(K_3,x)=g_3(x)$ for all $x\in[0,1]$, and this was proven by Razborov [17] (see also [5, 14, 16, 2, 8, 4]). As $i(K_3,x)=i_{\rm ord}(K_3,x)$ due to the structure of K_3, we have that $i_{\rm ord}(K_3,x)=g_3(x)$. Note that M is a subgraph of the ordered K_3, so $i_{\rm ord}(K_3,x)\leq i_{\rm ord}(M,x)$. Therefore,

$$g_3(x)=i_{\rm ord}(K_3,x)\leq i_{\rm ord}(M,x)$$

and we conclude that $i_{\rm ord}(M,x)=i_{\rm ord}(K_3,x)=g_3(x)$.

Determining $I_{\rm ord}(M,x)$ appears to be more difficult.

Construction 2.4. We construct a sequence of graphs $(P(n,x))_{n=1}^{\infty}$ for any $x\in[0,1]$. For each n, define $P(n,x)$ to be the ordered graph on $[n]=A\sqcup B\sqcup C$, where $|B|=\lfloor n(1-\sqrt{1-x})\rfloor$, $||A|-|C||\leq 1$, and $a<b<c$ for all $a\in A$, $b\in B$, $c\in C$ with edge set

$$\{ab:a\in A,b\in B\}\cup\{b_1b_2:b_1,b_2\in B\}\cup\{bc:b\in B,c\in C\}.$$

A short calculation shows that $\lim_{n\to\infty}\varrho(P(n,x))=x$ and

$$\lim_{n\to\infty}\varrho_{\rm ord}(M,P(n,x))=\frac{\eta(3-\eta^2)}{2}$$

for $\eta=1-\sqrt{1-x}$.

Construction 2.5. We construct a sequence of graphs $(Q(n,x))_{n=1}^{\infty}$ for any $x \in [0,1]$. For each n, define $Q(n,x)$ to be the ordered graph on $[n]$ with edge set

$$\{ij : j - i \leq \lfloor (1 - \sqrt{1-x})n \rfloor\}.$$

A short calculation shows that $\lim_{n\to\infty} \varrho(Q(n,x)) = x$ and that for $\eta = 1 - \sqrt{1-x}$,

$$\lim_{n\to\infty} \varrho_{\text{ord}}(M, Q(n,x)) = \begin{cases} 6\eta^3 + 6(1-2\eta)\eta^2 & \text{if } \eta \leq 1/2, \\ 2\eta^3 - 6\eta^2 + 6\eta - 1 & \text{if } \eta \geq 1/2. \end{cases}$$

Constructions 2.4 and 2.5 show that for $\eta = 1 - \sqrt{1-x}$,

$$I_{\text{ord}}(M,x) \geq \max\Big\{\lim_{n\to\infty} \varrho_{\text{ord}}(M, P(n,x)),\ \lim_{n\to\infty} \varrho_{\text{ord}}(M, Q(n,x))\Big\}.$$

It is an interesting open problem to determine if the inequality is sharp.

Problem 2.6. *Is*

$$I_{\text{ord}}(M,x) = \max\Big\{\lim_{n\to\infty} \varrho_{\text{ord}}(M, P(n,x)),\ \lim_{n\to\infty} \varrho_{\text{ord}}(M, Q(n,x))\Big\}$$

for any (possibly all) $x \in [0,1]$*?*

A short calculation shows that there exists $x_0 \in [0,1]$ such that $\lim_{n\to\infty} \varrho_{\text{ord}}(M, Q(n,x)) \leq \lim_{n\to\infty} \varrho_{\text{ord}}(M, P(n,x))$ iff $x < x_0$.

2.3 Colored graphs

A *q-colored graph* is a graph $G = (V,E)$ together with a coloring function $f : E \to C$, where $|C| = q$. Fix $q \in \mathbb{Z}^+$ and let $G = (V,E)$ be a q-colored complete graph with coloring function f. For $1 \leq i \leq q$, let $e_i(G) = |\{e \in E : f(e) = i\}|$ and let $\varrho_i(G) := e_i(G)/\binom{|V|}{2}$ be the *density* of color i. Given a q-colored complete graph F with $|V(F)| = s$ and coloring function g, a subset $X \subset V$ with $|X| = s$ is a *copy of F in G* if there is a bijection $\sigma : V(F) \to X$ such that $g(uv) = g(\sigma(u)\sigma(v))$ for all distinct $u, v \in V(F)$. Let $N_q(F,G)$ be the number of copies of F in G and let $\varrho_q(F,G) := N_q(F,G)/\binom{|V|}{s}$ be the *density of F in G*.

Let $(G_n)_{n=1}^{\infty}$ be a sequence of q-colored complete graphs with $|V(G_n)| \to \infty$. The sequence $(G_n)_{n=1}^{\infty}$ is *F-good* if $x_i = \lim_{n\to\infty} \varrho_i(G_n)$ exists for all $i \in [q]$ and $y = \lim_{n\to\infty} \varrho_q(F, G_n)$ exists. In this case, $(G_n)_{n=1}^{\infty}$ *realizes* $(x_1, \ldots, x_{q-1}, y)$. Note that we only list $x_1, \ldots, x_{q-1}$ since $x_q = 1 - (x_1 + \cdots + x_{q-1})$. Define

$$I_q(F, (x_1, \ldots, x_{q-1})) := \sup\{y : (x_1, \ldots, x_{q-1}, y) \in [0,1]^q \text{ is realized by some } F\text{-good } (G_n)_{n=1}^{\infty}\}.$$

For $2 \leq s \leq t$, let $K'_{s,t}$ be the 3-colored clique on vertex set $V = V_1 \sqcup V_2$ with $|V_1| = s$ and $|V_2| = t$ with coloring function f defined by

$$f(ij) := \begin{cases} \text{blue} & \text{if } i,j \in V_1, \\ \text{green} & \text{if } i,j \in V_2, \\ \text{red} & \text{otherwise,} \end{cases}$$

for all distinct $i, j \in [s+t]$ (see Figure 3).

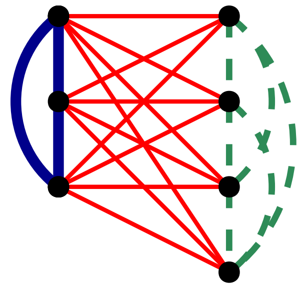

Figure 3: $K'_{3,4}$.

Theorem 2.7. *Let $2 \le s \le t$ and $x_b, x_g, x_r \in [0,1]$ such that $x_b + x_g + x_r = 1$. Then*

$$I_3(K'_{s,s},(x_b,x_g)) = \begin{cases} x_b^{s/2} x_g^{s/2} \binom{2s}{s} & \text{if } \sqrt{x_b}+\sqrt{x_g} \le 1, \\ \left(\frac{x_r}{2}\right)^s \binom{2s}{s} & \text{if } \sqrt{x_b}+\sqrt{x_g} \ge 1. \end{cases}$$

Furthermore, if $\sqrt{x_b}+\sqrt{x_g} \le 1$, then

$$I_3(K'_{s,t},(x_b,x_g)) = x_b^{s/2} x_g^{t/2} \binom{s+t}{s}.$$

Theorem 2.7 determines $I_3(K'_{s,s},(x_b,x_g))$ for all vectors (x_b,x_g) in the region $x_b, x_g \ge 0$ and $x_b + x_g \le 1$. Write $\mathrm{ind}(K'_{s,s})$ for the *inducibility of* $K'_{s,s}$ which is the maximum value of $I_3(K'_{s,s},(x_b,x_g))$. An easy optimization shows that

$$\mathrm{ind}(K'_{s,s}) = I_3(K'_{s,s},(1/4,1/4)) = \left(\frac{1}{4}\right)^s \binom{2s}{s}.$$

Note that $I_3(K'_{s,t},(x_b,x_g))$ is not known when $\sqrt{x_b}+\sqrt{x_g} > 1$ and $s \ne t$.

3 Proof of Theorem 2.1

In this section, we will correct the proof of Theorem 2.1 from [19]. This involves defining a new parameter (called $T(W)$) on graphons that we will optimize. Nevertheless, many parts of the argument are identical to those in [19], and we will indicate when this is the case in various lemmas, claims and propositions.

Given a measurable function $F : [0,1] \to \mathbb{R}$, a graphon W, and $\gamma \in [0,1]$, let

$$D(F,W) := \int_0^1 F(d_W(x))\, dx,$$
$$\mathrm{MAX}(\gamma,F) := \max\{(1-\sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma}),\ (1-\eta)F(\eta) + \eta F(1)\},$$

where $\eta = 1 - \sqrt{1-\gamma}$. A measurable function $F : [0,1] \to \mathbb{R}$ is *good* for W if

$$D(F,W) \le \mathrm{MAX}(\gamma,F),$$

where $\gamma = t(\,|\,,W)$ and $\eta = 1-\sqrt{1-\gamma}$; F is *bad* for W if it is not good for W. A collection of measurable functions is good (bad) for W if all its members are good (bad) for W. In [19], a set

$\mathscr{C}$ of twice differentiable convex functions $F : [0,1] \to \mathbb{R}$ satisfying certain conditions is defined. We do not need the details of the conditions here, so we do not state them, but we note that $\mathscr{C}$ contains the function $F(x) = x^k$.

The error in [19] appears in the proof of the following proposition.

Proposition 3.1 (corresponds to [19, Proposition 3.7])**.** *$\mathscr{C}$ is good for all step graphons.*

To prove this proposition, we introduce the following more refined notion of "good." For any $\delta > 0$, say that $F \in \mathscr{C}$ is *δ-good* for a graphon W if

$$D(F, W) < \mathrm{MAX}(\gamma, F) + \delta$$

for $\gamma = t(\,|\,, W)$ and $\eta = 1 - \sqrt{1-\gamma}$. Say that F is *δ-bad* for W if it is not δ-good for W.

We next show that several lemmas from [19] still apply when we replace "good" with "δ-good." The proofs are almost exact copies of those in [19], and are given in the Appendix.

Lemma 3.2 (corresponds to [19, Lemma 3.2])**.** *If all functions in $\mathscr{C}$ are δ-good for a graphon W, then the same is true for the graphon $1 - W$.*

Given a graphon W and a real number $\lambda \in [0,1]$, let $[\lambda, W]$ be the graphon satisfying

$$[\lambda, W](x,y) = \begin{cases} 0 & \text{if } 0 \le x < \lambda \text{ or } 0 \le y < \lambda, \\ W\left(\frac{x-\lambda}{1-\lambda}, \frac{y-\lambda}{1-\lambda}\right) & \text{otherwise.} \end{cases}$$

Lemma 3.3 (corresponds to [19, Lemma 3.5])**.** *If $\lambda \in [0,1]$ and the graphon W has the property that all functions in $\mathscr{C}$ are δ-good for it, then the same applies to $[\lambda, W]$.*

Similarly, let $[W, \lambda]$ be the graphon satisfying

$$[W, \lambda](x,y) = \begin{cases} W\left(\frac{x}{1-\lambda}, \frac{y}{1-\lambda}\right) & \text{if } 0 \le x \le 1-\lambda \text{ and } 0 \le y \le 1-\lambda, \\ 1 & \text{otherwise.} \end{cases}$$

The next lemma follows from the previous two lemmas and the observation that $[W, \lambda]$ is isomorphic to $1 - [\lambda, 1 - W]$.

Lemma 3.4 (corresponds to [19, Lemma 3.6])**.** *If all functions in $\mathscr{C}$ are δ-good for the graphon W and $\lambda \in [0,1]$, then all functions in $\mathscr{C}$ are good for $[W, \lambda]$ as well.*

We now prove Proposition 3.1. Let $\mathcal{G}$ be the collection of all step graphons and let $\mathcal{G}_i$ be the collection of all step graphons with i parts.

Proof of Proposition 3.1. Suppose for a contradiction that there exists $W' \in \mathcal{G}$ and $F' \in \mathscr{C}$ such that F' is bad for W'. This means that $D(F', W') > \mathrm{MAX}(\gamma, W')$. Then there exists $\delta > 0$ such that $D(F', W') \ge \mathrm{MAX}(\gamma, W') + \delta$. In other words, F' is δ-bad for W'. Let

$$\mathcal{S} = \mathcal{S}(\delta) := \{(F, W) \in \mathscr{C} \times \mathcal{G} : F \text{ is } \delta\text{-bad for } W\}.$$

Note that $(F', W') \in \mathcal{S}$, so $\mathcal{S} \neq \emptyset$. Partition $\mathcal{S}$ into $\bigcup_{i=1}^{\infty} \mathcal{S}_i$, where

$$\mathcal{S}_i := \{(F, W) \in \mathscr{C} \times \mathcal{G}_i : F \text{ is } \delta\text{-bad for } W\}.$$

Let k be the smallest integer such that $\mathcal{S}_k \neq \emptyset$. As all convex functions are good for all constant graphons (see [19, Observation 2.1]), we have $k \geq 2$. Pick $F \in \mathscr{C}$ such that $(F, W) \in \mathcal{S}_k$ for some W. Let

$$\mathcal{W} := \{W : (F, W) \in \mathcal{S}_k\}.$$

Let μ be the Lebesgue measure on $\mathbb{R}$. Each $W \in \mathcal{W}$ is a step function with respect to a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the unit interval with $\alpha_i = \mu(P_i)$ for all $i \in [k]$ and β_{ij} the value attained by W on $P_i \times P_j$ for $i, j \in [k]$. By the choice of k, we deduce $\alpha_1, \ldots, \alpha_k > 0$ for all $W \in \mathcal{W}$. Define

$$T(W) := \sum_{i=1}^{k} \alpha_i (d_W(i))^2,$$

where

$$d_W(i) = \int_0^1 W(i, y)\ dy = \sum_{j=1}^{k} \alpha_j \beta_{ij}$$

for all $i \in [k]$.

Claim 3.5. *$T := \sup_{W \in \mathcal{W}} T(W) = \max_{W \in \mathcal{W}} T(W)$.*

Proof. First we note that each $\mathcal{G}_\ell$ has a natural compact topology corresponding to convergence of all parameters α_i and β_{ij}. In particular, the space is homeomorphic to $\Delta^{\ell-1} \times [0,1]^{\binom{\ell+1}{2}}$, where $\Delta^{\ell-1}$ is the standard $(\ell - 1)$-simplex since $\sum_{i=1}^{\ell} \alpha_i = 1$. In this topology, it is straightforward to check that $T(W)$, $D(F, W)$ and $\mathrm{MAX}(\gamma_W, F)$ are continuous functions of W provided that F is continuous and that the property of being δ-bad is preserved under taking the limit. (This is why we needed to define the property of being δ-bad, as this would not be true if we replaced "δ-bad" with just "bad.") In particular, this implies that the set $\mathcal{W}$ is closed, hence compact. Thus, $T(\mathcal{W})$ must attain its maximum in the compact set $\mathcal{W}$. $\square$

Fix $W \in \mathcal{W}$ with partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ and parameters α_i and $\beta_{ij} \in [0, 1]$ for all $i, j \in [k]$ such that $T(W) = T$. Recall that by the minimality of k, we know that $\alpha_i \in (0, 1)$. By definition of $\mathcal{W}$, F is δ-bad for W. Set $d_i := d_W(i)$. To obtain the necessary contradiction to complete the proof of the proposition, we will show that F is δ-good for W, i.e. that

$$D(F, W) = \int_0^1 F(d_W(x))\ dx = \sum_{i=1}^{k} \alpha_i F(d_i) < \mathrm{MAX}(\gamma, F) + \delta,$$

for $\gamma = t(\,|\,, W) = \sum_{i=1}^{k} \alpha_i d_i$. Without loss of generality, we may assume $d_1 \leq d_2 \leq \cdots \leq d_k$.

Claim 3.6 (corresponds to [19, Claim 3.8])**.** *If $1 \leq r < s \leq k$ and $\beta_{ir} > 0$, then $\beta_{is} = 1$.*

Proof of Claim 3.6. Suppose, for contradiction, that $\beta_{ir} > 0$ and $\beta_{is} < 1$. Define $Q \in \mathcal{G}_k$ with the same partition $\mathcal{P}$ as follows: let δ_{ij} denote the Kronecker delta, and set, for $x \in P_m$ and $y \in P_n$,

$$Q(x, y) = \begin{cases} -(1 + \delta_{ir})\alpha_s & \text{if } \{m, n\} = \{i, r\}, \\ (1 + \delta_{is})\alpha_r & \text{if } \{m, n\} = \{i, s\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\varepsilon \geq 0$ be maximal such that $W_\varepsilon = W + \varepsilon Q$ still satisfies $W_\varepsilon(x, y) \in [0, 1]$ for all $x, y \in [0, 1]$. By our assumptions on β_{ir} and β_{is}, we know that $\varepsilon > 0$.

For all $j \in [k]$, let d'_j denote the value attained by $d_{W_\varepsilon}(x)$ for all $x \in P_j$. We have

$$d'_r - d_r = -(1+\delta_{ir})\alpha_i\alpha_s\varepsilon + \delta_{ir}(1+\delta_{is})\alpha_i\alpha_s\varepsilon = -\alpha_i\alpha_s\varepsilon,$$

and similarly

$$d'_s - d_s = \alpha_i\alpha_r\varepsilon.$$

Further, if $i \notin \{r, s\}$, then

$$d'_i - d_i = \alpha_r(\beta_{ir} - \alpha_s) + \alpha_s(\beta_{is} + \alpha_r) - (\alpha_r\beta_{ir} + \alpha_s\beta_{is}) = 0.$$

Therefore $d'_j = d_j$ for all $j \notin \{r, s\}$. Consequently,

$$\begin{aligned} T(W_\varepsilon) - T(W) = \sum_{j=1}^{k} \alpha_j (d'_j)^2 - \sum_{j=1}^{k} \alpha_j d_j^2 &= \alpha_r(d_r - \alpha_i\alpha_s\varepsilon)^2 - \alpha_r d_r^2 + \alpha_s(d_s + \alpha_i\alpha_r\varepsilon)^2 - \alpha_s d_s^2 \\ &= 2\varepsilon\alpha_i\alpha_r\alpha_s(d_s - d_r) + \varepsilon^2\alpha_i^2\alpha_r\alpha_s(\alpha_r + \alpha_s) \\ &\geq \varepsilon^2\alpha_i^2\alpha_r\alpha_s(\alpha_r + \alpha_s) > 0 \end{aligned}$$

since $\alpha_i, \alpha_r, \alpha_s, \varepsilon > 0$ and $d_s \geq d_r$ by assumption. Thus $T(W_\varepsilon) > T(W)$, and clearly $W_\varepsilon \in \mathcal{G}_k$, so we conclude by the choice of W that $W_\varepsilon \notin \mathcal{W}$. To complete the proof, we will now obtain the contradiction $W_\varepsilon \in \mathcal{W}$ by showing that F is δ-bad for W_ε. First, observe that

$$\int_{[0,1]^2} Q(x,y)\ dxdy = \alpha_i\alpha_r\alpha_s((1+\delta_{is})(2-\delta_{is}) - (1+\delta_{ir})(2-\delta_{ir})) = 0.$$

This implies that $t(\,|\,, W_\varepsilon) = t(\,|\,, W) = \gamma$. Next,

$$\begin{aligned} D(F, W_\varepsilon) - D(F, W) &= \int_0^1 F(d_{W_\varepsilon}(x))\ dx - \int_0^1 F(d_W(x))\ dx \\ &= \sum_{j=1}^{k} \alpha_j (F(d'_j) - F(d_j)) \\ &= \alpha_s(F(d_s + \alpha_i\alpha_r\varepsilon) - F(d_s)) + \alpha_r(F(d_r - \alpha_i\alpha_s\varepsilon) - F(d_r)) \\ &\geq \alpha_i\alpha_r\alpha_s\varepsilon(F'(d_s) - F'(d_r)) \geq 0 \end{aligned}$$

by convexity of F and $d_s \geq d_r$. Since F is δ-bad for W, we obtain

$$D(F, W_\varepsilon) \geq D(F, W) \geq \mathrm{MAX}(\gamma, F) + \delta.$$

This shows that F is δ-bad for W_ε and completes the proof of the claim. □

Claim 3.7 (corresponds to [19, Claim 3.9])**.** *$\beta_{1k} > 0$.*

Proof. If this does not hold, then $\beta_{1k} = 0$ and the previous claim implies $\beta_{1i} = 0$ for all $i \in [k-1]$. It follows that there exists $W' \in \mathcal{G}_{k-1}$ such that W is isomorphic to $[\alpha_1, W']$. Due to the minimality of k all functions in $\mathscr{C}$ are δ-good for W' and by Lemma 3.3 the same applies to the graphon W, contrary to its choice. □

Claim 3.8 (corresponds to [19, Claim 3.10])**.** *$\beta_{1k} < 1$.*

Proof. Suppose for contradiction that $\beta_{1k} = 1$. Then $\beta_{k1} = \beta_{1k} > 0$, and Claim 3.6 implies $\beta_{ki} = 1$ for all $1 \le i \le k$. So some $W' \in \mathcal{G}_{k-1}$ has the property that $[W', \alpha_k]$ is isomorphic to W. This implies that all functions in $\mathscr{C}$ are δ-good for W' and then Lemma 3.4 implies that all functions in $\mathscr{C}$ are δ-good for W, contradiction. ☐

By Claims 3.7 and 3.8, we have $0 < \beta_{1k} < 1$. Moreover, by Claim 3.6, $\beta_{1i} = 0$ for all $i \in [k-1]$ and $\beta_{jk} = 1$ for all j with $2 \le j \le k$. Divide P_k into two measurable subsets Q_k and Q_{k+1} satisfying $\mu(Q_k) = (1-\beta_{1k})\alpha_k$ and, consequently, $\mu(Q_{k+1}) = \beta_{1k}\alpha_k$. Set $Q_i = P_i$ for $i \in [k-1]$ and $\mathcal{Q} = \{Q_1, \dots, Q_{k+1}\}$. Let W' be the step graphon with respect to $\mathcal{Q}$ defined as follows: for $x \in Q_i$ and $y \in Q_j$,

$$W'(x,y) = \begin{cases} \beta_{ij} & \text{if } 2 \le i \le k \text{ and } 2 \le j \le k, \\ 0 & \text{if } i = 1 \text{ and } j \in [k] \text{ or vice versa,} \\ 1 & \text{if } i = k+1 \text{ or } j = k+1. \end{cases}$$

By the last two clauses W' is isomorphic to a graphon of the form $[[\alpha_1/(1-\beta_{1k}\alpha_k), W''], \beta_{1k}\alpha_k]$ for some graphon W'', and by the first clause $W'' \in \mathcal{G}_{k-1}$. So Lemmas 3.3 and 3.4 show that F is δ-good for W'. Set $d' := d_k - \alpha_1\beta_{1k}$ and $d'' := d_k + \alpha_1(1-\beta_{1k})$. Since $t(\,|\,, W') = t(\,|\,, W) = \gamma$ and $(1-\beta_{1k})d' + \beta_{1k}d'' = d_k$, Jensen's Inequality implies that

$$D(F,W) = \sum_{i=1}^{k} \alpha_i F(d_i) \le \sum_{i=1}^{k-1} \alpha_i F(d_i) + \alpha_k((1-\beta_{1k})F(d') + \beta_{1k}F(d'')) = D(F,W').$$

Therefore, $D(F,W) \le D(F,W') < \mathrm{MAX}(\gamma, F) + \delta$ so F is δ-good for W, a contradiction. ☐

The proof of Theorem 2.1 follows from Proposition 3.1 exactly as in [19].

The error in [19] is in the proof of their Claim 3.8 (which corresponds to our Claim 3.6). In [19], T is defined as the number of pairs $(i,j) \in [k]^2$ for which $\beta_{ij} \in \{0,1\}$ and $W \in \mathcal{G}_k$ is chosen to maximize T. Then Claim 3.8 in [19] states that $\beta_{ir} > 0$ and $\beta_{is} < 1$ together imply that W_ε is 0 or 1 on at least $T+1$ of the sets $P_i \times P_j$ by construction. However, this is not true if $\beta_{ir} = 1$ and $\beta_{is} = 0$. For example, consider the graphon W defined on the partition $P_1 = [0, 2/5), P_2 = [2/5, 4/5), P_3 = [4/5, 1]$ by

$$W(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (P_1 \times P_3) \cup (P_2 \times P_2) \cup (P_2 \times P_3) \\ 0 & \text{if } (x,y) \in (P_1 \times P_1) \cup (P_1 \times P_2) \cup (P_3 \times P_3) \end{cases}$$

(see Figure 4). We see that $d_1 = 1/5$, $d_2 = 3/5$, and $d_3 = 4/5$, so $d_1 \le d_2 \le d_3$ is satisfied. Setting $i = 3$, $r = 2$, and $s = 3$, we have that $\beta_{ir} = 1 > 0$ and $\beta_{is} = 0 < 1$. However, all entries are already 0 or 1 in W, so W_ε cannot equal 0 or 1 on more sets than W. Note that any step graphon with parts ordered by degree such that $\beta_{ir} = 1$ and $\beta_{is} = 0$ for some i and $r < s$ is also a counterexample; the other entries need not equal 0 or 1 as in this example.

4 Proof of Theorem 2.3

Recall that we are assuming $G = ([n], E)$. For a vertex i in G, the *right-degree* of i is $d_G^+(i) := |\{j > i : ij \in E\}|$. Recall that when we write uv for an edge, we implicitly mean $u < v$. The *left vertex* of $vw \in E$ is v. A vertex v has *full right-degree* if $E \supset \{vw : w > v\}$. Note that $N_{\mathrm{ord}}(S_L(k), G) = \sum_{i \in V(G)} \binom{d_G^+(i)}{k}$.

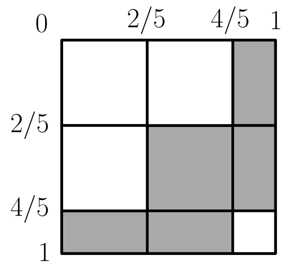

Figure 4: A counterexample to [19, Claim 3.8] (shaded squares have β value 1).

Proof of Theorem 2.3. If $m = 0$ or $m = \binom{n}{2}$, the theorem is trivial, so we assume that $0 < m < \binom{n}{2}$.

Let $G = ([n], E)$ with $|E| = m > 0$ such that G maximizes the number of copies of $S_L(k)$ among all ordered n vertex m edge graphs. Suppose that there exists $i \in [n-1]$ such that $d^+_G(i) < d^+_G(i+1)$. Let G_i be the ordered graph obtained from G by interchanging the positions of i and $i+1$ in the ordering of $V(G)$. We say that G_i is obtained from G by *swapping* i and $i+1$.

Since i and $i+1$ are consecutive in the ordering of G, we have that $d^+_G(j) = d^+_{G_i}(j)$ for all $j \in [n] \setminus \{i, i+1\}$. If there is no edge between i and $i+1$, then $d^+_G(i) = d^+_{G_i}(i)$ and $d^+_G(i+1) = d^+_{G_i}(i+1)$ as well, so $N(S_L(k), G_i) = N(S_L(k), G)$. If there is an edge between i and $i+1$, then $d^+_{G_i}(i) = d^+_G(i) - 1$ and $d^+_{G_i}(i+1) = d^+_G(i+1) + 1$. By convexity of the binomial coefficient and the fact that $d^+_G(i) < d^+_G(i+1)$,

$$\begin{aligned} N_{\text{ord}}(S_L(k), G_i) - N_{\text{ord}}(S_L(k), G) &= \sum_{j \in V(G_i)} \binom{d^+_{G_i}(j)}{k} - \sum_{j \in V(G)} \binom{d^+_G(j)}{k} \\ &= \left(\binom{d^+_{G_i}(i)}{k} + \binom{d^+_{G_i}(i+1)}{k}\right) - \left(\binom{d^+_G(i)}{k} + \binom{d^+_G(i+1)}{k}\right) \\ &= \left(\binom{d^+_G(i) - 1}{k} + \binom{d^+_G(i+1) + 1}{k}\right) - \left(\binom{d^+_G(i)}{k} + \binom{d^+_G(i+1)}{k}\right) \geq 0. \end{aligned}$$

We continue swapping adjacent vertices in this way until there is no $i \in [n-1]$ such that $d^+_G(i) < d^+_G(i+1)$. Call the resulting graph G'. Then $d^+_{G'}(v) \geq d^+_{G'}(w)$ for all $v < w$ and $N_{\text{ord}}(S_L(k), G') = N_{\text{ord}}(S_L(k), G)$ by our assumption on G.

Let $v_{G'} := \min\{i \in [n] : d^+_{G'}(i) < n - i\}$, which exists since $m < \binom{n}{2}$, and choose $w \in [n]$ such that $w > v_{G'}$ and $v_{G'}w \notin E(G')$. Let $x_{G'} := \max\{i \in [n] : d^+_{G'}(i) > 0\}$, which exists since $m > 0$. For brevity, set $v' := v_{G'}$ and $x' := x_{G'}$. Note that if $v' = 1$, then $x' \geq 1 = v'$; if $v' \geq 2$, then the definition of v' implies that $d^+_{G'}(v' - 1) = n - v' + 1 > 0$, so $x' \geq v' - 1$. Thus $v' \leq x' + 1$. In the next paragraph, we show that there is a graph G'' with the same number of copies of $S_L(k)$ as G' satisfying $x_{G''} \leq v_{G''} \leq x_{G''} + 1$.

Suppose that $v' < x'$ and fix $y \in [n]$ such that $x'y \in E(G')$ (recall that this notation implies $x' < y$). Let $H = ([n], (E(G') \cup \{v'w\}) \setminus \{x'y\})$. Then $d^+_H(v') = d^+_{G'}(v') + 1$ and $d^+_H(x') = d^+_{G'}(x') - 1$. By convexity of the binomial coefficient and the fact that $d^+_{G'}(v') \geq d^+_{G'}(x')$,

$$N_{\text{ord}}(S_L(k), H) \geq N_{\text{ord}}(S_L(k), G') = N_{\text{ord}}(S_L(k), G).$$

We continue adding and deleting edges in this way until we reach a new graph, call it G'', satisfying $v_{G''} \geq x_{G''}$. Notice that $x_{G''} \leq v_{G''} \leq x_{G''} + 1$ and $N_{\text{ord}}(S_L(k), G'') = N_{\text{ord}}(S_L(k), G)$.

Setting $v := v_{G''}$, we further see that

$$E(G'') = \{ij : i \in [v-1], j \in [n]\} \cup A$$

for some $A \subset \{vj : j \in \{v+1, v+2, \dots, n\}\}$ with $|A| = b$ for some $0 \le b < n - v$. Since $d^+_{G''}(v) = b$ and $d^+_{G''}(w) = 0$ for $w > v$,

$$N_{\text{ord}}(S_L(k), G) = N_{\text{ord}}(S_L(k), G'') = N_{\text{ord}}(S_L(k), S_L(n, m))$$

as required.

Let $G = ([n], E)$ with $|E| = m$ such that G minimizes the number of copies of $S_L(k)$ among all ordered n vertex m edge graphs. Suppose further that there are some $i, j \in V(G)$ with $i < j$ such that $d^+_G(j) < n - j$ and $d^+_G(i) > d^+_G(j)$. Then choose $v \in [n]$ such that $j < v$ and $jv \notin E$. As $d^+_G(i) > d^+_G(j) \ge 0$, we also choose $w \in [n]$ such that $i < w$ and $iw \in E$. Let $H = ([n], (E \cup \{jv\}) \setminus \{iw\})$. Then $d^+_H(i) = d^+_G(i) - 1$ and $d^+_H(j) = d^+_G(j) + 1$. By convexity of the binomial coefficient,

$$N_{\text{ord}}(S_L(k), H) \le N_{\text{ord}}(S_L(k), G).$$

By our assumption on G, we must have $N_{\text{ord}}(S_L(k), H) = N_{\text{ord}}(S_L(k), G)$. We repeatedly remove an edge iw and add an edge jv for $i < j$ satisfying $d^+(j) < n - j$ and $d^+(i) > d^+(j)$ until no such i and j exist. Call the resulting graph G' and note that G' has m edges. Then $N_{\text{ord}}(S_L(k), G') = N_{\text{ord}}(S_L(k), G)$ and

$$\text{for all } i < j \text{ in } G', \text{ either } d^+_{G'}(j) = n - j \text{ or } d^+_{G'}(i) \le d^+_{G'}(j) \qquad (*).$$

We note here that the right degree sequence of $S_R(n, m)$ is given by (not necessarily in the vertex order):

$$0, 1, \dots, a-1, \underbrace{a, \dots, a}_{n-a-b}, \underbrace{a+1, \dots, a+1}_{b}. \tag{1}$$

Our plan is to alter G' until its right degree sequence is the same as in (1). This will allow us to conclude that $N_{\text{ord}}(S_L(k), G') = N_{\text{ord}}(S_L(k), S_R(n, m))$.

Set $v_{G'} := \min\{j \in [n] : d^+_{G'}(j) = n - j\}$. For brevity, set $v' := v_{G'}$. Then $d^+_{G'}(i) \le d^+_{G'}(j)$ for all $i < j \le v' - 1$ by $(*)$. Furthermore, if there exists $i \ge v'$ such that $d^+_{G'}(i) < n - i$, then we can find consecutive vertices x, y such that $v' \le x < y$, $d^+_{G'}(y) < n - y$ and $d^+_{G'}(x) = n - x$. But this is impossible as $(*)$ implies that $d^+_{G'}(y) \ge d^+_{G'}(x) = n - x > n - y$. Consequently, $d^+_{G'}(i) = n - i$ for all $i \in \{v', v'+1, \dots, n\}$.

If $v' = n$, then $d^+_{G'}(n-1) = 0$ and $d^+_{G'}(i) \le d^+_{G'}(n-1) = 0$ for all $i \in [n-1]$, so $|E(G')| = 0$. This contradicts $m > 0$, so we must have $v' < n$. If $d^+_{G'}(i) \ge n - v' + 1$ for some $i \in [v' - 1]$, then since $i \le v' - 1$, we have $d^+_{G'}(v'-1) \ge d^+_{G'}(i) \ge n - v' + 1 = n - (v' - 1)$. This implies that $d^+_{G'}(v'-1) = n - (v'-1)$, contradicting the definition of v'. Therefore $v' < n$ and $d^+_{G'}(i) \le n - v'$ for all $i \in [v' - 1]$.

Next, suppose that $d^+_{G'}(1) < n - v' - 1$. Note that $d^+_{G'}(v') = n - v' > n - v' - 1 > d^+_{G'}(1)$ implies that $d^+_{G'}(v') \ge d^+_{G'}(1) + 2$. Choose $x \in [n] \setminus \{1\}$ such that $1x \notin E(G')$ and $y \in \{v'+1, \dots, n\}$ such that $v'y \in E(G')$. Let $H' = ([n], (E(G') \cup \{1x\}) \setminus \{v'y\})$. Then $d^+_{H'}(1) = d^+_{G'}(1) + 1$ and $d^+_{H'}(v') = d^+_{G'}(v') - 1$. By convexity of binomial coefficients and the fact that $d^+_{G'}(v') \ge d^+_{G'}(1) + 2$,

$$N_{\text{ord}}(S_L(k), H') \le N_{\text{ord}}(S_L(k), G') = N_{\text{ord}}(S_L(k), G).$$

We continue adding and deleting edges in this way until we reach a graph, call it G'', that satisfies $d^+_{G''}(1) \ge n - v_{G''} - 1$. Note that $n - v_{G''} - 1 \le d^+_{G''}(1) \le n - v_{G''}$.

Set $v := v_{G''}$. Since the right-degrees of vertices x are nondecreasing for $1 \le x \le v$, there is some $0 < b_0 \le v$ such that $d^+_{G''}(i) = n - v - 1$ for all $i \in [v - b_0]$, $d^+_{G''}(i) = n - v$ for the remaining

b_0 vertices $v - b_0 + 1 \leq i \leq v$; we have already observed that $d^+_{G''}(i) = n - i$ for all $i > v$. Thus the right degree sequence of G'' is given by

$$0, 1, \ldots, n-v-2, \underbrace{n-v-1, \ldots, n-v-1}_{v+1-b_0}, \underbrace{n-v, \ldots, n-v}_{b_0}. \tag{2}$$

If $b_0 = v$, set $b = 0$ and $a = n - v$. Otherwise, set $b = b_0$ and $a = n - v - 1$. In either case, the degree sequence of $S_R(n, m)$ in (1) is the same as that of G'' in (2) and therefore

$$N_{\mathrm{ord}}(S_L(k), G) = N_{\mathrm{ord}}(S_L(k), G'') = N_{\mathrm{ord}}(S_L(k), S_R(n, m))$$

as required. □

5 Proof of Theorem 2.7

Let $(G_n) := (G_n)_{n=1}^{\infty}$ be a sequence of graphs with $\lim_{n\to\infty} |V(G_n)| = \infty$. The sequence (G_n) is *F-good* if both $x = \lim_{n\to\infty} \varrho(G_n)$ and $y = \lim_{n\to\infty} \varrho_{\mathrm{ind}}(F, G_n)$ exist. In this case, we say that (G_n) *realizes* (x, y). Define

$$I_{\mathrm{ind}}(F, x) := \sup\{y : (x, y) \in [0, 1] \text{ is realized by some } F\text{-good } (G_n)\}.$$

The following two results will be needed to give upper bounds.

Theorem 5.1 (Kruskal–Katona [9, 10])**.** *Let $r \geq 2$ be an integer. Then for every $x \in [0, 1]$ we have $I_{\mathrm{ind}}(K_r, x) \leq x^{r/2}$.*

Theorem 5.2 (Liu, Mubayi, Reiher [11, Theorem 1.16])**.** *Let $s \geq 2$ be an integer. Then for every $x \in [0, 1]$ we have $I_{\mathrm{ind}}(K_{s,s}, x) \leq \binom{2s}{s} x^s / 2^s$.*

Proof of Theorem 2.7. We address the two cases separately.

Case 1: $\sqrt{x_b} + \sqrt{x_g} \leq 1$.
We first prove the upper bound. Suppose that $(G_n)_{n=1}^{\infty}$ be a $K'_{s,t}$-good sequence that realizes $((x_b, x_g), I_3(K'_{s,t}, (x_b, x_g)))$ and let f_n be the coloring function for G_n. Let $K'_s = ([s], \binom{s}{2})$ with coloring function $f_b \equiv$ blue. Similarly, let $K'_t = ([t], \binom{t}{2})$ have coloring function $f_g \equiv$ green. Since $K'_{s,t}$ contains exactly one copy of K'_s and one copy of K'_t,

$$N_3(K'_{s,t}, G_n) \leq N_3(K'_s, G_n) N_3(K'_t, G_n). \tag{3}$$

By deleting edges that are not colored blue and applying Theorem 5.1, we obtain

$$\limsup_{n\to\infty} \frac{N_3(K'_s, G_n)}{\binom{|V(G_n)|}{s}} \leq I_3(K'_s, (x_b, x_g)) \leq I_{\mathrm{ind}}(K_s, x_b) \leq x_b^{s/2}.$$

Similarly, by deleting edges not colored green, we obtain

$$\limsup_{n\to\infty} \frac{N_3(K'_t, G_n)}{\binom{|V(G_n)|}{t}} \leq I_3(K'_t, (x_b, x_g)) \leq I_{\mathrm{ind}}(K_t, x_g) \leq x_g^{t/2}.$$

Together with (3), we get

$$I_3(K'_{s,t}, (x_b, x_g)) \leq \lim_{n\to\infty} \frac{x_b^{s/2} \binom{|V(G_n)|}{s} \cdot x_g^{t/2} \binom{|V(G_n)|}{t}}{\binom{|V(G_n)|}{s+t}} = x_b^{s/2} x_g^{t/2} \binom{s+t}{s}.$$

We now prove the lower bound. As $\lfloor n\sqrt{x_b}\rfloor + \lfloor n\sqrt{x_g}\rfloor \le n\sqrt{x_b} + n\sqrt{x_g} \le n$, there exist disjoint subsets A_n and B_n of $[n]$ with $|A_n| = \lfloor n\sqrt{x_b}\rfloor$ and $|B_n| = \lfloor n\sqrt{x_g}\rfloor$ for all n. For each $n \ge 1$, let $G_n = ([n], \binom{[n]}{2})$ have coloring function f_n defined by

$$f_n(ij) = \begin{cases} \text{blue} & \text{if } i,j \in A_n, \\ \text{green} & \text{if } i,j \in B_n, \\ \text{red} & \text{otherwise.} \end{cases}$$

See Figure 5 for an illustration. Clearly, $\lim_{n\to\infty} \varrho_{\text{blue}}(G_n) = x_b$ and $\lim_{n\to\infty} \varrho_{\text{green}}(G_n) = x_g$. If

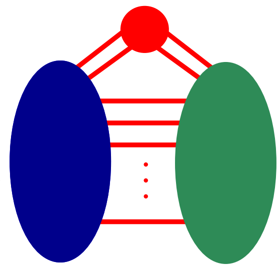

Figure 5: Construction maximizing $K'_{s,t}$-density when $\sqrt{x_b} + \sqrt{x_g} \le 1$.

A is a copy of K'_s in G_n and B is a copy of K'_t in G_n, then $A \cup B$ is a copy of $K'_{s,t}$ in G_n, as $f_n(ij) = \text{red}$ for all $i \in A_n$, $j \in B_n$. Thus, $(G_n)_{n=1}^{\infty}$ realizes (x_b, x_g, y), where

$$y = \lim_{n\to\infty} \varrho_3(K'_{s,t}, G_n) = \lim_{n\to\infty} \frac{\binom{\lfloor n\sqrt{x_b}\rfloor}{s} \cdot \binom{\lfloor n\sqrt{x_g}\rfloor}{t}}{\binom{n}{s+t}} = \lim_{n\to\infty} \frac{(s+t)!n^s x_b^{s/2} n^t x_g^{t/2}}{n^{s+t} s!t!} = x_b^{s/2} x_g^{t/2} \binom{s+t}{s}.$$

Consequently, $I_3(K'_{s,t}, (x_b, x_g)) \ge x_b^{s/2} x_g^{t/2} \binom{s+t}{s}$.

Case 2: $\sqrt{x_b} + \sqrt{x_g} \ge 1$.

We first prove the upper bound. Set $x_r := 1 - x_b - x_g$. Let $(G_n)_{n=1}^{\infty}$ be a $K'_{s,s}$-good sequence that realizes $((x_b, x_g), I_3(K'_{s,s}, (x_b, x_g)))$. Every copy of $K'_{s,s}$ contains a red copy of $K_{s,s}$. Consequently, Theorem 5.2 implies that

$$I_3(K'_{s,s}, (x_b, x_g)) = \lim_{n\to\infty} \frac{N_3(K'_{s,s}, G_n)}{\binom{|V(G_n)|}{2s}} \le I_{\text{ind}}(K_{s,s}, x_r) \le \left(\frac{x_r}{2}\right)^s \binom{2s}{s}.$$

We now prove the lower bound. Note that

$$\lfloor n\sqrt{x_b}\rfloor + \left\lfloor n\frac{x_r}{2\sqrt{x_b}}\right\rfloor \le \frac{2x_b + x_r}{2\sqrt{x_b}} n = \frac{1 + x_b - x_g}{2\sqrt{x_b}} n \le \frac{2\sqrt{x_b}}{2\sqrt{x_b}} n = n$$

since $\sqrt{x_b} + \sqrt{x_g} \ge 1$ implies $x_g \ge 1 - 2\sqrt{x_b} + x_b$. For each n, let $A_n \sqcup B_n \subseteq [n]$, where $|A_n| = \lfloor n\sqrt{x_b}\rfloor$ and $|B_n| = \lfloor nx_r/(2\sqrt{x_b})\rfloor$ and let $G_n = ([n], \binom{[n]}{2})$ have coloring function f_n defined by

$$f_n(ij) = \begin{cases} \text{blue} & \text{if } i,j \in A_n, \\ \text{red} & \text{if } i \in A_n, j \in B_n \text{ or } j \in A_n, i \in B_n, \\ \text{green} & \text{otherwise.} \end{cases}$$

See Figure 6 for an illustration. Now $\lim_{n\to\infty} \varrho_{\text{blue}}(G_n) = x_b$, and

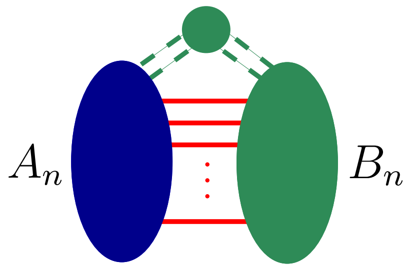

Figure 6: Construction maximizing $K'_{s,s}$-density when $\sqrt{x_b}+\sqrt{x_g}>1$.

$$\lim_{n\to\infty}\varrho_{\text{red}}(G_n)=\lim_{n\to\infty}\frac{\lfloor n\sqrt{x_b}\rfloor\cdot\lfloor nx_r/2\sqrt{x_b}\rfloor}{\binom{n}{2}}=\frac{2x_r\sqrt{x_b}}{2\sqrt{x_b}}=x_r.$$

We conclude that $\lim_{n\to\infty}\varrho_{\text{green}}(G_n)=1-x_b-x_r=x_g$. Every pair of s vertices in A_n and s vertices in B_n induces a copy of $K'_{s,s}$, so $(G_n)_{n=1}^{\infty}$ realizes (x_b,x_g,y), where

$$y=\lim_{n\to\infty}\frac{\binom{\lfloor n\sqrt{x_b}\rfloor}{s}\cdot\binom{\lfloor nx_r/2\sqrt{x_b}\rfloor}{s}}{\binom{n}{2s}}=\lim_{n\to\infty}\frac{n^s x_b^{s/2}n^s x_r^s x_b^{-s/2}}{n^{2s}2^s}\cdot\binom{2s}{s}=\left(\frac{x_r}{2}\right)^s\binom{2s}{s}.$$

Consequently, $I_3(F,(x_b,x_g))\geq\left(\frac{x_r}{2}\right)^s\binom{2s}{s}$. $\square$

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Appendix

Proof of Lemma 3.2. Let $F \in \mathscr{C}$ be the function that we want to prove δ-good for $1-W$. Using the fact that $G : [0,1] \to \mathbb{R}$ given by $G(x) := F(1-x)$ from [19, Lemma 3.1(ii)] is δ-good for W we find that

$$\gamma = t(\,|\,, 1-W) = 1 - t(\,|\,, W) \text{ and } \eta = 1 - \sqrt{1-\gamma}$$

satisfy

$$\begin{aligned} \int_0^1 F(d_{1-W}(x)) &= \int_0^1 G(d_W(x)) \\ &< \max\Big\{\Big(1-\sqrt{1-\gamma}\Big)G(0) + \sqrt{1-\gamma}\,G\Big(\sqrt{1-\gamma}\Big),\ \sqrt{\gamma}G(1-\sqrt{\gamma}) + (1-\sqrt{\gamma})G(1)\Big\} + \delta \\ &= \max\{(1-\eta)F(\eta) + \eta F(1),\ (1-\sqrt{\gamma})F(0) + \sqrt{\gamma}F(\sqrt{\gamma})\} + \delta \end{aligned}$$

as desired. □

Proof of Lemma 3.3. Let $F \in \mathscr{C}$ be any function that we want to prove δ-good for $[\lambda, W]$. By [19, Lemma 3.1(iii)], the function $H : [0,1] \to \mathbb{R}$ given by $H(x) = F((1-\lambda)x)$ for all $x \in [0,1]$ is in $\mathscr{C}$. Thus it is δ-good for W, which tells us that

$$\int_0^1 H(d_W(x))\ dx < \max\{(1-\sqrt{\gamma})H(0) + \sqrt{\gamma}H(\sqrt{\gamma}), (1-\eta)H(\eta) + \eta H(1)\} + \delta,$$

where $\gamma = t(\,|\,, W)$ and $\eta = 1 - \sqrt{1-\gamma}$. Since

$$\begin{aligned}\int_0^1 F(d_{[\lambda,W]}(x))\ dx &= \lambda F(0) + \int_\lambda^1 F\left((1-\lambda)d_W\left(\frac{x-\lambda}{1-\lambda}\right)\right)\ dx \\ &= \lambda F(0) + (1-\lambda)\int_0^1 H(d_W(x))\ dx,\end{aligned}$$

it follows that either

$$\begin{aligned}\int_0^1 F(d_{[\lambda,W]}(x))\ dx &< \lambda F(0) + (1-\lambda)((1-\sqrt{\gamma})F(0) + \sqrt{\gamma}F((1-\lambda)\sqrt{\gamma}) + \delta) \\ &= \lambda F(0) + (1-\lambda)(1-\sqrt{\gamma})F(0) + (1-\lambda)\sqrt{\gamma}F((1-\lambda)\sqrt{\gamma}) + (1-\lambda)\delta \\ &\le \lambda F(0) + (1-\lambda)(1-\sqrt{\gamma})F(0) + (1-\lambda)\sqrt{\gamma}F((1-\lambda)\sqrt{\gamma}) + \delta,\end{aligned}$$

or

$$\begin{aligned}\int_0^1 F(d_{[\lambda,W]}(x))\ dx &< \lambda F(0) + (1-\lambda)((1-\eta)F((1-\lambda)\eta) + \eta F(1-\lambda) + \delta) \\ &= \lambda F(0) + (1-\lambda)(1-\eta)F((1-\lambda)\eta) + (1-\lambda)\eta F(1-\lambda) + (1-\lambda)\delta \\ &\le \lambda F(0) + (1-\lambda)(1-\eta)F((1-\lambda)\eta) + (1-\lambda)\eta F(1-\lambda) + \delta.\end{aligned}$$

In the former case the right side simplifies to

$$\left(1-\sqrt{\gamma'}\right)F(0) + \sqrt{\gamma'}F\left(\sqrt{\gamma'}\right) + \delta,$$

where $\gamma' = (1-\lambda)^2\gamma = t(\,|\,, [\lambda, W])$, meaning that F is, in particular, δ-good for $[\lambda, W]$.

So we may assume that the second alternative occurs. Setting $x = \lambda$, $y = (1-\lambda)\eta$, and $z = (1-\lambda)(1-\eta)$ we thus get

$$\int_0^1 F(d_{[\lambda,W]}(x))\ dx < xF(0) + zF(y) + yF(y+z) + \delta.$$

Since $y^2 + 2yz = (1-\lambda)^2(2\eta - \eta^2) = (1-\lambda)^2\gamma = \gamma'$, it follows in view of [19, Lemma 3.3] that

$$\int_0^1 F(d_{[\lambda,W]}(x))\ dx < \max\left\{\left(1-\sqrt{\gamma'}\right)F(0) + \sqrt{\gamma'}F\left(\sqrt{\gamma'}\right),\ (1-\eta')F(\eta') + \eta' F(1)\right\} + \delta,$$

where $\eta' = 1 - \sqrt{1-\gamma'}$. This tells us that F is indeed δ-good for $[\lambda, W]$. □