Off-diagonal hypergraph Ramsey numbers

Dhruv Mubayi* Andrew Suk†

Abstract

The Ramsey number $r_k(s,n)$ is the minimum N such that every red-blue coloring of the k-subsets of $\{1,\ldots,N\}$ contains a red set of size s or a blue set of size n, where a set is red (blue) if all of its k-subsets are red (blue). Let $r_k(P_s,n)$ be the minimum N such that every red-blue coloring of the k-subsets of $\{1,\ldots,N\}$ results in a red ordered tight path with s vertices or a blue set of size n. The problem of estimating both $r_k(s,n)$ and $r_k(P_s,n)$ for $k=2$ goes back to the seminal work of Erdős and Szekeres from 1935, while the case $k \geq 3$ was first investigated by Erdős and Rado in 1952.

In this paper, we deduce the first quantitative relationship between multicolor variants of $r_k(P_s,n)$ and $r_k(n,n)$. This yields several consequences including the following:

- We determine the correct tower growth rate for both $r_k(s,n)$ and $r_k(P_s,n)$ for $s \geq k+3$. The question of determining the tower growth rate of $r_k(s,n)$ for all $s \geq k+1$ was posed by Erdős and Hajnal in 1972 and this is almost settled here.
- We show that determining the tower growth rate of $r_4(P_5,n)$ is equivalent to determining the tower growth rate of $r_k(n,n)$ for all $k \geq 3$, which is a notorious conjecture of Erdős, Hajnal and Rado from 1965 that remains open.

1 Introduction

A k-uniform hypergraph H with vertex set V is a collection of k-element subsets of V. We write $K_n^{(k)}$ for the complete k-uniform hypergraph on an n-element vertex set. The Ramsey number $r_k(s,n)$ is the minimum N such that every red-blue coloring of the edges of $K_N^{(k)}$ contains a monochromatic red copy of $K_s^{(k)}$ or a monochromatic blue copy of $K_n^{(k)}$. Due to its wide range of applications in logic, number theory, analysis, geometry, and computer science, estimating Ramsey numbers has become one of the most central problems in combinatorics.

1.1 Diagonal Ramsey numbers.

The *diagonal* Ramsey number, $r_k(n,n)$ where k is fixed and n tends to infinity, has been studied extensively over the past 80 years. Classic results of Erdős and Szekeres [16] and Erdős [12] imply

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Research partially supported by NSF grant DMS-1300138. Email: mubayi@uic.edu

†Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Supported by NSF grant DMS-1500153. Email: suk@uic.edu.

that $2^{n/2} < r_2(n,n) \leq 2^{2n}$ for every integer $n > 2$. These bounds have been improved by various authors (see [5, 25, 17]). However, the constant factors in the exponents have not changed over the last 70 years. For 3-uniform hypergraphs, a result of Erdős, Hajnal, and Rado [14] gives the best known lower and upper bounds for $r_3(n,n)$ which are of the form[1] $2^{\Omega(n^2)} < r_3(n,n) \leq 2^{2^{O(n)}}$. For $k \geq 4$, there is also a difference of one exponential between the known lower and upper bounds for $r_k(n,n)$, that is,

$$\mathrm{twr}_{k-1}(\Omega(n^2)) \leq r_k(n,n) \leq \mathrm{twr}_k(O(n)),$$

where the tower function $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$ (see [16, 15, 13]). A notoriously difficult conjecture of Erdős, Hajnal, and Rado states the following (Erdős offered \$500 for a proof).

Conjecture 1.1. (Erdős-Hajnal-Rado [14]) *For $k \geq 3$ fixed, $r_k(n,n) \geq \mathrm{twr}_k(\Omega(n))$.*

The study of $r_3(n,n)$ may be particularly important for our understanding of hypergraph Ramsey numbers. Any improvement on the lower bound for $r_3(n,n)$ can be used with a result of Erdős and Hajnal [13], known as the stepping-up lemma, to obtain improved lower bounds for $r_k(n,n)$, for all $k \geq 4$. In particular, proving that $r_3(n,n)$ grows at least double exponential in $\Omega(n)$, would imply that $r_k(n,n)$ does indeed grow as a tower of height k in $\Omega(n)$, settling Conjecture 1.1. In the other direction, any improvement on the upper bound for $r_3(n,n)$ can be used with a result of Erdős and Rado [15], to obtain improved upper bounds for $r_k(n,n)$, for all $k \geq 4$. It is widely believed that Conjecture 1.1 is true, based on the fact that such bounds are known for four colors. More precisely, the q-color Ramsey number,

$$r_k(\underbrace{n,\ldots,n}_{q \text{ times}})$$

is the minimum N such that every q-coloring of the edges of the complete N-vertex k-uniform hypergraph $K_N^{(k)}$, contains a monochromatic copy of $K_n^{(k)}$. A result of Erdős and Hajnal [13] shows that $r_3(n,n,n,n) > 2^{2^{\Omega(n)}}$, and this implies that

$$r_k(n,n,n,n) = \mathrm{twr}_k(\Theta(n)),$$

for all fixed $k \geq 4$. For three colors, Conlon, Fox, and Sudakov [7] showed that for fixed $k \geq 3$,

$$r_k(n,n,n) \geq \mathrm{twr}_k(\Omega(\log^2 n)).$$

1.2 Off-diagonal Ramsey numbers.

The *off-diagonal* Ramsey number, $r_k(s,n)$ with k,s fixed and n tending to infinity, has also been extensively studied. It is known [2, 19, 3, 4] that $r_2(3,n) = \Theta(n^2/\log n)$ and, for fixed $s > 3$,

$$c_1\left(\frac{n}{\log n}\right)^{\frac{s+1}{2}} \leq r_2(s,n) \leq c_2\frac{n^{s-1}}{\log^{s-2} n},$$

[1] We write $f(n) = O(g(n))$ if $|f(n)| \leq c|g(n)|$ for some fixed constant c and for all $n \geq 1$; $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$; and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold.

where c_1 and c_2 are absolute constants. For 3-uniform hypergraphs, a result of Conlon, Fox, and Sudakov [7] shows that for fixed $s \geq 4$

$$2^{\Omega(n \log n)} \leq r_3(s,n) \leq 2^{n^{s-2} \log n}.$$

For fixed $s > k \geq 4$, it is known that

$$r_k(s,n) \leq \mathrm{twr}_{k-1}(O(n^{s-2} \log n)).$$

By applying the Erdős-Hajnal stepping up lemma in the off-diagonal setting, it follows that $r_k(s,n) \geq \mathrm{twr}_{k-1}(\Omega(n))$, for $k \geq 4$ and for all $s \geq 2^{k-1} - k + 3$. In 1972, Erdős and Hajnal conjectured the following.

Conjecture 1.2. (Erdős-Hajnal [13]) *For $s \geq k+1 \geq 5$ fixed, $r_k(s,n) \geq \mathrm{twr}_{k-1}(\Omega(n))$.*

In fact, Erdős and Hajnal made a more general conjecture in [13] on the tower growth rate of the minimum $N = N(n,k,t)$, such that every red/blue coloring of the edges of $K_N^{(k)}$ yields either a blue clique of size n, or $k+1$ vertices that induces at least t red edges. For more results on their general conjecture, see [24].

In [6], Conlon, Fox, and Sudakov modified the Erdős-Hajnal stepping-up lemma to show that Conjecture 1.2 holds for all $s \geq \lceil 5k/2 \rceil - 3$. Using a result of Duffus et al. [9] (see also Moshkovitz and Shapira [22] and Milans et al. [21]), one can show that $r_k(s,n) \geq \mathrm{twr}_{k-2}(\Omega(n))$ for all $s \geq k+1$. In this paper, we prove the following result that nearly settles Conjecture 1.2 by determining the correct tower growth rate for $s \geq k+3$, and obtaining new bounds for the two remaining cases.[2]

Theorem 1.3. *There is a positive constant $c > 0$ such that the following holds. For $k \geq 4$ and $n > 3k$, we have*

1. $r_k(k+3,n) \geq \mathrm{twr}_{k-1}(cn)$,
2. $r_k(k+2,n) \geq \mathrm{twr}_{k-1}(c \log^2 n)$,
3. $r_k(k+1,n) \geq \mathrm{twr}_{k-2}(cn^2)$.

There are two novel ingredients to our constructions. First, we relate these problems to estimates for Ramsey numbers of tight-paths versus cliques, which we find of independent interest. Second, we use $(k-1)$-uniform diagonal Ramsey numbers for more than two colors to obtain constructions for k-uniform off-diagonal Ramsey numbers for two colors. This differs from the usual paradigm in this area, exemplified by the stepping up lemma, where the number of colors stays the same or goes up as the uniformity increases (see, e.g. [1, 6, 7, 9, 13, 14, 20, 23]). This topic has also been extensively studied in the context of partition relations for ordinals. It is quite possible that our constructions can also be applied to the infinite setting, though we have not explored this here.

[2] After this paper was written, we learned that a bound similar to Theorem 1.3 part 1 was recently claimed by Conlon, Fox, and Sudakov (unpublished), using the more traditional stepping-up argument of Erdős and Hajnal.

1.3 Tight-path versus clique.

Consider an *ordered* N-vertex k-uniform hypergraph H, that is, a hypergraph whose vertex set is $[N] = \{1, 2, \ldots, N\}$. A *tight path* of size s in H, denoted by $P_s^{(k)}$, comprises a set of s vertices $v_1, \ldots, v_s \in [N]$, $v_1 < \cdots < v_s$, such that $(v_j, v_{j+1}, \ldots, v_{j+k-1}) \in E(H)$ for $j = 1, 2, \ldots, s-k+1$. The *length* of $P_s^{(k)}$ is the number of edges, $s-k+1$.

Here, we obtain lower and upper bounds for Ramsey numbers for tight-paths versus cliques. To be more precise, we need the following definition. Given q ordred k-uniform hypergraphs $F_1, \ldots, F_q$, the Ramsey number $r(F_1, \ldots, F_q)$ is the minimum N such that every q-coloring of the edges of the complete N-vertex k-uniform hypergraph $K_N^{(k)}$, whose vertex set is $[N] = \{1, \ldots, N\}$, contains an i-colored copy of F_i for some i. In order to avoid the excessive use of superscripts, we use the simpler notation

$$r_k(P_s, P_n) = r(P_s^{(k)}, P_n^{(k)}) \qquad \text{and} \qquad r_k(P_s, n) = r(P_s^{(k)}, K_n^{(k)}).$$

The proofs of two famous theorems of Erdős and Szekeres in [16], known as the monotone subsequence theorem and the cups-caps theorem, imply that $r_2(P_s, P_n) = (n-1)(s-1)+1$ and $r_3(P_s, P_n) = \binom{n+s-4}{s-2} + 1$ (see [10]). Fox, Pach, Sudakov, and Suk [10] later extended their results to k-uniform hypergraphs, and gave a geometric application related to the Happy Ending Theorem.[3] See also [9, 22, 21] for related results.

The proof of the Erdős-Szekeres monotone subsequence theorem [16] (see also Dilworth's Theorem [8]) actually implies that $r_2(P_s, n) = (n-1)(s-1)+1$. For $k \geq 3$, estimating $r_k(P_s, n)$ appears to be more difficult. Clearly we have

$$r_k(P_s, n) \leq r_k(s, n) \leq \mathrm{twr}_{k-1}(O(n^{s-2} \log n)). \tag{1}$$

Our main result is a new connection between $r_k(P_s, n)$ and $r_k(n, n)$. Again, we will use the simpler notation

$$r_k(n; q) = r_k(\underbrace{n, \ldots, n}_{q \text{ times}}) \qquad \text{and} \qquad r_k(P_{s_1}, \ldots, P_{s_t}, n) = r(P_{s_1}^{(k)}, \ldots, P_{s_t}^{(k)}, K_n^{(k)}).$$

Theorem 1.4. (Main Result) *Let $k \geq 2$ and $s_1, \ldots, s_t \geq k+1$. Then for $q = (s_1 - k + 1) \cdots (s_t - k + 1)$, we have*

$$r_{k-1}(\lfloor n/q \rfloor; q) \leq r_k(P_{s_1}, \ldots, P_{s_t}, n) \leq r_{k-1}(n; q).$$

Theorem 1.4 when $t = 1$ reduces to the following simpler statement for all $s > k \geq 2$:

$$r_{k-1}\left(\left\lfloor \frac{n}{s-k+1} \right\rfloor; s-k+1\right) \leq r_k(P_s, n) \leq r_{k-1}(n; s-k+1).$$

This has several consequences, the first of which is a considerable improvement to the upper bound for $r_k(P_s, n)$ in (1).

[3] The main result in [16], known as the Happy Ending Theorem, states that for any positive integer n, any sufficiently large set of points in the plane in general position has a subset of n members that form the vertices of a convex polygon.

Corollary 1.5. *For fixed $k \geq 3$ and $s \geq k+1$, we have $r_k(P_s, n) \leq \mathrm{twr}_{k-1}(O(sn \log s))$.*

Indeed, using the standard Erdős-Szekeres recurrence [16], we have $r_2(n; q) < q^{nq} = \mathrm{twr}_2(O(qn \log q))$, and the upper bound argument of Erdős-Rado [15] then yields $r_{k-1}(n; q) < \mathrm{twr}_{k-1}(O(qn \log q))$. Applying Theorem 1.4 with $t = 1, s_1 = s$, and $q = s - k + 1 < s$, now implies Corollary 1.5.

Combining the lower bounds in Theorem 1.4 with the known lower bounds for $r_{k-1}(n, n, n, n)$ in [13], $r_{k-1}(n, n, n)$ in [7], and $r_{k-1}(n, n)$ in [13], we establish the following inequalities. There is an absolute constant $c > 0$ such that for all $k \geq 4$ and $n > 4k$

$$r_k(P_{k+3}, n) \geq r_{k-1}\left(\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}\right) \geq \mathrm{twr}_{k-1}(cn),$$

$$r_k(P_{k+2}, n) \geq r_{k-1}\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right) \geq \mathrm{twr}_{k-1}(c \log^2 n),$$

$$r_k(P_{k+1}, n) \geq r_{k-1}\left(\frac{n}{2}, \frac{n}{2}\right) \geq \mathrm{twr}_{k-2}(cn^2).$$

Summarizing, we have just proved parts 1–3 of the following theorem, which is a strengthening of Theorem 1.3 as $r_k(s, n) \geq r_k(P_s, n)$.

Theorem 1.6. *There is a positive constant $c > 0$ such that $r_3(P_4, n) > 2^{cn}$, and for $k \geq 4$ and $n > 3k$,*

1. $r_k(P_{k+3}, n) \geq \mathrm{twr}_{k-1}(cn)$,
2. $r_k(P_{k+2}, n) \geq \mathrm{twr}_{k-1}(c \log^2 n)$,
3. $r_k(P_{k+1}, n) \geq \mathrm{twr}_{k-2}(cn^2)$.

We conjecture the following strengthening of the Erdős-Hajnal conjecture.

Conjecture 1.7. *For $k \geq 4$ fixed, $r_k(P_{k+1}, n) \geq \mathrm{twr}_{k-1}(\Omega(n))$.*

For $t = 1$, $q = 2$, and $s_1 = k+1$ in Theorem 1.4, we have $r_{k-1}(n/2, n/2) \leq r_k(P_{k+1}, n) \leq r_{k-1}(n, n)$. Hence, we obtain the following corollary which relates $r_4(P_5, n)$ to the diagonal Ramsey number.

Corollary 1.8. *Conjecture 1.1 holds if and only if there is a constant $c > 0$ such that*

$$r_4(P_5, n) \geq 2^{2^{cn}}.$$

Corollary 1.8 shows that our lack of understanding of the Ramsey number $r_k(P_{k+1}, n)$ is due to our lack of understanding of the diagonal Ramsey number $r_{k-1}(n, n)$. However, if we add one additional color, then Theorem 1.4 with $t = 2$ implies that $r_k(P_{k+1}, P_{k+1}, n)$ does indeed grow as a tower of height $k - 1$ in $\Omega(n)$.

Corollary 1.9. *There is a positive constant $c > 0$ such that for $k \geq 4$ and $n > 3k$,*

$$r_k(k+1, k+1, n) \geq r_k(P_{k+1}, P_{k+1}, n) \geq r_{k-1}\left(\frac{n}{4}, \frac{n}{4}, \frac{n}{4}, \frac{n}{4}\right) \geq \mathrm{twr}_{k-1}(cn).$$

Note that by the results of Erdős and Rado [15], for every $k \geq 4$, there is an $c_k > 0$ such that $r_k(k+1, k+1, n) \leq \mathrm{twr}_{k-1}(n^{c_k})$.

In the next Section, we prove Theorem 1.4 and the inequality $r_3(P_4, n) > 2^{\Omega(n)}$ from Theorem 1.6. We sometimes omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation.

2 Ramsey numbers for tight paths versus cliques

In this section, we prove Theorem 1.4.

Proof of Theorem 1.4. Let us first prove the upper bound. Set $q_i = s_i - k + 1$ so that $q = q_1 \cdots q_t$, and $N = r_{k-1}(n; q)$. Let $\chi : \binom{[N]}{k} \rightarrow \{1, 2, \ldots, t+1\}$ be a $(t+1)$-coloring of the edges of $K_N^{(k)}$. We will show that χ must produce a monochromatic copy of $P_{s_i}^{(k)}$ in color i, for some i, or a monochromatic copy of $K_n^{(k)}$ in color $t+1$.

Define $\phi : \binom{[N]}{k-1} \rightarrow \mathbb{Z}^t$ by $\phi(v_1, \ldots, v_{k-1}) = (a_1, \ldots, a_t)$ for $v_1, \ldots, v_{k-1} \in [N]$ and $v_1 < \cdots < v_{k-1}$, where a_i is the *length* of the longest monochromatic tight-path in color i ending with $v_1, \ldots, v_{k-1}$. If $a_i \geq q_i$ for some i, then we have a monochromatic copy of $P_{s_i}^{(k)}$ in color i and are done. Therefore, we can assume $a_i \in \{0, 1, \ldots, q_i - 1\}$ for all i, and hence ϕ uses at most q colors.

Since $N = r_{k-1}(n; q)$, there is a subset $S \subset [N]$ of n vertices such that ϕ colors every $(k-1)$-tuple in S the same color, say with color $(b_1, \ldots, b_t)$. Then for every k-tuple $(v_1, \ldots, v_k) \in \binom{S}{k}$, we have $\chi(v_1, \ldots, v_k) = t+1$. Indeed, suppose there are k vertices $v_1, \ldots, v_k \in S$ such that $\chi(v_1, \ldots, v_k) = i$, where $i \leq t$. Since the longest monochromatic tight-path in color i ending with vertices $v_1, \ldots, v_{k-1}$ is b_i, the longest monochromatic tight-path in color i ending with vertices $v_2, \ldots, v_k$ is at least $b_i + 1$, a contradiction. Therefore, S induces a monochromatic copy of $K_n^{(k)}$ in color $t+1$. This concludes the proof of the upper bound.

We now prove the lower bound. Set $N = r_{k-1}(\lfloor n/q \rfloor; q) - 1$ and $q_i = s_i - k + 1$, so that $q = q_1 \cdots q_t$. Let $K_N^{(k-1)}$ be the complete N-vertex $(k-1)$-uniform hypergraph with vertex set $[N]$. Next, let

$$\phi : \binom{N}{k-1} \rightarrow [q_1] \times \cdots \times [q_t]$$

be a q-coloring on the edges of $K_N^{(k-1)}$, that does not produce a monochromatic copy of $K_{\lfloor n/q \rfloor}^{(k-1)}$. Such a coloring ϕ exists since $N = r_{k-1}(\lfloor n/q \rfloor; q) - 1$. We now define a $(t+1)$-coloring

$$\chi : \binom{[N]}{k} \rightarrow [t+1]$$

on the k-tuples of $[N]$ as follows. For $v_1, \ldots, v_k \in [N]$, where $v_1 < \cdots < v_k$, let $\chi(v_1, \ldots, v_k) = i$ if and only if for $\phi(v_1, \ldots, v_{k-1}) = (a_1, \ldots, a_t)$ and $\phi(v_2, \ldots, v_k) = (b_1, \ldots, b_t)$, i is the minimum index such that $a_i < b_i$. If no such i exists, then $\chi(v_1, \ldots, v_k) = t+1$. We will show that χ does not produce a monochromatic i-colored copy of $P_{s_i}^{(k)}$, for $i \leq t$, nor a monochromatic $(t+1)$-colored copy of $K_n^{(k)}$.

First, suppose that the coloring χ produces a monochromatic $P_{s_i}^{(k)}$ in color i. That is, there are s_i vertices $v_1, v_2, \ldots, v_{s_i} \in [N]$, $v_1 < \cdots < v_{s_i}$, such that $\chi(v_j, v_{j+1}, \ldots, v_{j+k-1}) = i$ for $j = 1, \ldots, s_i - k + 1$. Let $\phi(v_j, v_{j+1}, \ldots, v_{j+k-2}) = (a_{j,1}, \ldots, a_{j,t})$, for $j = 1, \ldots, s_i - k + 2$. Then we have

$$a_{1,i} < a_{2,i} < \cdots < a_{s_i-k+2,i},$$

which is a contradiction since $q_i < s_i - k + 2$. Hence, χ does not produce a monochromatic $P_{s_i}^{(k)}$ in color i for $i \leq t$.

Next, we show that χ does not produce a monochromatic copy of $K_n^{(k)}$ in color $t+1$. Again, for sake of contradiction, suppose there is a set $S \subset [N]$ where $S = \{v_1, \ldots, v_n\}$, $v_1 < \cdots < v_n$, such that χ colors every k-tuple of S with color $t+1$. We obtain a contradiction from the following claim.

Claim 2.1. *Let $S = \{v_1, \ldots, v_n\}$, χ, and ϕ be as above, and $1 \leq \ell \leq q$. If ϕ uses at most ℓ distinct colors on $\binom{S}{k-1}$, and if χ colors every k-tuple of S with color $t+1$, then there is a subset $T \subset S$ of size $\lfloor n/\ell \rfloor$ and a color $a = (a_1, \ldots, a_t)$ such that $\phi(T') = a$ for every $T' \in \binom{T}{k-1}$.*

The contradiction follows from the fact that $\lfloor n/\ell \rfloor \geq \lfloor n/q \rfloor$, and ϕ does not produce a monochromatic copy of $K_{\lfloor n/q \rfloor}^{(k-1)}$.

Proof of Claim. We proceed by induction on ℓ. The base case $\ell = 1$ is trivial. For the inductive step, assume that the statement holds for $\ell' < \ell$. Let $\mathcal{C}$ be the set of ℓ distinct colors defined by ϕ on $\binom{S}{k-1}$, and let $(a_1^*, \ldots, a_t^*) \in \mathcal{C}$ be the smallest element in $\mathcal{C}$ with respect to the lexicographic ordering. We set $S_1 = \{v_1, \ldots, v_{n - \lfloor n/\ell \rfloor}\}$ and $S_2 = \{v_{n-\lfloor n/\ell \rfloor + 1}, \ldots, v_n\}$. The proof now falls into two cases.

Case 1. Suppose there is a $(k-1)$-tuple $(u_1, \ldots, u_{k-1}) \in \binom{S_1}{k-1}$ such that $\phi(u_1, \ldots, u_{k-1}) = (a_1^*, \ldots, a_t^*)$. Then we have $\phi(T') = (a_1^*, \ldots, a_t^*)$ for all $T' \in \binom{S_2}{k-1}$. Indeed let $T' = (w_1, \ldots, w_{k-1}) \in \binom{S_2}{k-1}$. Since $\chi(u_1, \ldots, u_{k-1}, w_1) = t+1$, we have $\phi(u_2, \ldots, u_{k-1}, w_1) = (a_1^*, \ldots, a_t^*)$. Likewise, since we have $\chi(u_2, \ldots, u_{k-1}, w_1, w_2) = t+1$, we have $\phi(u_3, \ldots, u_{k-1}, w_1, w_2) = (a_1^*, \ldots, a_t^*)$. By repeating this argument $k-3$ more times, $\phi(w_1, \ldots, w_{k-1}) = (a_1^*, \ldots, a_t^*)$.

Case 2. If we are not in Case 1, then $\phi(T') \in \mathcal{C} \setminus \{(a_1^*, \ldots, a_t^*)\}$ for every $T' \in \binom{S_1}{k-1}$. Hence ϕ uses at most $\ell - 1$ distinct colors on $\binom{S_1}{k-1}$. By the induction hypothesis, there is a subset $T \subset S_1$ of size $(n - \lfloor n/\ell \rfloor)/(\ell - 1) \geq \lfloor n/\ell \rfloor$ and a color $a = (a_1, \ldots, a_t)$ such that $\phi(T') = a$ for every $T' \in \binom{T}{k-1}$. This concludes the proof of the claim and the theorem. $\square$

Lower bound construction for $r_3(P_4, n)$ in Theorem 1.6. Set $N = 2^{cn}$ where c will be determined later. Consider the coloring $\phi : \binom{[N]}{2} \to \{1, 2\}$, where each edge has probability $1/2$ of being a particular color independent of all other edges. Using ϕ, we define the coloring $\chi : \binom{[N]}{3} \to \{\text{red}, \text{blue}\}$, where the triple (v_1, v_2, v_3), $v_1 < v_2 < v_3$, is red if $\phi(v_1, v_2) < \phi(v_2, v_3)$, and is blue otherwise. It is easy to see that χ does not produce a monochromatic red copy of $P_4^{(3)}$.

Next we estimate the expected number of monochromatic blue copies of $K_n^{(k)}$ in χ. For a given triple $\{v_1, v_2, v_3\} \in \binom{[N]}{3}$, the probability that $\chi(v_1, v_2, v_3) =$ blue is $3/4$. Let $T = \{v_1, \ldots, v_n\}$ be a set of n vertices in $[N]$, where $v_1 < \cdots < v_n$. Let S be a partial Steiner $(n, 3, 2)$-system with vertex set T, that is, S is a 3-uniform hypergraph such that each 2-element set of vertices is contained in at most one edge in S. Moreover, S satisfies $|S| = c'n^2$. It is known that such a system exists. Then the probability that every triple in T is blue is at most the probability that every triple in S is blue. Since the edges in S are independent, that is no two edges have more than one vertex in common, the probability that T is a monochromatic blue clique is at most $\left(\frac{3}{4}\right)^{|S|} \leq \left(\frac{3}{4}\right)^{c'n^2}$. Therefore the expected number of monochromatic blue copies of $K_n^{(k)}$ produced by χ is at most

$$\binom{N}{n}\left(\frac{3}{4}\right)^{c'n^2} < 1,$$

for an appropriate choice for c. Hence, there is a coloring χ with no monochromatic red copy of $P_4^{(3)}$, and no monochromatic blue copy of $K_n^{(k)}$. Therefore $r_3(P_4, n) > 2^{\Omega(n)}$. $\square$

Acknowledgments. We thank David Conlon and Jacob Fox for comments that helped improve the presentation of this paper.

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