# The Poset of Hypergraph Quasirandomness

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**Abstract**

Chung and Graham began the systematic study of $k$-uniform hypergraph quasirandom properties soon after the foundational results of Thomason and Chung-Graham-Wilson on quasirandom graphs. One feature that became apparent in the early work on $k$-uniform hypergraph quasirandomness is that properties that are equivalent for graphs are not equivalent for hypergraphs, and thus hypergraphs enjoy a variety of inequivalent quasirandom properties. In the past two decades, there has been an intensive study of these disparate notions of quasirandomness for hypergraphs, and an open problem that has emerged is to determine the relationship between them.

Our main result is to determine the poset of implications between these quasirandom properties. This answers a recent question of Chung and continues a project begun by Chung and Graham in their first paper on hypergraph quasirandomness in the early 1990's.

# 1 Introduction

An important line of research in extremal combinatorics and computer science in the last few decades is the study of quasirandom or pseudorandom structures. This was initiated by Thomason [42, 43] and Chung, Graham, and Wilson [11], who studied explicitly constructed graphs which mimic the random graph. Applications of quasirandom structures have appeared in many situations in extremal combinatorics and computer science, for example in recent proofs of Szemerédi's Theorem [40] using the Strong Hypergraph Regularity Lemma [15, 30, 32, 33, 41] and in expander graphs [17] in computer science. For details on quasirandomness, we refer the reader to a survey of Krivelevich and Sudakov [24] for graph quasirandomness and recent papers of Gowers [14, 15, 16] for other quasirandom structures.

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Soon after the papers [42, 43] and [11], Chung and Graham [8] initiated the study of quasirandomness in hypergraphs. Since these early papers on the subject, there have been a variety of different notions of quasirandomness defined for hypergraphs, and the relationships between these quasirandom properties are not completely understood. Chung [4, 7] posed the following problem.

**Problem 1. (Chung [4, 7])** How is a given property placed in the quasirandom hierarchy and what is the lattice structure illustrating the relationship among quasirandom properties of hypergraphs?

Our main result is to answer this question for many $k$-uniform hypergraph quasirandom properties.

A *$k$-uniform hypergraph* is a pair of finite sets $(V(G), E(G))$ such that $E(G)$ is a collection of $k$-subsets of $V(G)$. The set $V(G)$ is the vertex set and $E(G)$ is the edge set. For a hypergraph $G$ and $U \subseteq V(G)$, the induced subhypergraph on $U$, denoted $G[U]$, is the hypergraph with vertex set $U$ and edge set $\{e \in E(G) : e \subseteq U\}$. A *graph* is a 2-uniform hypergraph. Let $\mathcal{G} = \{G_n\}_{n\to\infty}$ be a sequence of graphs with $|V(G_n)| = n$ and let $0 < p < 1$ be a fixed real. The graph sequence $\mathcal{G}$ is *$p$-quasirandom* if it satisfies the following properties.

- $\texttt{Disc}_p$: (short for discrepency) for every $U \subseteq V(G_n)$, $|E(G_n[U])| = p\binom{|U|}{2} + o(n^2)$.
- $\texttt{Expand}_p$: For every $S, T \subseteq V(G_n)$, $e(S,T) = p|S||T| + o(n^2)$, where $e(S,T)$ is the number of edges with one endpoint in $S$ and one endpoint in $T$, with edges inside $S \cap T$ counted twice.

The use of little-$o$ notation in the above definitions requires some explanation. The precise definition of $\texttt{Disc}_p$ is the property of graph sequences defined as follows: $\mathcal{G} = \{G_n\}_{n\to\infty}$ with $|V(G_n)| = n$ satisfies $\texttt{Disc}_p$ if there exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = o(n^2)$ (i.e. $\lim_{n\to\infty} f(n)n^{-2} = 0$) so that for all $n$ and all $U \subseteq V(G_n)$, $p\binom{|U|}{2} - f(n) \leq |E(G_n[U])| \leq p\binom{|U|}{2} + f(n)$. $\texttt{Expand}_p$ is defined similarly.

It is easy to see that $\texttt{Disc}_p$ and $\texttt{Expand}_p$ are equivalent; $\texttt{Expand}_p \Rightarrow \texttt{Disc}_p$ is trivial by letting $S = T = U$ and the converse is a simple inclusion/exclusion argument. In addition, $\texttt{Disc}_p$ and $\texttt{Expand}_p$ are both central properties of the random graph. Many more properties of graph sequences have been shown equivalent to $\texttt{Disc}_p$ and $\texttt{Expand}_p$, including counting subgraphs, counting induced subgraphs, spectral conditions, sizes of common neighborhoods, and counting even/odd subgraphs of cycles, see [11, 18, 27, 28, 29, 31, 34, 35, 36, 37, 38, 39, 44]. In addition, several researchers investigated the sparse case where $p$ is no longer a constant but $p = o(1)$, see [1, 5, 6, 21, 22]. In this paper, we will be concentrating only on the dense case when $p$ is a fixed constant.

For $k$-uniform hypergraphs, there are several obvious generalizations of the graph properties $\texttt{Disc}_p$ and $\texttt{Expand}_p$ which we discuss next. A *proper partition* $\pi$ of $k$ is an unordered list of at least two positive integers whose sum is $k$. For the partition $\pi$ of $k$ given by $k = k_1 + \cdots + k_t$, we will abuse notation by saying that $\pi = k_1 + \cdots + k_t$. Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of $k$-uniform hypergraphs with $|V(H_n)| = n$ and let $0 < p < 1$ be a fixed integer. For a proper partition $\pi = k_1 + \cdots + k_t$ of $k$, define the following properties of $\mathcal{H}$.

- $\texttt{Disc}_p$: for every $U \subseteq V(H_n)$, $|E(H_n[U])| = p\binom{|U|}{k} + o(n^k)$.
- $\texttt{Expand}_p[\pi]$: For all $S_i \subseteq \binom{V(H_n)}{k_i}$ where $1 \le i \le t$,

$$e(S_1, \dots, S_t) = p \prod_{i=1}^{t} |S_i| + o(n^k)$$

where $e(S_1, \dots, S_t)$ is the number of tuples $(s_1, \dots, s_t)$ such that $s_1 \cup \dots \cup s_t$ is a hyperedge and $s_i \in S_i$.

$\texttt{Expand}_p[1 + \dots + 1] \Rightarrow \texttt{Disc}_p$ is easy by letting $S_i = U$ and an inclusion/exclusion argument shows $\texttt{Disc}_p \Rightarrow \texttt{Expand}_p[1 + \dots + 1]$ (see Lemma 9 for a proof of a more general statement). One of the most important graph properties equivalent to $\texttt{Disc}_p$ is $\texttt{Count}_p[\text{All}]$, the property that for all graphs $F$, the number of labeled copies of $F$ in $G_n$ is $p^{|E(F)|}n^{|V(F)|} + o(n^{|V(F)|})$ and at first glance one might suspect this equivalence also holds for hypergraphs. However, Rödl observed that a three-uniform construction of Erdős and Hajnal [13] satisfies $\texttt{Disc}_{1/4}$ and fails $\texttt{Count}_{1/4}[\text{All}]$. In light of this construction, Frankl and Rödl suggested the following property which can be seen as an alternate generalization of $\texttt{Disc}_p$ from graphs to hypergraphs. Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of $k$-uniform hypergraphs with $|V(H_n)| = n$, let $0 < p < 1$ be a fixed integer, and let $1 \le \ell \le k - 1$ be an integer and define the following property.

- $\texttt{CliqueDisc}_p[\ell]$: for every $\ell$-uniform hypergraph $G$ where $V(G) = V(H_n)$, $|E(H_n) \cap \mathcal{K}_k(G)| = p|\mathcal{K}_k(G)| + o(n^k)$, where $\mathcal{K}_k(G)$ is set of *$k$-cliques of $G$*, the collection of $k$-sets $T \subseteq V(G)$ such that all $\ell$-subsets of $T$ are edges of $G$.

Note that for $k$-uniform hypergraphs and $\ell = 1$, $\texttt{CliqueDisc}_p[1] \Leftrightarrow \texttt{Disc}_p$ by definition so $\texttt{CliqueDisc}_p[\ell]$ is a generalization of $\texttt{Disc}_p$. Many hypergraph quasirandom properties are equivalent to $\texttt{CliqueDisc}_p[\ell]$ and $\texttt{Expand}_p[\pi]$ for some $\ell$ or $\pi$. See [3, 4, 7, 8, 9, 10, 12, 14, 19, 20, 23, 25] for the studies of these properties, which include counting subhypergraphs, counting induced subhypergraphs, spectral characterizations, and counting even/odd subgraphs.

There are two more hypergraph quasirandom properties that have been studied. First, Chung and Graham's [8] original property on even/odd subgraphs of the octahedron called $\texttt{Deviation}[\ell]$ and an extension of $\texttt{CliqueDisc}_p[\ell]$ recently proposed by Chung [4].

- For $2 \le \ell \le k$, define $\texttt{Deviation}[\ell]$ as follows:

$$\sum_{\substack{x_1,\dots,x_{k-\ell} \in V(H) \\ y_{1,0},y_{1,1},\dots,y_{\ell,0},y_{\ell,1} \in V(H)}} (-1)^{|\mathcal{O}[\vec{x},\vec{y}] \cap E(H)|} = o(n^{k+\ell}),$$

where $\mathcal{O}[\vec{x},\vec{y}]$ is the collection of hyperedges of the squashed octahedron. That is, $\mathcal{O}[\vec{x},\vec{y}] = \{\{x_1,\dots,x_{k-\ell},y_{1,i_1},\dots,y_{\ell,i_\ell}\} : 0 \le i_j \le 1\}$. Conceptually, $\texttt{Deviation}[\ell]$ states that the difference between the number of even and odd squashed octahedrons is negligible compared to the number of squashed octahedrons.

- For $1 \le \ell \le k-1$ and $1 \le s \le \binom{k}{\ell}$, define $\texttt{CliqueDisc}_p[\ell, s]$ as follows: for every $\ell$-uniform hypergraph $G$ where $V(G) \subseteq V(H_n)$,[1]

$$|\{T \in E(H_n) : |E(G[T])| \ge s\}| = p\left|\left\{T \in \binom{V(G)}{k} : |E(G[T])| \ge s\right\}\right| + o(n^k).$$

Although it is possible to extend the definition of $\texttt{Deviation}[\ell]$ to arbitrary $0 < p < 1$, the deviation property has been studied primarily for $p = \frac{1}{2}$, which is how we have stated it. Also, note that $\texttt{CliqueDisc}_p[\ell, \binom{k}{\ell}]$ is the same property as $\texttt{CliqueDisc}_p[\ell]$.

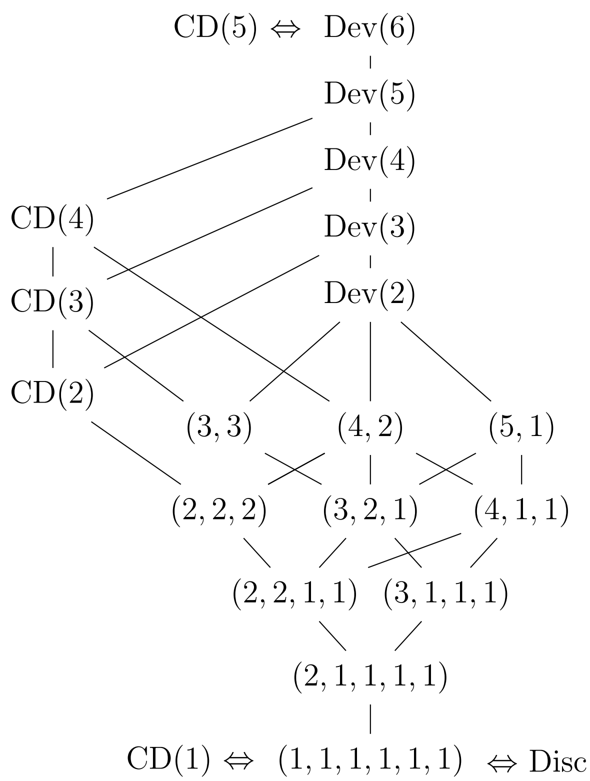


Figure 1: The Hasse diagram of quasirandom properties for $k = 6$.

In [4, 7], Chung made partial progress on Problem 1; see Table 1 for the exact results she proved. Our main result is to determine all relationships between $\texttt{Expand}_p[\pi]$, $\texttt{CliqueDisc}_p[\ell, s]$, and $\texttt{Deviation}[\ell]$ for all $\ell$, $s$, and $\pi$. As a consequence, our work also determines the relationships between other properties like counting and spectral conditions studied in the literature, since these have been shown equivalent to one of $\texttt{Expand}_p[\pi]$, $\texttt{CliqueDisc}_p[\ell, s]$, or $\texttt{Deviation}[\ell]$. While we use some basic ideas introduced by Chung,

[1] This is slightly different than Chung's [4] definition; she defined $\texttt{CliqueDisc}[\ell, s]$ only for spanning $G$. We believe the two definitions are equivalent and have proved this for several small cases.

most of our results require new constructions for the non-implications of quasirandom properties. The proofs of our two main positive results (Theorems 2 and 4) also use new techniques.

Our first result is that $\texttt{CliqueDisc}_p[\ell, s]$ is a superfluous property in the sense that $\texttt{CliqueDisc}_p[\ell, s] \Leftrightarrow \texttt{CliqueDisc}_p[\ell, s']$ for all $\ell, s, s'$. Since $\texttt{CliqueDisc}_p[\ell]$ is equivalent to $\texttt{CliqueDisc}_p[\ell, \binom{k}{\ell}]$, we can reduce to studying just $\texttt{CliqueDisc}_p[\ell]$.

**Theorem 2.** *Fix $k \geq 3$ and $2 \leq \ell < k$. Then $\texttt{CliqueDisc}_p[\ell, 1] \Leftrightarrow \texttt{CliqueDisc}_p[\ell, 2] \Leftrightarrow \cdots \Leftrightarrow \texttt{CliqueDisc}_p[\ell, \binom{k}{\ell}]$.*

The proof of Theorem 2 appears in Section 5. Our next result is that the expansion properties are arranged in a poset via partition refinement. In particular, $\texttt{Expand}_p[\pi]$ is a distinct property for each $\pi$.

**Definition.** *A partition $\pi' = m_1 + \cdots + m_r$ is a* refinement *of a partition $\pi = k_1 + \cdots + k_t$ if there is a surjection $\phi : \{1, \ldots, r\} \to \{1, \ldots, t\}$ such that for every $1 \leq i \leq t$, $k_i = \sum_{j:\phi(j)=i} m_j$. If $\pi'$ is a refinement of $\pi$, we write $\pi' \leq \pi$. Also, for $\pi = k_1 + \cdots + k_t$, let $\max \pi = \max_i k_i$.*

**Theorem 3.** *$\texttt{Expand}_p[\pi] \Rightarrow \texttt{Expand}_p[\pi']$ if and only if $\pi'$ is a refinement of $\pi$.*

Having determined the poset of implications for the properties $\texttt{Expand}_p[\pi]$, we now give the relationships between $\texttt{CliqueDisc}_p[\ell]$ and $\texttt{Expand}_p[\pi]$ for all $\ell$ and $\pi$.

**Theorem 4.** *Let $1 \leq \ell \leq k - 1$ and $\pi = k_1 + \cdots + k_t$. $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{Expand}_p[\pi]$ if and only if $k_i \leq \ell$ for all $i$. Also, $\texttt{Expand}_p[\pi] \Rightarrow \texttt{CliqueDisc}_p[1]$ for all $\pi$ but $\texttt{Expand}_p[\pi] \not\Rightarrow \texttt{CliqueDisc}_p[\ell]$ for any $\pi$ and $\ell \geq 2$.*

Next, we determine the relationships between $\texttt{Deviation}[\ell]$ and the other properties. Since $\texttt{Deviation}[\ell]$ has been studied primarily for $p = \frac{1}{2}$, we only study the relationships between $\texttt{Deviation}[\ell]$ and $\texttt{CliqueDisc}_{1/2}[\ell]$ and $\texttt{Expand}_{1/2}[\pi]$.

**Theorem 5.** *For all $2 \leq \ell \leq k$ and all $\pi$, we have $\texttt{Deviation}[\ell] \Rightarrow \texttt{Expand}_{1/2}[\pi]$. Furthermore, $\texttt{CliqueDisc}_{1/2}[k - 1] \Rightarrow \texttt{Deviation}[k]$ but no expansion and no other clique discrepency implies $\texttt{Deviation}[\ell]$ for any $\ell$.*

Lastly, we prove that $\texttt{Deviation}[\ell] \not\Rightarrow \texttt{Deviation}[\ell + 1]$. When combined with the implication $\texttt{Deviation}[\ell] \Rightarrow \texttt{Deviation}[\ell - 1]$ (Chung [7]), this proves that the properties $\texttt{Deviation}[\ell]$ form a chain of distinct hypergraph quasirandom properties.

**Proposition 6.** *For all $2 \leq \ell \leq k - 1$, we have $\texttt{Deviation}[\ell] \not\Rightarrow \texttt{Deviation}[\ell + 1]$.*

The proofs of Theorems 3, 4, 5, and Proposition 6 appear in Sections 2, 3, and 4. Together with results of Chung [7] and Chung and Graham [8], these theorems complete the characterization between $\texttt{Expand}_p[\pi]$, $\texttt{CliqueDisc}_p[\ell]$, and $\texttt{Deviation}[\ell]$ for all $\ell$ and $\pi$. Table 1 summarizes these results and states where each piece is proved. Figure 1 shows a diagram of the relationships for $k = 6$.

| Range | Result | Proof |
|---|---|---|
| $\pi' \leq \pi$ | $\texttt{Expand}_p[\pi] \Rightarrow \texttt{Expand}_p[\pi']$ | Theorem 3, Lemma 8 |
| $\pi' \not\leq \pi$ | $\texttt{Expand}_p[\pi] \not\Rightarrow \texttt{Expand}_p[\pi']$ | Theorem 3, Lemma 31 |
| all $\pi$ | $\texttt{Expand}_p[\pi] \Rightarrow \texttt{CliqueDisc}_p[1]$ | Theorem 4 |
| all $\pi$, $\ell \geq 2$ | $\texttt{Expand}_p[\pi] \not\Rightarrow \texttt{CliqueDisc}_p[\ell]$ | Theorem 4, Lemma 32 |
| all $\pi$, $\ell$ | $\texttt{Expand}_{1/2}[\pi] \not\Rightarrow \texttt{Deviation}[\ell]$ | Theorem 5, Lemma 33 |
| | | |
| $2 \leq \ell \leq k-1$ | $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{CliqueDisc}_p[\ell-1]$ | Chung [7] |
| $1 \leq \ell \leq k-2$ | $\texttt{CliqueDisc}_p[\ell] \not\Rightarrow \texttt{CliqueDisc}_p[\ell+1]$ | Chung [7] for $p = \frac{1}{2}$, Lemma 34 |
| $\max \pi \leq \ell$ | $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{Expand}_p[\pi]$ | Theorem 4, Lemma 9 |
| $\max \pi > \ell$ | $\texttt{CliqueDisc}_p[\ell] \not\Rightarrow \texttt{Expand}_p[\pi]$ | Theorem 4, Lemma 36 |
| | $\texttt{CliqueDisc}_{1/2}[k-1] \Rightarrow \texttt{Deviation}[k]$ | Chung and Graham [8] |
| all $\ell$ | $\texttt{CliqueDisc}_{1/2}[k-2] \not\Rightarrow \texttt{Deviation}[\ell]$ | Theorem 5, Lemma 35 |
| | | |
| $3 \leq \ell \leq k$ | $\texttt{Deviation}[\ell] \Rightarrow \texttt{Deviation}[\ell-1]$ | Chung [7], Lemma 45 |
| $2 \leq \ell \leq k-1$ | $\texttt{Deviation}[\ell] \not\Rightarrow \texttt{Deviation}[\ell+1]$ | Proposition 6, Lemma 38 |
| $2 \leq \ell \leq k$ | $\texttt{Deviation}[\ell] \Rightarrow \texttt{CliqueDisc}_{1/2}[\ell-1]$ | Chung [7], Lemma 46 |
| $2 \leq \ell \leq k-1$ | $\texttt{Deviation}[\ell] \not\Rightarrow \texttt{CliqueDisc}_{1/2}[\ell]$ | Chung [7], Lemma 37 |
| all $\pi$, $\ell$ | $\texttt{Deviation}[\ell] \Rightarrow \texttt{Expand}_{1/2}[\pi]$ | Theorem 5, Lemma 12 |

Table 1: Relationships between quasirandom properties

The remainder of this paper is organized as follows. In Section 2, we prove the implications in Table 1 (Lemmas 8, 9, and 12). In Section 3, we define three families of constructions which are used to show the separation of quasirandom properties, and in Section 4 we use these constructions to prove all the negative implications in Table 1. Section 5 contains the proof of Theorem 2. Lastly, Appendix A contains for completeness some proofs of results of Chung [7] that are used in this paper. The subscript $p$ on the quasirandom properties is dropped if it is clear from context.

# 2 Implications

In this section, we prove the implications in Table 1.

## 2.1 Expansion

Our goal in this subsection is to prove Lemma 8 below. First, we introduce a variant of $\texttt{Expand}[\pi]$ where the sets are disjoint. Let $\pi = k_1 + \cdots + k_t$ be a proper partition of $k$. If $H$ is a hypergraph and $S_1, \ldots, S_t$ are sets such that $S_i \subseteq \binom{V(H)}{k_i}$, denote by $V(S_i) = \cup_{s \in S_i} s$ and call $S_1, \ldots, S_t$ *disjoint* if $V(S_i) \cap V(S_j) = \emptyset$ for $i \neq j$. Let $\mathcal{H} = \{H_n\}_{n \to \infty}$ be a sequence of $k$-uniform hypergraphs such that $|V(H_n)| = n$. Define the following property of the sequence

$\mathcal{H}$.

- $\texttt{PartiteExpand}_p[\pi]$: For all $S_i \subseteq \binom{V(H_n)}{k_i}$ where $S_1, \ldots, S_t$ are disjoint,

$$e(S_1, \ldots, S_t) = p \prod_{i=1}^{t} |S_i| + o\left(n^k\right)$$

  where $e(S_1, \ldots, S_t)$ is the number of tuples $(s_1, \ldots, s_t)$ such that $s_i \in S_i$ for all $i$ and $s_1 \cup \cdots \cup s_t \in E(H_n)$.

**Lemma 7.** *$\texttt{PartiteExpand}_p[\pi] \Rightarrow \texttt{Expand}_p[\pi]$.*

*Proof.* Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of hypergraphs satisfying $\texttt{PartiteExpand}[\pi]$. Throughout this proof, for notational simplicity we drop the subscript $n$. Let $S_i \subseteq \binom{V(H)}{k_i}$ be given. Let $\mathcal{P} = (P_1, \ldots, P_t)$ be an ordered partition of $V(H)$ into $t$ non-empty parts. That is, $\mathcal{P}$ is an ordered tuple of $t$ non-empty vertex sets such that $P_i \cap P_j = \emptyset$ for $i \neq j$ and $\cup P_i = V(H)$. For $1 \leq i \leq t$, define $S_i[P_i]$ to be the collection of $k_i$-sets in $S_i$ which are subsets of $P_i$. Then

$$e(S_1, \ldots, S_t) = \frac{1}{t^{n-k}} \sum_{\mathcal{P}} e(S_1[P_1], \ldots, S_t[P_t]),$$

since in the sum over partitions, each $(s_1, \ldots, s_t) \in S_1 \times \cdots \times S_t$ with $s_1 \cup \cdots \cup s_t \in E(H)$ is counted $t^{n-k}$ times. That is, if $E = s_1 \cup \cdots \cup s_t$ is an edge with $s_i \in S_i$, then the partitions which count $(s_1, \ldots, s_t)$ are the partitions formed by starting with $P_1 = s_1, \ldots, P_t = s_t$ and adding the other $n - k$ vertices arbitrarily to the $t$ parts.

A similar argument shows that

$$|S_1| \cdots |S_t| = \frac{1}{t^{n-k}} \sum_{\mathcal{P}} |S_1[P_1]| \cdots |S_t[P_t]|. \tag{1}$$

Now apply $\texttt{PartiteExpand}[\pi]$ to $S_1[P_1], \ldots, S_t[P_t]$ to obtain

$$\begin{aligned} e(S_1, \ldots, S_t) &= \frac{1}{t^{n-k}} \sum_{\mathcal{P}} \Big(p|S_1[P_1]| \cdots |S_t[P_t]| + o(n^k)\Big) \\ &= \frac{p}{t^{n-k}} \sum_{\mathcal{P}} |S_1[P_1]| \cdots |S_t[P_t]| + \sum_{\mathcal{P}} o\left(\frac{n^k}{t^{n-k}}\right) \\ &= p|S_1| \cdots |S_t| + o\left(\frac{n^k t! S(n,t)}{t^{n-k}}\right). \end{aligned}$$

The last equality combines (1) with the fact that the number of partitions in the sum is $t!S(n,t)$ where $S(n,t)$ is the Stirling number of the second kind. Trivially, $\frac{t^{n-t}}{t!} \leq S(n,t) \leq t^n$ so that $S(n,t) = \Theta(t^n)$. Since $t$ and $k$ are fixed, $\frac{n^k t! S(n,t)}{t^{n-k}} = \Theta(n^k)$ implying that $e(S_1, \ldots, S_t) = p|S_1| \cdots |S_t| + o(n^k)$, completing the proof. □

**Lemma 8.** *If $\pi'$ is a refinement of $\pi$, then* $\texttt{Expand}_p[\pi] \Rightarrow \texttt{Expand}_p[\pi']$.

*Proof.* Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of hypergraphs and let $\pi = k_1 + \cdots + k_t$ and $\pi' = m_1 + \cdots + m_r$. Let $\phi : \{1, \ldots, r\} \to \{1, \ldots, t\}$ be the surjection for the refinement of $\pi'$ of $\pi$. That is, $k_i = \sum_{j:\phi(j)=i} m_j$. By Lemma 7, we only need to show that $\texttt{PartiteExpand}[\pi']$ holds, so let $S'_1, \ldots, S'_t$ be disjoint sets with $S'_i \subseteq \binom{V(H_n)}{m_i}$. For $1 \leq i \leq t$, define

$$S_i = \{X_{j_1} \cup \cdots \cup X_{j_\ell} : \ \{j_1, \ldots, j_\ell\} = \{j : \phi(j) = i\} \text{ and } \forall a, X_{j_a} \in S'_{j_a}\}.$$

In other words, $S_i$ consists of all vertex sets formed by combining via the refinement sets from $S'_1, \ldots, S'_t$. Since $S'_1, \ldots, S'_t$ are disjoint,

$$e(S_1, \ldots, S_t) = e(S'_1, \ldots, S'_r) \qquad \text{and} \qquad |S_1| \cdots |S_t| = |S'_1| \cdots |S'_r|. \tag{2}$$

Since $\texttt{Expand}[\pi]$ holds for $\mathcal{H}$,

$$e(S_1, \ldots, S_t) = p|S_1| \cdots |S_t| + o(n^k).$$

Combining this with (2) shows that $\texttt{PartiteExpand}[\pi']$ holds for $\mathcal{H}$. □

## 2.2 Clique Discrepency

Our goal in this subsection is to discuss and prove all the implications in Table 1 involving $\texttt{CliqueDisc}[\ell]$. In particular, Lemma 9 below states that $\texttt{CliqueDisc}[\ell] \Rightarrow \texttt{Expand}[\pi]$ if $\max \pi \leq \ell$.

The implication $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{CliqueDisc}_p[\ell - 1]$ is easy to see directly from the definitions: given an $(\ell-1)$-uniform hypergraph $G$, let $F$ be the $\ell$-uniform hypergraph whose hyperedges consist of the $\ell$-cliques in $G$. Then $\mathcal{K}_k(G) = \mathcal{K}_k(F)$, so applying $\texttt{CliqueDisc}_p[\ell]$ to $F$ implies that $\texttt{CliqueDisc}_p[\ell - 1]$ holds for $G$.

As part of their initial investigation of hypergraph quasirandomness, Chung and Graham [8] proved that $\texttt{CliqueDisc}_{1/2}[k - 1] \Rightarrow \texttt{Deviation}[k]$. Since the reverse implication also holds, these properties are equivalent. Indeed, they have also both been shown equivalent to $\texttt{Count}[\text{All}]$, the property that for every $k$-uniform hypergraph $F$, the number of labeled copies of $F$ in $\mathcal{H}$ is $(1/2)^{|E(F)|} n^{|V(F)|} + o(n^{|V(F)|})$.

The final implication involving $\texttt{CliqueDisc}_p[\ell]$ in Table 1 is that if $\pi = k_1 + \cdots + k_t$ is a proper partition of $k$ where $k_i \leq \ell$ for all $i$, then $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{Expand}_p[\pi]$.

**Lemma 9.** *Let $k \geq 3$, let $2 \leq \ell < k$, and let $\pi$ be a proper partition of $k$ where* $\max \pi \leq \ell$. *Then* $\texttt{CliqueDisc}_p[\ell] \Rightarrow \texttt{Expand}_p[\pi]$.

*Proof.* First, view $\pi$ as an ordered partition $\vec{\pi} = (k_1, \ldots, k_t)$ where $\sum k_i = k$. Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of hypergraphs satisfying $\texttt{CliqueDisc}[\ell]$. Throughout this proof, for notational simplicity we drop the subscript $n$. By Lemma 7, we only need to show that $\texttt{PartiteExpand}[\pi]$ holds, so let $S_i \subseteq \binom{V(H)}{k_i}$ be given such that $S_1, \ldots, S_t$ are disjoint. Define

$$\mathcal{M} = \left\{ (m_1, \ldots, m_t) : 0 \leq m_i \leq k, \sum_{i=1}^{t} m_i = k \right\}.$$

For $\vec{m} \in \mathcal{M}$, define the *cliques of type* $\vec{m}$ as the following set:

$$\mathcal{T}_{\vec{m}} = \left\{ A \in \binom{V(S_1) \cup \cdots \cup V(S_t)}{k} : |A \cap V(S_i)| = m_i \text{ and } \binom{A \cap V(S_i)}{k_i} \subseteq S_i \right\}.$$

That is, the cliques of type $\vec{m}$ are the $k$-sets of vertices which have exactly $m_i$ vertices in $V(S_i)$ and if $m_i \geq k_i$ then all $k_i$ subsets of $A \cap V(S_i)$ are elements of $S_i$ (since if $m_i < k_i$ then $\binom{A \cap V(S_i)}{k_i} = \emptyset$). Depending on $\vec{\pi}$, $k$, and $\ell$ some of the collections $\mathcal{T}_{\vec{m}}$ could be empty. Now define an equivalence relation $\sim$ on $\mathcal{M}$ as follows. For $\vec{m}, \vec{m}' \in \mathcal{M}$,

$$\begin{aligned} \vec{m} \sim \vec{m}' \text{ if and only if } & \{i : m_i < \ell\} = \{i : m'_i < \ell\} \\ & \text{and } \forall i \in \{i : m_i < \ell\}, m_i = m'_i \end{aligned}$$

In other words, $\vec{m} \sim \vec{m}'$ if they are equal in coordinates which are smaller than $\ell$ and have the same sum of coordinates at least $\ell$. For example, if $\vec{\pi} = (3, 3, 2, 2, 2)$ and $\ell = 3$, then $(6, 4, 1, 1, 0) \sim (5, 5, 1, 1, 0)$. For $\vec{m} \in \mathcal{M}$, define $[\vec{m}] = \{\vec{m}' \in \mathcal{M} : \vec{m} \sim \vec{m}'\}$. It is trivial to see that this is an equivalence relation on $\mathcal{M}$.

**Claim 1.** $[(k_1, \ldots, k_t)] = \{(k_1, \ldots, k_t)\}$.

*Proof.* Assume that $\vec{m} \sim (k_1, \ldots, k_t)$. For indices $i$ where $k_i < \ell$, $m_i = k_i$ and for indices where $k_i = \ell$, $m_i \geq \ell = k_i$. But since $\sum m_i = \sum k_i = k$, we must have $m_i = k_i$ for all $i$. □

Define $\mathcal{T}_{[\vec{m}]} = \bigcup_{\vec{m}' \sim \vec{m}} \mathcal{T}_{\vec{m}'}$. Note that $\mathcal{T}_{[(k_1,\ldots,k_t)]} \cap E(H) = \mathcal{T}_{(k_1,\ldots,k_t)} \cap E(H)$ is exactly the set of edges we would like to count; $\mathcal{T}_{(k_1,\ldots,k_t)}$ is isomorphic to the collection of ordered tuples $(s_1, \ldots, s_t)$ such that $s_i \in S_i$ since $S_1, \ldots, S_t$ is disjoint. Thus the following claim completes the proof.

**Claim 2.** *For all* $[\vec{m}] \in \mathcal{M}/\!\sim$, $|\mathcal{T}_{[\vec{m}]} \cap E(H)| = p \left|\mathcal{T}_{[\vec{m}]}\right| + o(n^k)$.

*Proof.* The proof is by induction; define a partial order on the equivalence classes in $\mathcal{M}/\!\sim$ as follows: $[\vec{m}'] < [\vec{m}]$ if one of the following holds:

- $|\{i : m'_i = 0\}| > |\{i : m_i = 0\}|$, or
- $|\{i : m'_i = 0\}| = |\{i : m_i = 0\}|$ and $\{i : m'_i \geq \ell\} \subsetneq \{i : m_i \geq \ell\}$, or
- $|\{i : m'_i = 0\}| = |\{i : m_i = 0\}|$ and $\{i : m'_i \geq \ell\} = \{i : m_i \geq \ell\}$ and

$$\sum_{\substack{1 \leq i \leq t \\ m'_i < \ell}} m'_i < \sum_{\substack{1 \leq i \leq t \\ m_i < \ell}} m_i.$$

Note that the definition is well defined since any vector in $[\vec{m}]$ has the same set of indices $i$ where $m_i = 0$ and the same set of indices where $m_i \geq \ell$. We prove Claim 2 by induction on this partial order. The base case proves the statement for all minimum elements in the partial order and the inductive argument applies the claim only for elements smaller in the partial order.

For the base case we consider vectors with exactly one non-zero $m_i$ which equals $k$ since the total sum of the entries of $\vec{m}$ is $k$. Note that the equivalence class of $(0,\ldots,0,k,0,\ldots,0)$ has size one, so the base case is to show that $|\mathcal{T}_{(0,\ldots,0,k,0,\ldots,0)} \cap E(H)| = p|\mathcal{T}_{(0,\ldots,0,k,0,\ldots,0)}| + o(n^k)$. Assume that $m_i = k$ and for $j \neq i$, $m_j = 0$. Since $\texttt{CliqueDisc}[\ell] \Rightarrow \texttt{CliqueDisc}[k_i]$, $\texttt{CliqueDisc}[k_i]$ holds for $\mathcal{H}$. Now apply $\texttt{CliqueDisc}[k_i]$ to the $k_i$-uniform hypergraph $S_i$. By definition, $\mathcal{T}_{\vec{m}} = \mathcal{K}_k(S_i)$ since both are the $k$-sets all of whose $k_i$-subsets are elements of $S_i$. Therefore, $\texttt{CliqueDisc}[k_i]$ applied to $S_i$ implies $|\mathcal{K}_k(S_i) \cap E(H)| = p\,|\mathcal{K}_k(S_i)| + o(n^k)$.

For the inductive step, define an $\ell$-uniform hypergraph $W_{[\vec{m}]}$ as follows. The vertex set of $W_{[\vec{m}]}$ is the same as the vertex set of $H$. The edge set is

$$E(W_{[\vec{m}]}) = \Big\{ B \in \binom{V(S_1) \cup \cdots \cup V(S_t)}{\ell} : \forall i,\ m_i < \ell \Longrightarrow |B \cap V(S_i)| \leq m_i, \\ \text{and } \forall i,\ \binom{B \cap V(S_i)}{k_i} \subseteq S_i \Big\}.$$

Note that the definition is well defined since any vector in $[\vec{m}]$ has the same entries for indices smaller than $\ell$.

**Claim 3.** $\mathcal{K}_k(W_{[\vec{m}]}) \subseteq \cup_{\vec{m}' \in \mathcal{M}} \mathcal{T}_{\vec{m}'}$.

*Proof.* Let $A \in \mathcal{K}_k(W_{[\vec{m}]})$ and define $m'_i = |A \cap V(S_i)|$. Since $A \subseteq V(S_1) \cup \cdots \cup V(S_t)$ and $S_1, \ldots, S_t$ are disjoint, $\sum m'_i = k$. Lastly, pick any $C \in \binom{A \cap V(S_i)}{k_i}$ and let $B$ be any $\ell$-subset of $A$ containing $C$. Such a $B$ exists since $\max \pi \leq \ell$. Since $A$ is a clique of $W_{[\vec{m}]}$, $B$ is an edge of $W_{[\vec{m}]}$ which implies that all $k_i$-subsets of $B \cap V(S_i)$ are elements of $S_i$. But $C \subseteq B \cap V(S_i)$ so $C \in S_i$, implying that $A \in \mathcal{T}_{\vec{m}'}$. □

**Claim 4.** $\mathcal{T}_{[\vec{m}]} \subseteq \mathcal{K}_k(W_{[\vec{m}]})$.

*Proof.* Let $A \in \mathcal{T}_{[\vec{m}]}$ and let $B$ be any $\ell$-subset of $A$. Then $|B \cap V(S_i)| \leq |A \cap V(S_i)| = m_i$ for all $i$. Also, for any $C \subseteq B \cap V(S_i)$ with $|C| = k_i$, $C \subseteq A \cap V(S_i)$ so $C \in S_i$. □

**Claim 5.** *For $\vec{m} \neq \vec{m}'$, $\mathcal{T}_{\vec{m}} \cap \mathcal{T}_{\vec{m}'} = \emptyset$.*

*Proof.* Let $A \in \mathcal{T}_{\vec{m}} \cap \mathcal{T}_{\vec{m}'}$. Then $|A \cap V(S_i)| = m_i$ and $|A \cap V(S_i)| = m'_i$ for all $i$ so $m_i = m'_i$ for all $i$. □

**Claim 6.** *There exists a collection $\mathcal{M}' \subseteq \mathcal{M}/\!\sim$ such that*

$$\mathcal{K}_k(W_{[\vec{m}]}) = \mathcal{T}_{[\vec{m}]} \,\dot\cup \dot{\bigcup}_{[\vec{m}'] \in \mathcal{M}'} \mathcal{T}_{[\vec{m}']}.$$

*Proof.* By Claims 3, 4, and 5 we only need to prove that for every $[\vec{m}'] \in \mathcal{M}/\!\sim$, either $\mathcal{K}_k(W_{[\vec{m}]})$ contains $\mathcal{T}_{[\vec{m}']}$ or is disjoint from $\mathcal{T}_{[\vec{m}']}$.

Let $A_1, A_2 \in \mathcal{T}_{[\vec{m}']}$ such that $A_1 \in \mathcal{K}_k(W_{[\vec{m}]})$. Let $B_2$ be any $\ell$-subset of $A_2$. We would like to show that $B_2$ is in $E(W_{[\vec{m}]})$ to imply that $A_2 \in \mathcal{K}_k(W_{[\vec{m}]})$. For $i$ with $m_i < \ell$, let $f_i = |B_2 \cap V(S_i)|$ and let $B_1$ be an $\ell$-subset of $A_1$ which takes any $f_i$ elements of $A_1 \cap V(S_i)$ for each $i$ with $m_i < \ell$ and takes vertices arbitrarily from $V(S_i)$ for $i$ where $m_i \geq \ell$. There exists such a set $B_1$ since for coordinates $i$ where $m_i < \ell$, $f_i \leq |A_2 \cap V(S_i)| = m_i = |A_1 \cap V(S_i)|$ so there exists a subset of $A_1 \cap V(S_i)$ of size $f_i$ and this subset can be used for $B_1 \cap V(S_i)$. Also, once these vertices are picked, $B_1$ can be extended to an $\ell$-set by taking vertices only from the other coordinates since $\sum\{|A_1 \cap V(S_i)| : m_i \geq \ell\} = \sum\{|A_2 \cap V(S_i)| : m_i \geq \ell\}$. In addition, this argument showing the existence of $B_1$ does not depend on the representatives $\vec{m}$ and $\vec{m}'$ chosen for the equivalence classes $[\vec{m}]$ and $[\vec{m}']$.

Now that we have defined $B_1$, since $A_1$ is a clique of $W_{[\vec{m}]}$ and $B_1$ is an $\ell$-subset of $A_1$, $B_1$ must be an element of $E(W_{[\vec{m}]})$. This implies for $i$ with $m_i < \ell$ that $f_i = |B_1 \cap V(S_i)| \leq m_i$ so that $|B_2 \cap V(S_i)| \leq m_i$. Lastly, if $C$ is a $k_i$-subset of $B_2 \cap V(S_i)$, then $C$ is a $k_i$-subset of $A_2 \cap V(S_i)$ so $C \in S_i$. □

The actual description of which equivalence classes are in $\mathcal{M}'$ is complicated and depends on the relationships between $k_i$, $m_i$ and $\ell$. Fortunately, we don't need the exact description; we just require that every $[\vec{m}'] \in \mathcal{M}'$ appears below $[\vec{m}]$ in the partial ordering. Assume that $\mathcal{M}'$ is defined so that for each $[\vec{m}'] \in \mathcal{M}'$, $\mathcal{T}_{[\vec{m}']} \neq \emptyset$. Recall that it is possible for $\mathcal{T}_{\vec{m}'}$ to be empty for certain $\vec{m}'$ depending on the interaction between the hypergraph $H$, $\pi$, $k$, and $\ell$. Therefore, in the remainder of this proof we just ignore the collections $\mathcal{T}_{\vec{m}'}$ which are empty.

**Claim 7.** *For every $[\vec{m}'] \in \mathcal{M}'$, $[\vec{m}'] < [\vec{m}]$.*

*Proof.* Let $A \in \mathcal{T}_{[\vec{m}']}$. First, we prove that $\{i : m'_i = 0\} \supseteq \{i : m_i = 0\}$. Assume for contradiction there exists some $i$ with $m'_i \neq 0$ and $m_i = 0$. Since $m'_i \neq 0$, $A$ contains a vertex $x$ inside $V(S_i)$. But now let $B$ be any $\ell$-subset of $A$ containing $x$. Since $m_i = 0$ and $x \in V(S_i)$, this $\ell$-subset $B$ is not in $E(W_{[\vec{m}]})$ so $A$ is not a $k$-clique of $W_{[\vec{m}]}$ contradicting $[\vec{m}'] \in \mathcal{M}'$. If $\{i : m'_i = 0\} \supsetneq \{i : m_i = 0\}$, then $[\vec{m}'] < [\vec{m}]$. Therefore, assume that $\{i : m'_i = 0\} = \{i : m_i = 0\}$.

Next, we prove that for $i$ with $m_i < \ell$, $m'_i \leq m_i$. Assume for contradiction that there exists an $i$ such that $m'_i > m_i$ and $m_i < \ell$. Let $A \in \mathcal{T}_{[\vec{m}']}$. Then $|A \cap V(S_i)| = m'_i \geq m_i + 1$ so let $B$ be an $\ell$-subsets of $A$ which has at least $m_i + 1$ elements of $A \cap V(S_i)$. There exists such a $B$ since $m_i + 1 \leq \ell$ and $m_i + 1 \leq |A \cap V(S_i)|$. But now $B$ is not in $E(W_{[\vec{m}]})$ since $|B \cap V(S_i)| > m_i$ and $m_i < \ell$ and this contradicts that $[\vec{m}'] \in \mathcal{M}'$.

Since for every $i$ with $m_i < \ell$, $m'_i \leq m_i$ we must have $\{i : m'_i \geq \ell\} \subseteq \{i : m_i \geq \ell\}$. If $\{i : m'_i \geq \ell\} \subsetneq \{i : m_i \geq \ell\}$, then $[\vec{m}'] < [\vec{m}]$. Therefore, assume that $\{i : m'_i \geq \ell\} = \{i : m_i \geq \ell\}$. This implies that

$$\sum_{\substack{1 \leq i \leq t \\ m'_i < \ell}} m'_i \leq \sum_{\substack{1 \leq i \leq t \\ m_i < \ell}} m_i \tag{3}$$

since $m_i < \ell$ if and only if $m'_i < \ell$ and for these indices, $m'_i \leq m_i$. If (3) is a strict inequality, then $[\vec{m}'] < [\vec{m}]$ so assume that (3) is an equality which implies $m_i = m'_i$ for all $i$ with $m_i < \ell$. Combining this with $\{i : m'_i \geq \ell\} = \{i : m_i \geq \ell\}$ implies that $\vec{m} \sim \vec{m}'$ which is a contradiction, since the union in Claim 6 is a disjoint union. □

Claims 6 and 7 combine to finish the proof of Claim 2. By induction, we know the size of $\mathcal{T}_{[\vec{m}']} \cap E(H)$ for all $[\vec{m}'] \in \mathcal{M}'$ and `CliqueDisc`$[\ell]$ implies that $|\mathcal{K}_k(W_{[\vec{m}]}) \cap E(H)| = p|\mathcal{K}_k(W_{[\vec{m}]})| + o(n^k)$. A simple subtraction counts the size of $\mathcal{T}_{[\vec{m}]} \cap V(H)$ as follows:

$$\begin{aligned}
\left|\mathcal{T}_{[\vec{m}]} \cap E(H)\right| &= \left|\mathcal{K}_k(W_{[\vec{m}]}) \cap E(H)\right| - \sum_{[\vec{m}'] \in \mathcal{M}'} \left|\mathcal{T}_{[\vec{m}']} \cap E(H)\right| \\
&= p\left|\mathcal{K}_k(W_{[\vec{m}]})\right| - \sum_{[\vec{m}'] \in \mathcal{M}'} p\left|\mathcal{T}_{[\vec{m}']}\right| + o(n^k) \\
&= p\left(\left|\mathcal{K}_k(W_{[\vec{m}]})\right| - \sum_{[\vec{m}'] \in \mathcal{M}'} \left|\mathcal{T}_{[\vec{m}']}\right|\right) + o(n^k) \\
&= p\left|\mathcal{T}_{[\vec{m}]}\right| + o(n^k).
\end{aligned}$$

□

By Claims 1 and 2, the proof of the lemma is now complete. □

## 2.3 Deviation

In this section, we discuss the implications involving `Deviation`$[\ell]$ in Table 1. The implications `Deviation`$[\ell] \Rightarrow$ `Deviation`$[\ell-1]$ and `Deviation`$[\ell] \Rightarrow$ `CliqueDisc`$[\ell-1]$ were both proved by Chung [7]. The remaining implication is `Deviation`$[\ell] \Rightarrow$ `Expand`$[\pi]$ for all $\ell$ and $\pi$. The proof uses several similar techniques to the other deviation implications proved by Chung [7].

**Definition.** *Let $A_1, \ldots, A_k \subseteq V(H)$ be subsets of vertices such that $|A_i| \in \{1, 2\}$. Define*

$$\mathcal{O}[A_1; \ldots; A_k] = \left\{(x_1, \ldots, x_k) \in V(H)^k : x_i \in A_i\right\}.$$

*That is, $\mathcal{O}[A_1; \ldots; A_k]$ is the collection of tuples of the squashed octahedron using the vertices from $A_1, \ldots, A_k$. Next, define*

$$\tilde{\mathcal{O}}[A_1; \ldots; A_k] = \Big\{\{x_1, \ldots, x_k\} : (x_1, \ldots, x_k) \in \mathcal{O}[A_1; \ldots; A_k] \text{ and } |\{x_1, \ldots, x_k\}| = k\Big\}$$

*so that $\tilde{\mathcal{O}}[A_1; \ldots; A_k]$ are the $k$-sets which come from tuples of distinct vertices of the squashed octahedron. Lastly, define*

$$\eta_H(A_1; \ldots; A_k) = \begin{cases} 1, & \text{if } |\tilde{\mathcal{O}}[A_1; \ldots; A_k] \cap E(H)| \text{ is even,} \\ -1, & \text{otherwise} \end{cases}$$

*For notational convenience, the braces defining $A_i$ are usually dropped. For example, we will write $\mathcal{O}[x; y_0, y_1; z_0, z_1]$ for $\mathcal{O}[\{x\}; \{y_0, y_1\}; \{z_0, z_1\}]$.*

**Definition.** *Let $P \subseteq V(H)^k$, let $0 \leq \ell \leq k$, and let $H$ be a $k$-uniform hypergraph. Define*

$$\mathrm{dev}_{\ell,P}(H) := \sum_{\substack{x_1,\dots,x_{k-\ell},y_{1,0},y_{1,1},\dots,y_{\ell,0},y_{\ell,1} \in V(H) \\ \mathcal{O}[x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1}] \subseteq P}} \eta_H(x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1}).$$

*and let* $\mathrm{dev}_\ell(H) := \mathrm{dev}_{\ell,V(H)^k}(H)$.

Note that by definition, $\texttt{Deviation}[\ell]$ is the property that $\mathrm{dev}_\ell(H_n) = o(n^{k+\ell})$. Also, $\texttt{CliqueDisc}_{1/2}[\ell]$ is the property that for all $\ell$-uniform hypergraphs $G$, $\mathrm{dev}_{0,P}(H_n) = o(n^k)$, where $P$ is the collection of tuples which form $k$-cliques of $G$. Finally, $\texttt{Expand}_{1/2}[k_1 + \cdots + k_t]$ is the property that for all $S_i \subseteq \binom{V(H)}{k_i}$, $\mathrm{dev}_{0,P}(H_n) = o(n^k)$ where $P$ is now the collection of $k$-tuples which are formed by taking one element of $S_i$ for each $i$.

**Definition.** *A set $P \subseteq V(H)^k$ is called* complete in coordinate $i$ *if there exists a $P' \subseteq V(H)^{k-1}$ such that $P = \{(x_1,\dots,x_k) : (x_1,\dots,x_{i-1},x_{i+1},\dots,x_k) \in P', x_i \in V(H)\}$.*

The following two lemmas are the heart of Chung's [7] proof that $\texttt{Deviation}[\ell] \Rightarrow \texttt{CliqueDisc}_{1/2}[\ell-1]$, although the lemmas aren't stated separately; they appear implicitly in the proof.

**Lemma 10.** *(Chung [7]) Let $H$ be a $k$-uniform hypergraph, let $P, Q \subseteq V(H)^k$, and let $2 \leq \ell \leq k$. If $Q$ is complete in coordinate $i$ where $k - \ell + 1 \leq i \leq k$, then*

$$\mathrm{dev}_{\ell,P\cap Q}(H) \leq \mathrm{dev}_{\ell,P}(H).$$

**Lemma 11.** *(Chung [7]) Let $1 \leq \ell \leq k$, let $P \subseteq V(H)^k$, and let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of hypergraphs with $|V(H_n)| = n$. If $\mathrm{dev}_{\ell,P}(H_n) = o(n^{k+\ell})$, then $\mathrm{dev}_{\ell-1,P}(H_n) = o(n^{k+\ell-1})$.*

The fact that $\texttt{Deviation}[\ell] \Rightarrow \texttt{Deviation}[\ell-1]$ follows from Lemma 11 for $P = V(H)^k$. The fact that $\texttt{Deviation}[\ell] \Rightarrow \texttt{CliqueDisc}_{1/2}[\ell-1]$ follows from a combination of Lemmas 10 and 11: given an $(\ell-1)$-uniform hypergraph $G$, define $P$ to be the tuples which are $k$-cliques in $G$ and write $P$ as an intersection of sets complete in a coordinate (for details, see Lemma 46). Combining Lemmas 10 and 11 in a slightly different way proves that $\texttt{Deviation}[\ell] \Rightarrow \texttt{Expand}_{1/2}[\pi]$.

**Lemma 12.** *For all $2 \leq \ell \leq k$ and all proper partitions $\pi$, $\texttt{Deviation}[\ell] \Rightarrow \texttt{Expand}_{1/2}[\pi]$.*

*Proof.* Since $\texttt{Deviation}[\ell] \Rightarrow \texttt{Deviation}[\ell-1]$, by Lemma 8 we just need to prove that $\texttt{Deviation}[2] \Rightarrow \texttt{Expand}[k_1 + k_2]$ for every $k_1, k_2 \geq 1$ with $k_1 + k_2 = k$. Indeed, every $\pi$ is a refinement of $k_1 + k_2$ for some choice of $k_1$ and $k_2$. Given $S_1 \subseteq \binom{V(H)}{k_1}$ and $S_2 \subseteq \binom{V(H)}{k_2}$, define

$$P_1 = \left\{(x_1,\dots,x_{k-2},y,z) \in V(H)^k : \{x_1,\dots,x_{k_1-1},y\} \in S_1\right\},$$
$$P_2 = \left\{(x_1,\dots,x_{k-2},y,z) \in V(H)^k : \{x_{k_1},\dots,x_{k-2},z\} \in S_2\right\}.$$

$P_1$ is complete in coordinate $k$ as we can let $P_1' = \{(x_1, \dots, x_{k-2}, y) : \{x_1, \dots, x_{k_1-1}, y\} \in S_1\}$ and similarly $P_2$ is complete in coordinate $k-1$. Thus by Lemma 10 and the fact that `Deviation`[2] holds,

$$\mathrm{dev}_{2,P_1 \cap P_2}(H_n) \le \mathrm{dev}_{2,V(H)^k}(H_n) = o(n^{k+2}).$$

Now apply Lemma 11 to show that

$$\mathrm{dev}_{0,P_1 \cap P_2}(H_n) = o(n^k). \tag{4}$$

But by definition,

$$\begin{aligned}\mathrm{dev}_{0,P_1 \cap P_2}(H_n) &= \sum_{\substack{x_1,\dots,x_k \in V(H) \\ \mathcal{O}[x_1;\dots;x_k] \subseteq P_1 \cap P_2}} \eta(x_1; \dots; x_k) \\ &= \sum_{(x_1,\dots,x_k) \in P_1 \cap P_2} \eta(x_1; \dots; x_k) \qquad (5)\end{aligned}$$

Let $(x_1, \dots, x_k) \in P_1 \cap P_2$, let $s_1 = \{x_1, \dots, x_{k_1-1}, x_{k-1}\}$ and let $s_2 = \{x_{k_1}, \dots, x_{k-2}, x_k\}$. Since $(x_1, \dots, x_k) \in P_1 \cap P_2$, we have $s_1 \in S_1$ and $s_2 \in S_2$. Also, $\eta(x_1; \dots; x_k) = -1$ if $s_1 \cup s_2 \in E(H)$ and is 1 otherwise. Each $(s_1, s_2) \in S_1 \times S_2$ is counted $k_1!k_2!$ times in (5), since the number of $(x_1, \dots, x_k) \in P_1 \cap P_2$ with $\{x_1, \dots, x_{k_1-1}, x_{k-1}\} = s_1$ and $\{x_{k_1}, \dots, x_{k-2}, x_k\} = s_2$ is $k_1!k_2!$. Thus equations (4) and (5) combine to show that

$$k_1!k_2!\Big( \left|\{(s_1,s_2) \in S_1 \times S_2 : s_1 \cup s_2 \notin E(H)\}\right| - \left|\{(s_1,s_2) \in S_1 \times S_2 : s_1 \cup s_2 \in E(H)\}\right| \Big)$$

is $o(n^k)$. Since $k_1$ and $k_2$ are constants, this implies that the number of tuples $(s_1, s_2) \in S_1 \times S_2$ which are edges of $H$ is $\frac{1}{2}$ of all such tuples (up to $o(n^k)$), so `Expand`[$k_1 + k_2$] holds and the proof is complete. □

# 3 Constructions

To show a property $P$ does not imply a property $Q$, we construct a hypergraph sequence that satisfies $P$ but fails $Q$. While Table 1 states several results of this form, we only use three constructions. This section defines these constructions and proves several facts about them. The constructions are built from random graphs and random hypergraphs, so we are actually defining three probability distributions over $n$-vertex, $k$-uniform hypergraphs which we call $A_\ell(n,p)$, $B_{\vec{\pi}}(n,p)$ and $D(n, 1/2)$. As is typical in random graph theory, we will abuse notation by also writing $A_\ell(n,p)$, $B_{\vec{\pi}}(n,p)$ and $D(n, 1/2)$ for a particular hypergraph drawn from these distributions. Most likely, these constructions can be made explicit by replacing the use of the random hypergraph with a quasirandom hypergraph.

Before stating the constructions, we briefly state two well known concentration bounds on sums of indicator random variables. For more details, see [2].

**Lemma 13.** ***(Chebyshev's Inequality)*** *Let $X_1, \dots, X_n$ be indicator random variables, let $X = \sum X_i$, and let $\mu = \mathbb{E}[X]$ and $\sigma^2 = Var(X) = \mathbb{E}[(X-\mu)^2]$. For every $\epsilon > 0$,*

$$\mathbb{P}\left[|X - \mu| > \epsilon\mu\right] \leq \frac{\sigma^2}{\epsilon^2\mu^2}.$$

**Lemma 14.** ***(Chernoff Bound)*** *Let $0 < p < 1$, let $X_1, \dots, X_n$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all $i$, let $X = \sum X_i$, and let $\mu = \mathbb{E}[X] = pn$. Then for all $a > 0$,*

$$\mathbb{P}[|X - \mu| > a] \leq 2e^{-a^2/2n}.$$

**Construction of $A_\ell(n,p)$.** *For $n \in \mathbb{N}$, $2 \leq \ell \leq k-1$, and $0 < p < 1$ with $p \in \mathbb{Q}$ so $p = \frac{a}{b}$ with $a, b \in \mathbb{Z}^+$, define a probability distribution $A_\ell(n,p)$ on $k$-uniform, $n$-vertex hypergraphs as follows. Let $c : E(K_n^{(\ell)}) \to \{0, \dots, b-1\}$ be a random $b$-coloring of the edges of the complete $\ell$-uniform hypergraph $K_n^{(\ell)}$ where each edge receives each color with equal probability independently of all other edges. Let the vertex set of $A_\ell(n,p)$ be $V(K_n^{(\ell)})$ and make $W \subseteq V(A_\ell(n,p))$ a hyperedge of $A_\ell(n,p)$ if $|W| = k$ and*

$$\sum_{\{x_1,\dots,x_\ell\}\subseteq W} c(\{x_1, \dots, x_\ell\}) < a \pmod{b}.$$

**Lemma 15.** *For every $\epsilon > 0$, with probability going to one as $n$ goes to infinity,*

$$\left| |E(A_\ell(n,p))| - p\binom{n}{k} \right| \leq \epsilon n^k.$$

*Proof.* Let $W \subseteq V(A_\ell(n,p))$ with $|W| = k$ and let $\{x_1, \dots, x_\ell\} \subseteq W$. Define

$$\Delta = \sum_{\substack{\{y_1,\dots,y_\ell\}\subseteq W \\ \{x_1,\dots,x_\ell\}\neq\{y_1,\dots,y_\ell\}}} c(\{y_1, \dots, y_\ell\}).$$

Now thinking of $\Delta$ as fixed, there are exactly $a$ choices for $c(\{x_1, \dots, x_\ell\})$ such that $\Delta + c(\{x_1, \dots, x_\ell\}) < a \pmod{b}$. Since the edge $\{x_1, \dots, x_\ell\}$ receives each color with equal probability, the probability that $W$ is an edge of $A_\ell(n,p)$ is $\frac{a}{b} = p$. Therefore, the expected number of edges of $A_\ell(n,p)$ is $p\binom{n}{k}$.

By the second moment method, with probability going to one as $n$ goes to infinity, $\left||E(A_\ell(n,p))| - p\binom{n}{k}\right| \leq \epsilon n^k$. Indeed, for each $k$-set $W$ in $V(A_\ell(n,p))$, define an indicator random variable $X_W$ where $X_W = 1$ if $W$ is an edge of $A_\ell(n,p)$. Let $X = \sum X_W$ so that $X = |E(A_\ell(n,p))|$ and $\mu = \mathbb{E}[X] = p\binom{n}{k}$. Let $\hat{\epsilon} = \frac{\epsilon}{p}$ so that $|X - \mu| \leq \hat{\epsilon}\mu$ implies that $|E(A_\ell(n,p)) - p\binom{n}{k}| \leq \epsilon n^k$. Since $X$ is the sum of indicator random variables, the variance $Var(X) = \sum_W Var(X_W) + 2\sum_{W,W'} Cov(X_W, X_{W'})$. The event "$X_W = 1$" will depend on "$X_{W'} = 1$" if and only if $W$ and $W'$ intersect in at least $\ell$ vertices, so there are at most $n^{2k-1}$ dependent pairs $(X_W, X_{W'})$. This implies that there are at most $n^{2k-1}$ pairs $(X_W, X_{W'})$ with $Cov(X_W, X_{W'}) \neq 0$ so that $Var(X) = o(n^{2k})$. Since $\mu^2 = \Omega(n^{2k})$, Chebyshev's Inequality (Lemma 13) implies that $\mathbb{P}\left[|X - \mu| > \hat{\epsilon}\mu\right] \to 0$ as $n \to \infty$, completing the proof. For more details on the second moment method, see [2]. $\square$

**Construction of** $B_{\vec{\pi}}(n,p)$**.** *Let* $\vec{\pi} = (k_1, \ldots, k_t)$ *be a proper ordered partition of* $k$*, let* $n \in \mathbb{N}$*, and let* $0 < p < 1$ *with* $p \in \mathbb{Q}$ *so* $p = \frac{a}{b}$ *with* $a, b \in \mathbb{Z}^+$*. Define a probability distribution* $B_{\vec{\pi}}(n,p)$ *on* $k$*-uniform hypergraphs with vertex set* $[n]$ *as follows. For* $1 \le i \le t$*, let* $c_i : \binom{[n]}{k_i} \to \{0, \ldots, b-1\}$ *be a random* $b$*-coloring of the hyperedges of the complete* $n$*-vertex,* $k_i$*-uniform hypergraph where each hyperedge receives each color with equal probability independently. Form a* $k$*-uniform hypergraph* $B_{\vec{\pi}}(n,p)$ *on vertex set* $[n]$ *as follows. Let* $W \subseteq [n]$ *with* $|W| = k$ *and partition* $W$ *into* $W_1, \ldots, W_t$ *such that* $|W_j| = k_j$ *and for all* $j < \ell$*, every element of* $W_j$ *is smaller than every element of* $W_\ell$*. In other words,* $W_1$ *is the set of first* $k_1$ *vertices of* $W$ *in the ordering,* $W_2$ *is the set of next* $k_2$ *vertices, and so on. Make* $W$ *a hyperedge of* $B_{\vec{\pi}}(n,p)$ *if*

$$\sum_{i=1}^{t} c_i(W_i) < a \pmod{b}.$$

**Lemma 16.** *For every* $\epsilon > 0$*, with probability going to one as* $n$ *goes to infinity,*

$$\left| |E(B_{\vec{\pi}}(n,p))| - p\binom{n}{k} \right| \le \epsilon n^k.$$

*Proof.* Let $W \in \binom{[n]}{k}$ and let $W_1, \ldots, W_t$ be the partition of $W$ as in the construction. Let $\Delta = \sum_{i=1}^{t-1} c(W_i)$. There are exactly $a$ choices for $c(W_t)$ such that $\Delta + c(W_t) < a \pmod{b}$ so the probability that $W$ is a hyperedge is $\frac{a}{b} = p$. Since two $k$-sets will depend on each other only if they share at least one vertex, the second moment method implies that with high probability, $|E(B_{\vec{\pi}}(n,p))| = p\binom{n}{k} \pm \epsilon n^k$. ☐

**Construction of** $D(n, \frac{1}{2})$**.** *For* $k \ge 3$ *and* $n \in \mathbb{N}$*, define a probability distribution* $D(n, \frac{1}{2})$ *on* $k$*-uniform,* $n$*-vertex hypergraphs as follows. Let* $G = G^{(k-1)}(n, \frac{1}{2})$ *be the random* $(k-1)$*-uniform hypergraph with edge probability* $\frac{1}{2}$*. For each* $T \in \binom{V(D(n,\frac{1}{2}))}{k}$*, select a* $(k-2)$*-subset of* $T$ *uniformly at random from among all* $(k-2)$*-subsets of* $T$*. Call the chosen* $(k-2)$*-subset of* $T$ *the* head *of* $T$ *and note that the choice of the head of* $T$ *is selected independently of all other choices for heads for other* $k$*-sets. If* $T = \{x_1, \ldots, x_{k-2}, y, z\}$ *where* $\{x_1, \ldots, x_{k-2}\}$ *is the head, make* $T$ *a hyperedge of* $D(n, \frac{1}{2})$ *if either both or neither of* $\{x_1, \ldots, x_{k-2}, y\}, \{x_1, \ldots, x_{k-2}, z\}$ *are edges of* $G$*.*

**Lemma 17.** *For every* $\epsilon > 0$*, with probability going to one as* $n$ *goes to infinity,*

$$\left| |E(D(n,1/2))| - \frac{1}{2}\binom{n}{k} \right| \le \epsilon n^k.$$

*Proof.* Let $T \in \binom{V(D(n,1/2))}{k}$. Conditioning on the choice of head $\{x_1, \ldots, x_{k-2}\}$ and the behavior of $\{x_1, \ldots, x_{k-2}, y\}$ in $G$, the set $\{x_1, \ldots, x_{k-2}, z\}$ is a hyperedge of $G$ with probability $\frac{1}{2}$, so the probability that $T$ is a hyperedge of $D(n,1/2)$ is $\frac{1}{2}$. Since two $k$-sets will depend on each other only if they share at least $k-1$ vertices, the second moment method implies that $|E(D(n,1/2))| = \frac{1}{2}\binom{n}{k} \pm \epsilon n^k$ with high probability. ☐

The next few sections prove that with high probability, $A(n,p)$, $B_{\vec{\pi}}(n,p)$, and $D(n,\frac{1}{2})$ satisfy and fail the following properties.

- $A_\ell(n,p)$
  - Satisfies: $\texttt{Expand}_p[\pi]$ for all $\pi$, $\texttt{CliqueDisc}_p[\ell-1]$, and $\texttt{Deviation}[\ell]$.
  - Fails: $\texttt{CliqueDisc}_p[\ell]$ and $\texttt{Deviation}[\ell+1]$.
- $B_{\vec{\pi}}(n,p)$
  - Satisfies: $\texttt{Expand}_p[\pi']$ for $\pi \not\leq \pi'$ and $\texttt{CliqueDisc}_p[\ell]$ for $\ell < \max\pi$.
  - Fails: $\texttt{Expand}_p[\pi]$ and $\texttt{Deviation}[2]$
- $D(n,\frac{1}{2})$
  - Satisfies: $\texttt{Expand}_p[\pi]$ for all $\pi$,
  - Fails: $\texttt{Deviation}[2]$

## 3.1 Failure of quasirandom properties

**Lemma 18.** *($A_\ell(n,p)$ fails $\texttt{CliqueDisc}[\ell]$) For $2 \leq \ell \leq k-1$, with probability going to one as $n$ goes to infinity, there exists an $\ell$-uniform hypergraph $G$ on vertex set $V(A_\ell(n,p))$ such that*

$$\Big||\mathcal{K}_k(G) \cap A_\ell(n,p)| - p|\mathcal{K}_k(G)|\Big| > \frac{1-p}{2b^{k^\ell}}\binom{n}{k}.$$

*Proof.* Let $G$ be the $\ell$-uniform hypergraph with vertex set $V(A_\ell(n,p))$ and edge set the set of edges of $K_n^{(\ell)}$ colored zero in the definition of $A_\ell(n,p)$. With high probability, the second moment method implies that the number of $k$-cliques in $G$ is $b^{-\binom{k}{\ell}}\binom{n}{k} + o(n^k)$. By definition, $A_\ell(n,p)$ will intersect all of the $k$-cliques of $G$ so

$$\Big||\mathcal{K}_k(G) \cap A_\ell(n,p)| - p|\mathcal{K}_k(G)|\Big| = (1-p)|\mathcal{K}_k(G)| = (1-p)b^{-\binom{k}{\ell}}\binom{n}{k} + o(n^k)$$

with high probability. □

**Lemma 19.** *($A_\ell(n,\frac{1}{2})$ fails $\texttt{Deviation}[\ell+1]$) For $2 \leq \ell \leq k-1$, there exists a constant $C > 0$ such that* $\mathrm{dev}_{\ell+1}(A_\ell(n,1/2)) > Cn^{k+\ell+1}$.

*Proof.* We will prove that every non-degenerate squashed octahedron induces an even number of hyperedges of $A_\ell(n,1/2)$. Let $x_1,\dots,x_{k-\ell-1},y_{1,0},y_{1,1},\dots,y_{\ell+1,0},y_{\ell+1,1} \in V(A_\ell(n,1/2))$ be distinct vertices. We claim that $|\mathcal{O}[x_1;\dots;x_{k-\ell-1};y_{1,0},y_{1,1};\dots;y_{\ell+1,0},y_{\ell+1,1}] \cap E(H)|$ is always even. Define $P_1=\{x_1\},\dots,P_{k-\ell-1}=\{x_{k-\ell-1}\},P_{k-\ell}=\{y_{1,0},y_{1,1}\},\dots,P_k=\{y_{\ell+1,0},y_{\ell+1,1}\}$

so that $P_1, \ldots, P_k$ are the parts of the squashed octahedron. Let $c : \binom{V(A_\ell(n,1/2))}{\ell} \to \{0,1\}$ be the random coloring used in the definition of $A_\ell(n, 1/2)$. For a $k$-set $T$, define

$$c(T) = \sum_{\substack{Z \subseteq T \\ |Z|=\ell}} c(Z) \pmod 2.$$

Lastly, define $\mathcal{T}$ to be the collection of $k$-sets which take exactly one vertex from each $P_i$.

**Claim**: $\sum_{T \in \mathcal{T}} c(T) = 0 \pmod 2$.

*Proof.* Expand the definition of $c(T)$ to obtain

$$\sum_{T \in \mathcal{T}} c(T) = \sum_{T \in \mathcal{T}} \sum_{\substack{Z \subseteq T \\ |Z|=\ell}} c(Z) \pmod 2. \tag{6}$$

Let $\Gamma_Z = \{k - \ell \leq i \leq k : Z \cap P_i = \emptyset\}$ and notice that $c(Z)$ appears $2^{|\Gamma_Z|}$ times in (6). Indeed, to form a $k$-set $T$ containing $Z$, there is a choice between $y_{i,0}$ and $y_{i,1}$ for each $i \in \Gamma_Z$. Since there are $\ell + 1$ parts with two vertices and $|Z| = \ell$, $|\Gamma_Z| \geq 1$. This implies that each $c(Z)$ appears an even number of times in (6), finishing the proof of the claim. □

By definition, $T$ is a hyperedge of $A_\ell(n, 1/2)$ if and only if $c(T) = 0 \pmod 2$. Thus the claim implies that the number of $T$s which are not hyperedges is even, but since the squashed octahedron has an even number of edges total, the number of $T$s which are hyperedges is then also even. Thus for every squashed octahedron using distinct vertices, the number of hyperedges appearing is even. There are $(k + \ell + 1)!\binom{n}{k+\ell+1}$ squashed octahedrons using distinct vertices and the number of degenerate squashed octahedrons is $o(n^{k+\ell+1})$, completing the proof of the lemma. □

**Lemma 20.** *($B_{\vec{\pi}}(n,p)$ fails* **Expand**$[\pi]$*) For all ordered partitions $\vec{\pi}$ of $k$, with probability going to one as $n$ goes to infinity, there exists $S_1 \subseteq \binom{[n]}{k_1}, \ldots, S_t \subseteq \binom{[n]}{k_t}$ such that*

$$\Big| e(S_1, \ldots, S_t) - p|S_1| \cdots |S_t| \Big| > \frac{1}{2} \binom{k}{k_1, \ldots, k_t} \frac{p}{b^t t^k} \binom{n}{k}.$$

*Proof.* Divide $V(B_{\vec{\pi}}(n,p)) = [n]$ into $t$ almost equal parts $X_1 = \{1, \ldots, \lfloor \frac{n}{t} \rfloor\}$, $X_2 = \{\lfloor \frac{n}{t} \rfloor + 1, \ldots, \lfloor \frac{2n}{t} \rfloor\}$, and so on. For $1 \leq i \leq t-1$, let $S_i \subseteq \binom{X_i}{k_i}$ be the set of hyperedges on $X_i$ colored zero under $c_i$ in the definition of $B_{\vec{\pi}}(n,p)$. Let $S_t \subseteq \binom{X_t}{k_t}$ be the set of hyperedges on $X_t$ colored $a$ under $c_t$ in the definition of $B_{\vec{\pi}}(n,p)$.

A $k$-set formed by taking a $k_i$-set from $S_i$ for each $i$ has color sum $a$, so is not a hyperedge of $B_{\vec{\pi}}(n,p)$. Thus $e(S_1, \ldots, S_t) = 0$. The second moment method implies that with high probability $|S_i| = \frac{1}{b}\binom{n/t}{k_i} + o(n^{k_i})$. Therefore,

$$\begin{aligned} \left| e(S_1, \ldots, S_t) - p \prod_{i=1}^{k} |S_i| \right| = p \prod_{i=1}^{k} |S_i| &= \frac{p}{b^t} \prod_{i=1}^{k} \binom{n/t}{k_i} + o(n^k) \\ &= \binom{k}{k_1, \ldots, k_t} \frac{p}{b^t t^k} \binom{n}{k} + o(n^k) \end{aligned}$$

with high probability, completing the proof. $\square$

**Lemma 21.** *($B_{(k-1,1)}(n,1/2)$ fails* **`Deviation`***[2]) Fix $k \geq 3$ and let $\vec{\pi} = (k-1,1)$. There exists a constant $C > 0$ such that with probability going to one as $n$ goes to infinity,*

$$\mathrm{dev}_2\left(B_{\vec{\pi}}(n,1/2)\right) > Cn^{k+2}.$$

*Proof.* Let $x_1,\ldots,x_{k-2},y_0,y_1,z_0,z_1$ be distinct vertices and recall that the vertex set of $B_{\vec{\pi}}(n,1/2)$ is $[n]$. There are several cases depending on how the vertices $x_1,\ldots,x_{k-2},y_0,y_1,z_0,z_1$ are ordered in $[n]$. Let $\mathcal{O} = \mathcal{O}[x_1;\ldots;x_{k-2};y_0,y_1;z_0,z_1]$ and let $c_1 : \binom{[n]}{k-1} \to \{0,1\}$ and $c_2 : [n] \to \{0,1\}$ be the two random colorings used in the definition of $B_{\vec{\pi}}(n,1/2)$.

- Case 1: $z_0$ and $z_1$ appear last. In this case, $|\mathcal{O} \cap E(H)|$ is always even as follows. If $c_2(z_0) = c_2(z_1)$, then either $x_1 \ldots x_{k-2}y_0z_0$ and $x_1 \ldots x_{k-2}y_0z_1$ are both hyperedges of $B_{\vec{\pi}}(n,1/2)$ or neither are hyperedges depending on the value of $c_1(x_1 \ldots x_{k-2}y_0)$. Similarly, either $x_1 \ldots x_{k-2}y_1z_0$ and $x_1 \ldots x_{k-2}y_1z_1$ are both hyperedges or neither are hyperedges so the total number of hyperedges induced by $\mathcal{O}$ is even. If $c_2(z_0) \neq c_2(z_1)$, then exactly one of $x_1 \ldots x_{k-2}y_0z_0$ and $x_1 \ldots x_{k-2}y_0z_1$ is a hyperedge and exactly one of $x_1 \ldots x_{k-2}y_1z_0$ and $x_1 \ldots x_{k-2}y_1z_1$ is a hyperedge. Thus the total number of hyperedges induced by $\mathcal{O}$ is even.

- Case 2: $y_0$ and $y_1$ appear last. This case is symmetric to Case 1: the total number of hyperedges induced by $\mathcal{O}$ is even.

- Case 3: Some $x_i$ appears after $y_0$ and $z_0$. In this case, the probability that $|\mathcal{O} \cap E(H)|$ is even is $\frac{1}{2}$. Assume that $x_i$ is the largest vertex among $x_1,\ldots,x_{k-2}$. The set $x_1 \ldots x_{k-2}y_0z_0$ is a hyperedge of $B_{\vec{\pi}}(n,1/2)$ if $c_1(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k-2},y_0,z_0) + c_2(x_i) = 0 \pmod 2$. Also, $x_1 \ldots x_{k-2}y_0z_0$ is the only hyperedge of $\mathcal{O}$ which tests the value of $c_1(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k-2},y_0,z_0)$, since this is the only hyperedge of $\mathcal{O}$ which includes both $y_0$ and $z_0$. Therefore, conditioning on the other hyperedges of $\mathcal{O}$ and also conditioning on $c_2(x_i)$, with probability $\frac{1}{2}$, $c_1(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{k-2},y_0,z_0) = 0$ so with probability $\frac{1}{2}$, $x_1 \ldots x_{k-2}y_0z_0$ is a hyperedge, so with probability $\frac{1}{2}$, $|\mathcal{O} \cap E(H)|$ is even.

- Cases 4-6: Some $x_i$ appears after $y_0,z_1$, some $x_i$ appears after $y_1,z_0$, and some $x_i$ appears after $y_1,z_1$. These three cases are symmetric to Case 3: the probability that $|\mathcal{O} \cap E(H)|$ is even is $\frac{1}{2}$.

Now consider the sum $\mathrm{dev}_2(H_n)$:

$$\begin{aligned}\mathrm{dev}_2(H_n) = &\sum_{x_1,\ldots,x_{k-2},y_0,y_1,z_0,z_1 \in \text{Cases 1,2}} \eta(x_1;\ldots;x_{k-2};y_0,y_1;z_0,z_1) \\ &+ \sum_{x_1,\ldots,x_{k-2},y_0,y_1,z_0,z_1 \in \text{Cases 3-6}} \eta(x_1;\ldots;x_{k-2};y_0,y_1;z_0,z_1)\end{aligned}$$

In the sum over Cases 1 and 2, $\eta$ is always $+1$ so the sum is at least $cn^{k+2}$, where $c$ is the fraction of octahedrons in Cases 1 and 2. Dividing the vertices in half, choosing $z_0, z_1$ from the second half and all other vertices from the first half is a lower bound on $c$, so $c > 2^{-k-3}$. The expected value of the sum over Cases 3-6 is zero by linearity of expectation. Since two octahedrons will depend on each other only if they share a vertex, the second moment method implies that with high probability the sum over Cases 3-6 is at most $\frac{c}{2}n^{k+2}$ in absolute value. Thus with high probability, $\text{dev}_2(H_n) > \frac{c}{2}n^{k+2}$. □

**Lemma 22.** *($D(n, 1/2)$ fails* `Deviation`*[2]). Fix $k \geq 3$. There exists a constant $C > 0$ such that, with probability going to one as $n$ goes to infinity,*

$$\text{dev}_2 (D(n, 1/2)) > Cn^{k+2}.$$

*Proof.* This proof is very similar to the proof of Lemma 21. Let $x_1, \ldots, x_{k-2}, y_0, y_1, z_0, z_1$ be distinct vertices and let $G = G^{(k-1)}(n, 1/2)$ be the random hypergraph used in the definition of $D(n, 1/2)$.

- Case 1: $\{x_1, \ldots, x_{k-2}\}$ is the head of every $k$-tuple in $\mathcal{O}$. In this case, $|\mathcal{O} \cap E(H)|$ is always even. Indeed, let $\vec{x} = x_1 \ldots x_{k-2}$ and consider the tuples of $\mathcal{O}$ ordered as $\vec{x}y_0z_0$, $\vec{x}y_0z_1$, $\vec{x}y_1z_1$, and $\vec{x}y_1z_0$. (In a drawing of the octahedron with $\vec{x}$ at the center and $y_0, z_0, y_1, z_1$ as the corners of a box around the center, these are the edges ordered cyclically.) These tuples will be hyperedges depending on if $\vec{x}y_0$, $\vec{x}z_1$, $\vec{x}y_1$, and $\vec{x}z_0$ are edges of $G$ or not. Considering these tuples in this order, each transition between edge and non-edge of $G$ implies a missing hyperedge of $\mathcal{O}$ and each transition between edge and edge or between non-edge and non-edge of $G$ implies a hyperedge of $\mathcal{O}$. Since there are an even number of transitions, $|\mathcal{O} \cap E(H)|$ is always even.

- Case 2: $\{x_1, \ldots, x_{k-2}\}$ is not the head of some tuple in $\mathcal{O}$. In this case, $|\mathcal{O} \cap E(H)|$ is even with probability $\frac{1}{2}$. Assume by symmetry that $y_0$ is included in the head of $x_1 \ldots x_{k-2}y_0z_0$. Then $x_1 \ldots x_{k-2}y_0z_0$ is a hyperedge depending on if two $(k-1)$-sets are in $G$, and at least one of these $(k-1)$-sets include both $y_0$ and $z_0$. This $(k-1)$-set including both $y_0$ and $z_0$ is only tested as part of deciding if $x_1 \ldots x_{k-2}y_0z_0$ is a hyperedge, since this is the only tuple of $\mathcal{O}$ which includes both $y_0$ and $z_0$. Thus conditioning on all other tuples of $\mathcal{O}$, $x_1 \ldots x_{k-2}y_0z_0$ is a hyperedge with probability $\frac{1}{2}$ so the number of tuples of $\mathcal{O}$ which are hyperedges is even with probability $\frac{1}{2}$.

Similar to the proof of Lemma 21, divide the sum $\text{dev}_2(H_n)$ into two sums by case. The sum over Case 1 is at least $cn^{k+2}$ for some $c > 0$ and the expected value of the sum over Case 2 is zero. Thus using the second moment method, with high probability, $\text{dev}_2(H_n) > \frac{c}{2}n^{k+2}$. □

## 3.2 Expansion

In this section, we show that with high probability $A_2(n, p)$, $B_{\vec{\pi}'}(n, p)$, and $D(n, \frac{1}{2})$ satisfy `Expand`$[\pi]$ if $\pi'$ is not a refinement of $\pi$. The proof generalizes to show that $A_\ell(n, p)$ satisfies

$\texttt{Expand}[\pi]$ for all $\ell$, but this is not required so the proof is omitted. To show these constructions satisfy $\texttt{Expand}[\pi]$, we take advantage of a theorem of the current authors [25] which shows that two properties on counting subgraphs are equivalent to $\texttt{Expand}[\pi]$. Counting subgraphs is easier than showing $\texttt{Expand}[\pi]$ holds, so using [25] simplifies the proof.

**Definition.** *Let $k \geq 2$ and let $\pi = k_1 + \cdots + k_t$ be a proper partition of $k$. A $k$-uniform hypergraph $F$ is* $\pi$-linear *if there exists an ordering $E_1, \ldots, E_m$ of the edges of $F$ such that for every $i$, there exists a partition of the vertices of $E_i$ into $A_{i,1}, \ldots, A_{i,t}$ such that for $1 \leq s \leq t$, $|A_{i,s}| = k_s$ and for every $j < i$, there exists an $s$ such that $E_j \cap E_i \subseteq A_{i,s}$.*

**Definition.** *Let $k \geq 2$ and let $\pi = k_1 + k_2$ be a partition of $k$ into two parts. The cycle $C_{\pi,4}$ of type $\pi$ and length four is the following hypergraph. Let $X_1, X_2, Y_1, Y_2$ be disjoint sets with $|X_1| = |X_2| = k_1$ and $|Y_1| = |Y_2| = k_2$. The vertex set of $C_{\pi,4}$ is $X_1 \cup X_2 \cup Y_1 \cup Y_2$ and the edge set is $\{X_i \cup Y_j : 1 \leq i, j \leq 2\}$.*

Among other things, the current authors [25, 26] proved that the properties $\texttt{Count}[\pi\text{-linear}]$ and $\texttt{Cycle}_4[\pi]$ (defined below) are equivalent to $\texttt{Expand}[\pi]$. If $F$ and $H$ are hypergraphs, a *labeled copy of $F$ in $H$* is an edge-preserving injection $V(F) \to V(H)$, i.e. an injection $\alpha : V(F) \to V(H)$ such that if $E$ is an edge of $F$, then $\{\alpha(x) : x \in E\}$ is an edge of $H$.

**Theorem 23.** *([25, 26]) Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of $k$-uniform hypergraphs where $|V(H_n)| = n$ and $|E(H_n)| \geq p\binom{n}{k} + o(n^k)$. Let $\pi$ be any proper partition of $k$. Then $\mathcal{H}$ satisfies $\texttt{Expand}_p[\pi]$ if and only if $\mathcal{H}$ satisfies*

- *$\texttt{Count}_p[\pi\text{-linear}]$: For all $f$-vertex, $m$-edge, $k$-uniform, $\pi$-linear hypergraphs $F$, the number of labeled copies of $F$ in $H_n$ is $p^m n^f + o(n^f)$.*

*In addition, if $\pi = k_1 + k_2$ is a partition into two parts, $\mathcal{H}$ satisfies $\texttt{Expand}_p[\pi]$ if and only if $\mathcal{H}$ satisfies*

- *$\texttt{Cycle}_{p,4}[\pi]$: The number of labeled copies of $C_{\pi,4}$ in $H_n$ is at most $p^4 n^{2k} + o(n^{2k})$.*

Note that [25, 26] actually defines a cycle $C_{\pi,2\ell}$ for any proper partition $\pi$ and any $\ell \geq 2$ and equates counting cycles with $\texttt{Expand}[\pi]$, but the full definition of $C_{\pi,2\ell}$ is complicated and not required in this paper. Therefore, we only state the definition of cycles and the equivalence between counting cycles and expansion for partitions into two parts.

**Lemma 24.** *($A_2(n,p)$ satisfies $\texttt{Cycle}_4[k_1, k_2]$) For $k = k_1 + k_2$ with $k_i \geq 1$, $\epsilon > 0$, and $0 < p < 1$, with probability going to one as $n$ goes to infinity, the number of labeled copies of $C_{k_1+k_2,4}$ in $A_2(n,p)$ satisfies*

$$\left| \#\{C_{k_1+k_2,4} \text{ in } A_2(n,p)\} - p^4 n^{2k} \right| < \epsilon n^{2k}. \tag{7}$$

*Proof.* Let $c : E(K_n) \to \{0, \dots, b-1\}$ be the random coloring used in the construction of $A_2(n,p)$. The cycle $C_{\pi,4}$ has four edges with four vertex groups $X_1, X_2, Y_1, Y_2$ where $|X_i| = k_1$ and $|Y_i| = k_2$ for all $i$ and $X_i \cup Y_j$ are hyperedges for all $i, j$. Let us pick disjoint sets $X_1, X_2, Y_1, Y_2$ of vertices of $A_2(n,p)$ and compute the probability that each $X_i \cup Y_j$ is a hyperedge of $A_2(n,p)$. We claim that the probability that $X_i \cup Y_j$ is a hyperedge of $A_2(n,p)$ is $p$ independently of if the other pairs are hyperedges or not. The only possible dependence between the events "$X_i \cup Y_j$ is a hyperedge of $A_2(n,p)$" and the event "$X_{i'} \cup Y_{j'}$ is a hyperedge of $A_2(n,p)$" would come from the edges of $K_n$ appearing in the intersection of the two hyperedges. But even conditioned on exactly the behavior of the colors of $\binom{X_i}{2}$ and the colors of $\binom{Y_i}{2}$, $X_i \cup Y_j$ is an edge with probability $p$. This is because if $x_i y_j$ is a pair of vertices with one endpoint in $X_i$ and one endpoint in $Y_j$, even conditioning on the color of every pair in $\binom{X_i \cup Y_j}{2}$ besides $\{x_i, y_j\}$, the set $X_i \cup Y_j$ is a hyperedge of $A_2(n,p)$ or not depending only on the color of $x_i y_j$ (similar to the proof of Lemma 15).

Thus since each $X_i \cup Y_j$ is a hyperedge of $A_2(n,p)$ with probability $p$, the expected number of labeled cycles $C_{\pi,4}$ in $A_2(n,p)$ is $p^4 (2k)! \binom{n}{2k}$. Since two cycles will depend on each other only if they share at least one vertex, the second moment method implies that with high probability, the number of labeled cycles in $A_2(n,p)$ is $p^4 n^{2k} \pm \epsilon n^{2k}$. □

The above proof generalizes in a straightforward manner to show that $A_\ell(n,p)$ satisfies `Cycle`$_4[k_1 + k_2]$, although we do not require this fact in this paper. Also, since every partition $\pi$ is a refinement of $k_1 + k_2$ for some $k_1$ and $k_2$, $A_\ell(n,p)$ satisfies `Expand`$[\pi]$ for all $\pi$.

**Lemma 25.** *($B_{\vec{\pi}'}(n,p)$ satisfies **`Count`**$[\pi$-linear$]$) Let $\pi$ and $\pi'$ be proper partitions of $k$ such that $\pi'$ is not a refinement of $\pi$. Let $\vec{\pi}'$ be any ordering of the entries of $\pi'$. For any $\pi$-linear, $v$-vertex, $m$-edge hypergraph $F$, any $\epsilon > 0$, and any $0 < p < 1$, with probability going to one as $n$ goes to infinity,*

$$|\#\{F \text{ in } B_{\vec{\pi}'}(n,p)\} - p^m n^v| < \epsilon n^v.$$

*Proof.* Let $F$ be a $\pi$-linear hypergraph with $v$ vertices, labeled $f_1, \dots, f_v$, and let $x_1, \dots, x_v \in V(B_{\vec{\pi}'}(n,p))$ be a list of $v$ distinct vertices of $B_{\vec{\pi}'}(n,p)$. Define an indicator random variable

$$X(F; x_1, \dots, x_v) = \begin{cases} 1, & \text{if } B_{\vec{\pi}'}[\{x_1, \dots, x_v\}] \text{ is a labeled copy of } F \text{ with } x_i \text{ mapped to } f_i, \\ 0 & \text{otherwise.} \end{cases}$$

*Claim:* For any $\pi$-linear hypergraph $F$, and any distinct vertices $x_1, \dots, x_v \in V(B_{\vec{\pi}}(n,p))$, we have $\mathbb{P}[X(F; x_1, \dots, x_v) = 1] = p^{|E(F)|}$.

*Proof.* The claim is proved by induction on the number of edges of $F$. If $F$ has no edges, then $X(F; x_1, \dots, x_v)$ is always one. For the inductive step, let $E$ be the last edge of $F$ in the ordering provided by the $\pi$-linearity of $F$, and let $m = |E(F)|$. We may assume that the vertices of $F$ are labeled so that $E = \{f_1, \dots, f_k\}$. Let $F'$ be the hypergraph with $V(F') = V(F)$ and $E(F') = E(F) - E$. Then

$$\begin{aligned}\mathbb{P}[X(F; x_1, \dots, x_v) = 1] =& \mathbb{P}[\{x_1, \dots, x_k\} \in E(B_{\vec{\pi}'}(n,p)) \Big| X(F'; x_1, \dots, x_v) = 1] \\ & \cdot \mathbb{P}[X(F'; x_1, \dots, x_v) = 1]. \end{aligned} \tag{8}$$

By induction, $\mathbb{P}[X(F'; x_1, \dots, x_v) = 1]$ is $p^{m-1}$ so let us investigate the probability that $\{x_1, \dots, x_k\}$ forms an edge of $B_{\vec{\pi}'}(n,p)$ conditioned on $x_1, \dots, x_v$ forming a copy of $F'$. The way we test if $\{x_1, \dots, x_k\}$ forms an edge of $B_{\vec{\pi}'}(n,p)$ is to sort the $x_i$s according to the underlying ordering of $V(B_{\vec{\pi}'}(n,p))$ and test the color sum of the $\pi'$-groups. More precisely, let $\eta$ be the permutation of $[k]$ such that $x_{\eta(1)} < x_{\eta(2)} < \cdots < x_{\eta(k)}$ and let $\vec{\pi}' = (k'_1, \dots, k'_t)$. Divide the $x_{\eta(i)}$s up into blocks $D_1, \dots, D_t$ so that $D_1$ consists of $x_{\eta(1)}, \dots, x_{\eta(k'_1)}$, $D_2$ consists of the next $k'_2$ vertices, and so on. The set $\{x_1, \dots, x_k\}$ will be an edge of $B_{\vec{\pi}'}(n)$ if $\sum c_i(D_i) < a \pmod b$.

Since $\pi'$ is not a refinement of $\pi$ and $F$ is $\pi$-linear, there is some block $D_i$ such that no edge of the copy of $F'$ on $x_1, \dots, x_v$ completely contains the block $D_i$. To see this, assume for contradiction that every block is completely contained in some edge of $F'$. Since $F$ is $\pi$-linear, there exists a partition of the vertices of $E$ into groups according to the partition $\pi$ such that every edge of $F'$ intersects at most one of these parts. If every block $D_i$ (which came from the $\pi'$ partition) was completely contained inside some edge, it would be completely contained inside the corresponding part of the $\pi$-partition of $E$. This assignment of blocks to parts of the $\pi$-partition of $E$ shows that $\pi'$ is a refinement of $\pi$, which is a contradiction.

Thus there exists some block $D_i$ which is not contained inside any edge of the copy of $F'$ on $x_1, \dots, x_v$, so the event "$D_i \in E(G_i)$" is independent of the event "$X(F', x_1, \dots, x_v) = 1$". Moreover, the event "$\{x_1, \dots, x_k\}$ forms an edge of $B_{\vec{\pi}'}(n,p)$" can be written in terms of the event "$D_i \in E(G_i)$," since no matter what happens to $D_j$ for $j \neq i$, $D_i$ has probability $p = \frac{a}{b}$ of making the total color sum in $\{0, \dots, a-1\}$ (similar to the proof of Lemma 16). Combining this with (8) and induction on the number of edges finishes proves the claim. ☐

By linearity of expectation, the expected number of labeled copies of $F$ in $B_{\vec{\pi}'}(n,p)$ is $p^m v! \binom{n}{v}$. Since two events will depend on each other only if the copies of $F$ share at least two vertices, the second moment method implies that with high probability, the number of labeled copies of $F$ is $p^m n^v \pm \epsilon n^v$. ☐

**Lemma 26.** *($D(n, \frac{1}{2})$ satisfies $\texttt{Cycle}_4[k_1 + k_2]$). For $k = k_1 + k_2$ with $k_i \geq 1$, $\epsilon > 0$, and $0 < p < 1$, with probability going to one as n goes to infinity, the number of labeled copies of $C_{k_1+k_2,4}$ in $D(n, \frac{1}{2})$ satisfies*

$$\left| \# \left\{ C_{k_1+k_2,4} \text{ in } D\left(n, 1/2\right) \right\} - \left(1/2\right)^4 n^{2k} \right| < \epsilon n^{2k}. \tag{9}$$

*Proof.* This proof is very similar to the proof of Lemma 24; we will show that the probability that some $2k$ vertices form a labeled copy of $C_{\pi,4}$ in $D(n, 1/2)$ is $2^{-4}$ and use the second moment method for concentration. Let $G = G^{(k-1)}(n, 1/2)$ be the random hypergraph used in the definition of $D(n, 1/2)$. Let $X_1, X_2, Y_1, Y_2$ be disjoint sets of vertices of $D(n, 1/2)$

with $|X_i| = k_1$ and $|Y_i| = k_2$. We claim that the probability that $X_i \cup Y_j$ is a hyperedge of $D(n, 1/2)$ is $\frac{1}{2}$ independently of if the other pairs are hyperedges or not. Indeed, let $R$ be the head of $X_i \cup Y_j$ and $z_1$ and $z_2$ the other two vertices of $X_i \cup Y_j$, and notice that since $|R| = k-2$ either $R \cup \{z_1\}$ or $R \cup \{z_2\}$ (or both) intersect both $X_i$ and $Y_j$. Say $R \cup \{z_1\}$ intersects both $X_i$ and $Y_j$. Since the sets $X_1, X_2, Y_1, Y_2$ are disjoint, $X_i \cup Y_j$ is the only hyperedge of the cycle to test if $R \cup \{z_1\}$ is in $G$ or not. Since $R \cup \{z_1\}$ is an edge of $G$ with probability $\frac{1}{2}$, $X_i \cup Y_j$ is a hyperedge of $D(n, 1/2)$ with probability $\frac{1}{2}$ independently of the other hyperedges of the cycle.

Thus the expected number of labeled four-cycles in $D(n, 1/2)$ is $2^{-4}(2k)!\binom{n}{2k}$ and by the second moment method, with probability going to one as $n$ goes to infinity, the number of labeled copies of $C_{\pi,4}$ in $D(n, 1/2)$ is $2^{-4}n^{2k} \pm \epsilon n^{2k}$. □

## 3.3 Clique Discrepency

In this section, we show that $A_{\ell+1}(n,p)$ and $B_{\vec{\pi}}(n,p)$ satisfy `CliqueDisc`$[\ell]$ for $\vec{\pi} = (\ell + 1, 1, \dots, 1)$. The proof generalizes to show that $B_{\vec{\pi}}(n,p)$ satisfies `CliqueDisc`$[\ell]$ for all $\pi$ with $\max \pi > \ell$, but this generalization is not required for Table 1 so the proof is omitted.

**Lemma 27.** *($A_{\ell+1}(n,p)$ satisfies `CliqueDisc`$[\ell]$) Let $1 \le \ell < k-1$ and $0 < p < 1$. For every $\epsilon > 0$, with probability going to one as $n$ goes to infinity, for every $\ell$-uniform hypergraph $F$ on vertex set $[n]$,*

$$\Big| |\mathcal{K}_k(F) \cap E(A_{\ell+1}(n,p))| - p|\mathcal{K}_k(F)| \Big| \le \epsilon n^k. \tag{10}$$

*Proof.* Let $A = A_{\ell+1}(n,p)$ and let $c : \binom{V(A)}{\ell+1} \to \{0, \dots, b-1\}$ be the random coloring used in the definition of $A$. Fix some $\ell$-uniform hypergraph $F$ on vertex set $V(A)$ and let us compute the probability that $F$ is bad, where *bad* means that (10) fails. Let $x_1, \dots, x_{k-\ell-1}$ be distinct vertices and define

$$W_{x_1,\dots,x_{k-\ell-1}} = \left\{ w \in \binom{V(A)}{\ell+1} : w \cup \{x_1, \dots, x_{k-\ell-1}\} \in \mathcal{K}_k(F) \right\},$$
$$Y_{x_1,\dots,x_{k-\ell-1}} = \left\{ w \cup \{x_1, \dots, x_{k-\ell-1}\} : w \in W_{x_1,\dots,x_{k-\ell-1}} \right\}.$$

Call $x_1, \dots, x_{k-\ell-1}$ *bad* if

$$\left| |Y_{x_1,\dots,x_{k-\ell-1}} \cap E(A)| - p|Y_{x_1,\dots,x_{k-\ell-1}}| \right| > \epsilon n^{\ell+1}. \tag{11}$$

If $F$ is bad, then some $x_1, \dots, x_{k-\ell-1}$ is bad. Indeed, if every $x_1, \dots, x_{k-\ell-1}$ was good, then using that $e < 2.8$ we have that

$$\begin{aligned}
&\Big| |\mathcal{K}_k(F) \cap E(A)| - p|\mathcal{K}_k(F)| \Big| \\
&\qquad = \binom{k}{k-\ell-1}^{-1} \left| \sum_{\{x_1,\dots,x_{k-\ell-1}\} \in \binom{V(A)}{k-\ell-1}} \Big( |Y_{x_1,\dots,x_{k-\ell-1}} \cap E(A)| - p|Y_{x_1,\dots,x_{k-\ell-1}}| \Big) \right| \\
&\qquad \le \binom{k}{k-\ell-1}^{-1} \sum_{\{x_1,\dots,x_{k-\ell-1}\} \in \binom{V(A)}{k-\ell-1}} \Big| |Y_{x_1,\dots,x_{k-\ell-1}} \cap E(A)| - p|Y_{x_1,\dots,x_{k-\ell-1}}| \Big| \\
&\qquad \le \binom{k}{k-\ell-1}^{-1} \sum_{\{x_1,\dots,x_{k-\ell-1}\} \in \binom{V(A)}{k-\ell-1}} \epsilon n^{\ell+1} \\
&\qquad = \frac{\binom{n}{k-\ell-1}}{\binom{k}{k-\ell-1}} \epsilon n^{\ell+1} \\
&\qquad \le \frac{\left(\frac{2.8n}{k-\ell-1}\right)^{k-\ell-1}}{\left(\frac{k}{k-\ell-1}\right)^{k-\ell-1}} \epsilon n^{\ell+1} \\
&\qquad = \left(\frac{2.8}{k}\right)^{k-\ell-1} \epsilon n^{k}.
\end{aligned}$$

Since $1 \le \ell < k-1$, we have $k \ge 3$ so that $\left(\frac{2.8}{k}\right)^{k+\ell-1} \le 1$. Thus if every $x_1, \dots, x_{k-\ell-1}$ was good, $F$ would be good. Therefore, using the union bound, the probability that $F$ is bad is $\binom{n}{k-\ell-1}$ times the probability that $x_1, \dots, x_{k-\ell-1}$ is bad.

Let us compute the probability that $x_1, \dots, x_{k-\ell-1}$ is bad. For each $w \in W_{x_1,\dots,x_{k-\ell-1}}$, define $X_w$ as the following indicator random variable:

$$X_w = \begin{cases} 1 & \text{if} \displaystyle\sum_{r \in \binom{w \cup \{x_1,\dots,x_{k-\ell-1}\}}{\ell+1}} c(r) < a \pmod b \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the event "$X_w = 1$" is mutually independent of the events "$X_{w'} = 1$" for $w' \neq w$, since $X_w$ is the only event to test the color assigned to $w$. Indeed, conditioning on all other $r \in \binom{w \cup \{x_1,\dots,x_{k-\ell-1}\}}{\ell+1}$ with $r \neq w$, the probability that $X_w = 1$ is $p$. Let $X = \sum X_w$ and $\mu = \mathbb{E}[X]$ and notice that $X = |Y_{x_1,\dots,x_{k-\ell-1}} \cap E(A)|$ and $\mu = p|W_{x_1,\dots,x_{k-\ell-1}}| = p|Y_{x_1,\dots,x_{k-\ell-1}}|$. Thus (11) becomes $|X - \mu| > \epsilon n^{\ell+1}$.

Next, by Chernoff's Bound (Lemma 14), there exists a constant $c$ depending only on $\epsilon$ such that

$$\mathbb{P}\left[|X-\mu| > \epsilon n^{\ell+1}\right] \le e^{-cn^{\ell+1}}. \tag{12}$$

Indeed, let $a = \epsilon n^{\ell+1}$ and let $n' = |W_{x_1,\ldots,x_{k-\ell-1}}| \leq n^{\ell+1}$ be the number of indicator random variables. Then by Chernoff's Bound, $\mathbb{P}[|X - \mu| > a] \leq 2e^{-a^2/2n'} \leq 2e^{-\epsilon^2 n^{\ell+1}/2} \leq e^{-cn^{\ell+1}}$ if $c = \epsilon^2/4$.

If $F$ is bad, then some $x_1, \ldots, x_{k-\ell-1}$ is bad, so the union bound implies that the probability that $F$ is bad is at most

$$\binom{n}{k-\ell-1} e^{-cn^{\ell+1}} \leq e^{-cn^{\ell+1}/2}.$$

Apply the union bound again to compute the probability that some $F$ is bad. There are at most $2^{n^\ell}$ choices for $F$ so the probability that some $F$ is bad is at most

$$2^{n^\ell} e^{-cn^{\ell+1}/2} \leq e^{-cn^{\ell+1}/4}$$

which goes to zero as $n$ goes to infinity, completing the proof. □

**Lemma 28.** *($B_{\ell+1,1,\ldots,1}(n,p)$ satisfies* **CliqueDisc[$\ell$]***) Let $1 \leq \ell < k-1$, $0 < p < 1$, and let $\vec{\pi} = (\ell+1, 1, \ldots, 1)$. For every $\epsilon > 0$, with probability going to one as $n$ goes to infinity, for every $\ell$-uniform hypergraph $F$ on vertex set $[n]$,*

$$\Big| |\mathcal{K}_k(F) \cap B_{\vec{\pi}}(n,p)| - p|\mathcal{K}_k(F)| \Big| \leq \epsilon n^k. \tag{13}$$

*Proof.* This proof is similar to the proof of the previous lemma. Fix some $\ell$-uniform hypergraph $F$ on vertex set $V(B_{\vec{\pi}}(n,p)) = [n]$ and let us compute the probability that $F$ is bad, where *bad* means that (13) fails. Recall that since $\vec{\pi} = (\ell+1, 1, \ldots, 1)$, $B_{\vec{\pi}}(n,p)$ is built from a random coloring $c_1$ of the complete $(\ell+1)$-uniform hypergraph and $k-\ell-1$ random colorings $c_2, \ldots, c_{k-\ell}$ of the complete one-uniform hypergraph.

Fix $k-\ell-1$ distinct vertices $x_2, \ldots, x_{k-\ell}$ with $x_2 < \cdots < x_{k-\ell}$ and let $W$ be the collection of $(\ell+1)$-sets which contain elements earlier than $x_2$ in the ordering and also form a clique of size $k$ in $F$ when added to $x_2, \ldots, x_{k-\ell}$. More precisely,

$$W_{x_2,\ldots,x_{k-\ell}} = \left\{ w : w \in \binom{[x_2 - 1]}{\ell+1}, w \cup \{x_2, \ldots, x_{k-\ell}\} \in \mathcal{K}_k(F) \right\}.$$

Notice that we define $W_{x_2,\ldots,x_{k-\ell}}$ as $(\ell+1)$-sets of elements smaller than $x_2$, so that asking if $w \cup \{x_2, \ldots, x_{k-\ell}\}$ is an edge of $B_{\vec{\pi}}(n,p)$ consists of asking about the color of $w$ in $c_1$ and the colors of $x_2, \ldots, x_{k-\ell}$ in $c_2, \ldots, c_{k-\ell}$. Since $x_2, \ldots, x_{k-\ell}$ are fixed, define $\Delta = \sum_{j=2}^{k-\ell} c_j(x_j)$. For each $w \in W_{x_2,\ldots,x_{k-\ell}}$, define a random variable $X_w$ as follows.

$$X_w = \begin{cases} 1 & \text{if } c_1(w) + \Delta < a \pmod{b}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Delta$ is fixed, the expectation $\mathbb{E}[X_w] = \frac{a}{b} = p$. Also, all these indicator random variables are mutually independent. Define $\hat{G}_\Delta$ to be the $(\ell+1)$-uniform hypergraph on vertex set

$V(B_{\vec{\pi}}(n,p))$ whose hyperedges are the $(\ell+1)$-sets receiving colors $\{-\Delta, -\Delta+1, \ldots, -\Delta+a-1\} \pmod b$. Define $X = \sum X_w$ and $\mu = \mathbb{E}[X] = p|W_{x_2,\ldots,x_{k-\ell}}|$. Consider an $(\ell+1)$-set $w$ in $E(\hat{G}_\Delta) \cap W_{x_2,\ldots,x_{k-\ell}}$. Then the color sum of $w \cup \{x_2, \ldots, x_{k-\ell}\}$ is between 0 and $a-1 \pmod b$ so that $w \cup \{x_2, \ldots, x_{k-\ell}\}$ is a hyperedge of $B_{\vec{\pi}}(n,p)$ and $X_w = 1$. In the other direction, if $X_w = 1$ then the color sum of $w \cup \{x_2, \ldots, x_{k-\ell}\}$ is between 0 and $a-1 \pmod b$ which implies that $w \in E(\hat{G}_\Delta)$. Therefore, $X = |E(\hat{G}_\Delta) \cap W_{x_2,\ldots,x_{k-\ell}}|$.

By a similar argument as in the proof of Lemma 27, the Chernoff Bound (Lemma 14) implies that there exists a constant $c > 0$ such that

$$\mathbb{P}\left[|X-\mu| > \epsilon n^{\ell+1}\right] < e^{-cn^{\ell+1}}. \tag{14}$$

Call $x_2, \ldots, x_{k-\ell}$ *bad* if

$$|X-\mu| = \left| \left| E(\hat{G}_\Delta) \cap W_{x_2,\ldots,x_{k-\ell}} \right| - p \left| W_{x_2,\ldots,x_{k-\ell}} \right| \right| > \epsilon n^{\ell+1}.$$

Next, we claim that if $F$ is bad then there exists some choice of $x_2, \ldots, x_{k-\ell}$ which is bad. To see this, notice that the $k$-cliques of $F$ can be partitioned based on their $k-\ell-1$ largest vertices. Let $Y_{x_2,\ldots,x_{k-\ell}} = \{w \cup \{x_2, \ldots, x_{k-\ell}\} : w \in W_{x_2,\ldots,x_{k-\ell}}\}$ so that $\mathcal{K}_k(F) = \dot{\bigcup} Y_{x_2,\ldots,x_{k-\ell}}$. There are at most $\binom{n}{k-\ell-1} \le n^{k-\ell-1}$ sets $Y_{x_2,\ldots,x_{k-\ell}}$ and they are all disjoint, so if $F$ is bad then there is some $x_2, \ldots, x_{k-\ell}$ such that

$$\Big| |E(B_{\vec{\pi}}(n,p)) \cap Y_{x_2,\ldots,x_{k-\ell}}| - p|Y_{x_2,\ldots,x_{k-\ell}}| \Big| > \epsilon n^{\ell+1}. \tag{15}$$

But $|E(B_{\vec{\pi}}(n,p)) \cap Y_{x_2,\ldots,x_{k-\ell}}| = |E(\hat{G}_\Delta) \cap W_{x_2,\ldots,x_{k-\ell}}|$ and $|Y_{x_2,\ldots,x_{k-\ell}}| = |W_{x_2,\ldots,x_{k-\ell}}|$, so that (15) implies that $x_2, \ldots, x_{k-\ell}$ is bad.

By the union bound, the probability that $F$ is bad is the number of choices for $x_2, \ldots, x_{k-\ell}$ times the probability that $x_2, \ldots, x_{k-\ell}$ is bad, which is bounded by (14). Thus the probability that $F$ is bad is at most

$$\binom{n}{k-\ell-1} e^{-cn^{\ell+1}} \le e^{-cn^{\ell+1}/2}.$$

We apply the union bound again to compute the probability that some $F$ is bad. There are at most $2^{n^\ell}$ choices for $F$ so the probability that some $F$ is bad is at most

$$2^{n^\ell} e^{-cn^{\ell+1}/2} \le e^{-cn^{\ell+1}/4}$$

which goes to zero as $n$ goes to infinity, completing the proof. □

While not required in this paper, the above proof generalizes to prove that $B_{\vec{\pi}}(n,p)$ satisfies $\texttt{CliqueDisc}[\ell]$ for all $\pi$ with $\max \pi > \ell$. If $\max \pi = k_i$, then the Chernoff Bound will imply a bound of $e^{-cn^{k_i}}$ which is enough to dominate the term $2^{n^\ell}$ from the number of $\ell$-uniform hypergraphs $F$.

## 3.4 Deviation

**Lemma 29.** *($A_\ell(n,1/2)$ satisfies* `Deviation`*[$\ell$]) For every $\epsilon > 0$, with probability going to one as $n$ goes to infinity,*

$$\mathrm{dev}_\ell\left(A_\ell(n,1/2)\right) \le \epsilon n^{k+\ell}.$$

*Proof.* This proof is similar to the proofs of Lemmas 21 and 22 except that in all cases the probability that a squashed octahedron is even is $\frac{1}{2}$. Let $x_1,\dots,x_{k-\ell},y_{1,0},y_{1,1},\dots,y_{\ell,0},y_{\ell,1}$ be distinct vertices and let $c : \binom{V(A_\ell(n,1/2))}{\ell} \to \{0,1\}$ be the random coloring used in the definition of $A_\ell(n,1/2)$. Let $G$ be the $\ell$-uniform hypergraph whose hyperedges are those $\ell$-sets colored one. Note that by definition, a set $T$ of $k$ vertices is a hyperedge of $A_\ell(n,1/2)$ if $|E(G[T])|$ is even.

Let $\mathcal{O} = \mathcal{O}[x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1}]$. We will show that with probability $\frac{1}{2}$, $|\mathcal{O} \cap E(A_\ell(n,1/2))|$ is even. Consider the tuple $(x_1,\dots,x_{k-\ell},y_{1,0},\dots,y_{\ell,0}) \in \mathcal{O}$ and let $Y = \{y_{1,0},\dots,y_{\ell,0}\}$. Note that $\{x_1,\dots,x_{k-\ell},y_{1,0},\dots,y_{\ell,0}\}$ is a hyperedge of $A_\ell(n,1/2)$ if the number of edges of $G$ induced by $\{x_1,\dots,x_{k-\ell},y_{1,0},\dots,y_{\ell,0}\}$ is even. But this tuple is the only tuple of $\mathcal{O}$ to test if $Y$ is an edge of $G$ or not, since no other tuple in $\mathcal{O}$ contains $Y$. Thus conditioning on all other tuples in $\mathcal{O}$ and all other $\ell$-subsets of $\{x_1,\dots,x_{k-\ell},y_{1,0},\dots,y_{\ell,0}\}$, $Y \in E(G)$ with probability $\frac{1}{2}$ so $x_1 \dots x_{k-\ell} y_{1,0} \dots y_{\ell,0}$ is a hyperedge of $A_\ell(n,1/2)$ with probability $\frac{1}{2}$ so $|\mathcal{O} \cap E(A_\ell(n,1/2))|$ is even with probability $\frac{1}{2}$. The expected value of the sum $\mathrm{dev}_\ell(A_\ell(n,1/2))$ is zero, and the second moment method shows that, with high probability, $\mathrm{dev}_\ell(A_\ell(n,1/2)) \le \epsilon n^{k+\ell}$. □

# 4 Non-implications in Table 1

This section completes the proof of Table 1 using the hypergraphs constructed in the previous section. The proof technique to construct hypergraph sequences satisfying some property and failing some other property is a diagonalization argument and is similar for all the results in Table 1. First we use the probabilistic method to prove that for every $\epsilon > 0$ and every $n \ge n_0$, with high probability there exists a hypergraph satisfying some property and failing another. We then construct a hypergraph sequence by creating a hypergraph for each $\epsilon = 1/n$ via diagonalization. Since all the proofs are very similar, we only give the full proof of `Expand`$[\pi] \not\Rightarrow$ `Expand`$[\pi']$ when $\pi'$ is not a refinement of $\pi$.

**Lemma 30.** *Let $0 < p < 1$ with $p \in \mathbb{Q}$ and let $\pi$ and $\pi'$ be proper partitions of $k$ such that $\pi'$ is not a refinement of $\pi$. For every $\epsilon > 0$, there exists a $N_0$ such that for $n \ge N_0$ there exists a hypergraph $B$ on $n$ vertices such that*

- $\left| |E(B)| - p\binom{n}{k} \right| \le \epsilon n^k$,
- *If $\pi' = k_1' + \dots + k_t'$, then there exists $S_1',\dots,S_t'$ with $S_i' \subseteq \binom{V(B)}{k_i'}$ and a constant $C > 0$*

*depending only on $p$, $k$, $\pi'$, and $t$ such that*

$$\left|e(S_1', \ldots, S_t') - p|S_1'| \ldots |S_t'|\right| > C\binom{n}{k},$$

- *For every $\pi$-linear hypergraph $F$ with $v$ vertices and $e$ edges where $v \leq \epsilon^{-1}$,*

$$|\#\{F \text{ in } B\} - p^e n^v| < \epsilon n^v.$$

*Proof.* Let $v_0 = \epsilon^{-1}$ and let $\eta = \frac{1}{2}2^{-2^{v_0}}$. By Lemma 16, $N_0$ can be chosen large enough so that for $n \geq N_0$, with probability at most $\eta$, $B = B_{\vec{\pi}'}(n, p)$ has $|E(B) - p\binom{n}{k}| \geq \epsilon n^k$. There are $2^{2^{v_0}}$ hypergraphs with $v_0$ vertices. Thus, for each of the at most $2^{2^{v_0}}$ $\pi$-linear hypergraphs $F$ with at most $v_0$ vertices, Lemma 25 shows that we can choose $N_0$ large enough so that for $n \geq N_0$, with probability at most $\eta$, $|\#\{F \text{ in } B\} - p^e n^v| \geq \epsilon n^v$. Lastly, by Lemma 20, we can choose $N_0$ large enough so that for $n \geq N_0$, with probability at least $1 - \eta$, there exists a constant $C > 0$ and sets $S_1', \ldots, S_t'$ so that $|e(S_1', \ldots, S_t') - p|S_1'| \ldots |S_t'|| > C\binom{n}{k}$.

Now fix $N_0$ to be the maximum of the $N_0$ from Lemma 16, the $N_0$ from Lemma 20, and the at most $2^{v_0}$ constants $N_0$ from Lemma 25. Note that the definition of $N_0$ depends only on $\epsilon$. Now consider $n \geq N_0$ and let $B = B_{\vec{\pi}'}(n, p)$. With probability at most $\eta$, $||E(B)| - p\binom{n}{k}| \geq \epsilon n^k$. Also, with probability at most $2^{2^{v_0}}\eta = \frac{1}{2}$, there exists some $F$ with at most $v_0$ vertices with $|\#\{F \text{ in } B\} - p^e n^v| \geq \epsilon n^v$. By Lemma 20, with probability at least $1 - \eta$, there exists a constant $C$ and sets $S_1', \ldots, S_t'$ such that $|e(S_1', \ldots, S_t') - p|S_1'| \ldots |S_t'|| > C\binom{n}{k}$. Therefore, for all $n \geq N_0$, with positive probability a hypergraph drawn from the distribution $B_{\vec{\pi}'}(n, p)$ satisfies all three conditions in the lemma. □

**Lemma 31.** *If $0 < p < 1$ and if $\pi'$ is not a refinement of $\pi$ then* $\texttt{Expand}_p[\pi] \not\Rightarrow \texttt{Expand}_p[\pi']$.

*Proof.* Form a sequence of hypergraphs $\{B_{n_q}\}_{q \to \infty}$ as follows. Let $\epsilon = 1/q$ and let $p_q$ be a rational with $|p_q - p| < \epsilon$. Apply Lemma 30 to $p_q$ and $\epsilon$ to produce an $N_0(1/q)$. Let $n_q$ be the maximum of $N_0$ and $|V(B_{n_{q-1}})| + 1$. Now since $n_q \geq N_0(1/q)$, Lemma 30 guarantees a hypergraph $B_{n_q}$ on $n_q$ vertices. The sequence $\{B_{n_q}\}_{q \to \infty}$ will satisfy $\texttt{Count}[\pi\text{-linear}]$; indeed, given any $\delta > 0$ and given any $\pi$-linear hypergraph $F$ with $v$ vertices and $e$ edges, let $q_0 = \max\{\frac{2e}{\delta}, v\}$. For all $q \geq q_0$, we have that $|p_q - p| \leq \frac{1}{q} \leq \frac{\delta}{2e}$. In addition, since $v \leq q_0 \leq q$, we have that the number of labeled copies of $F$ in $B_{n_q}$ differs from $p_q^e n_q^v$ by at most $\frac{\delta}{2e} n_q^v$. Using that $|p_q^e - p^e| \leq e|p_q - p|$,[2] we have $|p_q^e - p^e| \leq \frac{\delta}{2}$. Therefore, the number of labeled copies of $F$ differs from $p^e n_q^v$ by at most $(\frac{\delta}{2} + \frac{\delta}{2e})n_q^v$. Thus the number of labeled copies of $F$ differs from $p^e n_q^v$ by at most $\delta n_q^v$, which implies that $\texttt{Count}[\pi\text{-linear}]$ holds for the sequence $\{B_{n_q}\}_{q \to \infty}$. By Lemma 30, there exists a constant $C > 0$ depending only on $p$, $k$, $\pi'$, and $t$ such that the hypergraph $B$ from Lemma 30 has $|e(S_1', \ldots, S_t') - p|S_1'| \ldots |S_t'|| > C\binom{n}{k}$. This implies that the sequence $\{B_{n_q}\}_{q \to \infty}$ fails $\texttt{Expand}[\pi']$.

Note that the sequence can be extended to have a hypergraph on $n$ vertices for every $n$. If there is a gap between $N_0(1/q)$ and $n_{q-1} + 1$, Lemma 30 can be applied many times for

[2] For $0 \leq a < b \leq 1$, $b^e - a^e = (b-a)(b^{e-1} + ab^{e-2} + a^2 b^{e-3} + \cdots + a^{e-2}b + a^{e-1}) \leq (b-a)(1 + \cdots + 1) = (b-a)e$.

$\epsilon = 1/(q-1)$ to fill in the gap. Since $n_{q-1}$ was chosen bigger than $N_0(1/(q-1))$, Lemma 30 guarantees a hypergraph for every $n$ bigger than $n_{q-1}$, in particular the integers between $n_{q-1}$ and $N_0(1/q)$. □

**Lemma 32.** *For all $0 < p < 1$ and all $\pi$, `Expand`[$\pi$] $\not\Rightarrow$ `CliqueDisc`[2].*

*Proof.* Use a diagonalization argument similar to Lemmas 30 and 31 based on $A_2(n,p)$. By Lemma 24, with high probability $A_2(n,p)$ satisfies `Expand`[$\pi$] and by Lemma 18 fails `CliqueDisc`[2]. □

**Lemma 33.** *For $p = \frac{1}{2}$ and all $\pi$, `Expand`[$\pi$] $\not\Rightarrow$ `Deviation`[2].*

*Proof.* Use a diagonalization argument similar to Lemmas 30 and 31 based on $D(n,1/2)$. By Lemma 26, with high probability $D(n,1/2)$ satisfies `Expand`[$\pi$] and by Lemma 22 fails `Deviation`[2]. □

**Lemma 34.** *For $0 < p < 1$ and $1 \leq \ell \leq k-2$, `CliqueDisc`[$\ell$] $\not\Rightarrow$ `CliqueDisc`[$\ell+1$].*

*Proof.* This was proved by Chung [7] for $p = \frac{1}{2}$ using a construction similar to $A_{\ell+1}(n,1/2)$ except the random hypergraph is replaced by the Paley hypergraph. We expand the proof to all $0 < p < 1$ using a diagonalization argument similar to Lemmas 30 and 31 based on $A_{\ell+1}(n,p)$. By Lemma 27, with high probability $A_{\ell+1}(n,p)$ satisfies `CliqueDisc`[$\ell$] and by Lemma 18 fails `CliqueDisc`[$\ell+1$]. □

**Lemma 35.** *For $p = \frac{1}{2}$, `CliqueDisc`[$k-2$] $\not\Rightarrow$ `Deviation`[2].*

*Proof.* Let $\vec{\pi} = (k-1, 1)$. Use a diagonalization argument similar to Lemmas 30 and 31 based on $B_{\vec{\pi}}(n,1/2)$. By Lemma 28, with high probability $B_{\vec{\pi}}(n,1/2)$ satisfies `CliqueDisc`[$k-2$] and by Lemma 21 fails `Deviation`[2]. □

**Lemma 36.** *For $0 < p < 1$, $2 \leq \ell \leq k-1$, and $\pi = k_1 + \cdots + k_t$ with $k_i > \ell$ for some $i$, we have `CliqueDisc`[$\ell$] $\not\Rightarrow$ `Expand`[$\pi$].*

*Proof.* Let $\vec{\pi}' = (\ell+1, 1, \ldots, 1)$. Use a diagonalization argument similar to Lemmas 30 and 31 based on $B_{\vec{\pi}'}(n,p)$. By Lemma 28, with high probability $B_{\vec{\pi}'}(n,p)$ satisfies `CliqueDisc`[$\ell$] and by Lemma 20 (for $B_{\vec{\pi}'}(n,p)$) fails `Expand`[$\pi'$]. Since $\pi'$ is a refinement of $\pi$, this implies that with high probability $B_{\vec{\pi}'}(n,p)$ fails `Expand`[$\pi$]. □

**Lemma 37.** *(Chung [7]) For $p = \frac{1}{2}$ and $2 \leq \ell \leq k-1$, `Deviation`[$\ell$] $\not\Rightarrow$ `CliqueDisc`[$\ell$].*

*Proof.* This was originally proved by Chung [7] using a construction similar to $A_\ell(n,1/2)$ except the random hypergraph was replaced by the Paley hypergraph. Lemma 29 shows that with high probability $A_\ell(n,1/2)$ satisfies `Deviation`[$\ell$] and Lemma 18 shows that $A_\ell(n,1/2)$ fails `CliqueDisc`[$\ell$], so a diagonalization argument similar to Lemmas 30 and 31 shows that $A_\ell(n,1/2)$ provides an alternate construction proving that `Deviation`[$\ell$] $\not\Rightarrow$ `CliqueDisc`[$\ell$]. □

**Lemma 38.** *For $p = \frac{1}{2}$ and $2 \leq \ell \leq k-1$, `Deviation`[$\ell$] $\not\Rightarrow$ `Deviation`[$\ell+1$].*

*Proof.* Use a diagonalization argument similar to Lemmas 30 and 31 based on $A_\ell(n,1/2)$. By Lemma 29, with high probability $A_\ell(n,1/2)$ satisfies `Deviation`[$\ell$] and by Lemma 19 fails `Deviation`[$\ell+1$]. □

# 5 $\texttt{CliqueDisc}[\ell, s]$

In this section, we prove Theorem 2. Initially, in [7], Chung claimed that $\texttt{CliqueDisc}[\ell-1] \Leftrightarrow \texttt{Deviation}[\ell]$ for all $\ell$. This fact is true for $\ell = k$ since both are equivalent to $\texttt{Count}[\text{All}]$, but the claimed proof that $\texttt{CliqueDisc}[\ell-1] \Rightarrow \texttt{Deviation}[\ell]$ for $\ell < k$ was found to contain an error. In [4], Chung discussed the error, proposed the property $\texttt{CliqueDisc}[\ell, s]$, and claimed that $\texttt{Deviation}[s] \Leftrightarrow \texttt{CliqueDisc}[k-1, s]$. As our results (and Lemma 39 below) will show, $\texttt{Deviation}[s] \not\Rightarrow \texttt{CliqueDisc}[k-1, s]$ and so there is an error in [4] (the error is in the second to last equality in the equation at the end of Section 3 in [4]). In fact, the following counterexample was essentially discovered by Chung but our use of the random graph instead of the Paley graph makes the construction simpler to analyze. We note here that Chung's definition of $\texttt{CliqueDisc}[\ell, s]$ considered spanning hypergraphs while we consider not necessarily spanning hypergraphs. The counterexample below works with either definition.

**Lemma 39.** *For $k = 3$, $\texttt{Deviation}[2] \not\Rightarrow \texttt{CliqueDisc}_{1/2}[2, 2]$*

*Proof Sketch.* Consider $A_2(n, 1/2)$. By Lemma 29, with high probability $A_2(n, 1/2)$ satisfies $\texttt{Deviation}[2]$. On the other hand, with high probability $A_2(n, 1/2)$ will fail $\texttt{CliqueDisc}[2, 2]$ as follows. Let $G$ be the (spanning) graph consisting of the edges colored one in the definition of $A_2(n, 1/2)$, so that a triple $T \in \binom{V(A_2(n,1/2))}{3}$ is a hyperedge if and only if $|E(G[T])|$ is even. The probability that $G[T]$ has no edges is $\frac{1}{8}$, has one edge is $\frac{3}{8}$, has two edges is $\frac{3}{8}$, and has all three edges is $\frac{1}{8}$. Thus w.h.p. there are a total of $(1 + o(1))\frac{1}{2}\binom{n}{3}$ triples which induce at least two edges of $G$. But of these, only the ones with exactly two edges are hyperedges, so there are $(1 + o(1))\frac{3}{8}\binom{n}{3}$ hyperedges of $A_2(n, 1/2)$ inducing at least two edges of $G$. But $\frac{3}{8}$ is not one-half of $\frac{1}{2}$, implying that $\texttt{CliqueDisc}[2, 2]$ does not hold for $A_2(n, 1/2)$. $\square$

We now turn to proving Theorem 2, which states that the properties $\texttt{CliqueDisc}[\ell, s]$ are equivalent for fixed $k$ and $\ell$ as $s$ ranges between 1 and $\binom{k}{\ell}$. The proof occurs in two stages; first we prove that $\texttt{CliqueDisc}[\ell, s+1] \Rightarrow \texttt{CliqueDisc}[\ell, s]$ and secondly prove that $\texttt{CliqueDisc}[\ell, 1] \Rightarrow \texttt{CliqueDisc}[\ell, \binom{k}{\ell}]$. The former proof is the difficult one, and the main tool used in the proof is inclusion/exclusion.

**Theorem 40.** *(Inclusion/Exclusion) Let $U$ be a finite set and let $f, g : 2^U \to \mathbb{N}$. If for all $A \subseteq U$,*

$$g(A) = \sum_{B : B \subseteq A} f(B),$$

*then for all $A \subseteq U$,*

$$f(A) = \sum_{B : B \subseteq A} (-1)^{|A|-|B|} g(B).$$

**Definition.** *Let $G$ be an $\ell$-uniform hypergraph, let $\mathcal{P} = (P_1, \dots, P_k)$ be an ordered partition of $V(G)$ into $k$ parts, and let $\mathcal{R} \subseteq \binom{[k]}{\ell}$. Define an $\ell$-uniform hypergraph $G_{\mathcal{P},\mathcal{R}}$ as follows. $V(G_{\mathcal{P},\mathcal{R}}) = V(G)$ and*

$$E(G_{\mathcal{P},\mathcal{R}}) = \{X \in E(G) : \{i : X \cap P_i \neq \emptyset\} \in \mathcal{R}\}.$$

Conceptually, $G_{\mathcal{P},\mathcal{R}}$ is the subhypergraph of $G$ consisting of those edges with at most one vertex in each part of the $k$-partition $\mathcal{P}$ where in addition the intersection pattern of the edge appears in $\mathcal{R}$. Our proof that $\texttt{CliqueDisc}[\ell, s+1] \Rightarrow \texttt{CliqueDisc}[\ell, s]$ works as follows: given some $\ell$-uniform hypergraph $G$, we modify $G$ so that the $k$-sets inducing at least $s$ edges of $G$ transition to $k$-sets inducing at least $s+1$ edges in the modification of $G$. The complexity in the proof is that the modification of $G$ must be carefully chosen so that there is a strong relationship between the $k$-sets inducing $s+1$ edges in the modification and the $k$-sets inducing at least $s$ edges of $G$. This modification uses $G_{\mathcal{P},\mathcal{R}}$ as follows: pick some $I \in \binom{[k]}{\ell}$ with $I \notin \mathcal{R}$ and define $F$ to be the $\ell$-uniform hypergraph $G_{\mathcal{P},\mathcal{R}}$ plus the complete $\ell$-partite, $\ell$-uniform hypergraph with edges whose intersection pattern on $\mathcal{P}$ is given by $I$.

Now consider applying $\texttt{CliqueDisc}[\ell, s+1]$ to $F$, which tells us about the $k$-sets inducing at least $s+1$ edges of $F$. The $k$-sets which contain exactly one vertex in each part of $\mathcal{P}$ are well behaved. Indeed, if $|\mathcal{R}| = s$ and $T$ is a $k$-set with exactly one vertex in each part of $\mathcal{P}$, then $T$ will induce at least $s$ edges of $G_{\mathcal{P},\mathcal{R}}$ if and only if $T$ induces exactly $s$ edges of $G_{\mathcal{P},\mathcal{R}}$ since there are only $s$ intersection patterns in $\mathcal{R}$. In this case, $T$ will induce exactly $s+1$ edges of $F$ since $F$ added the complete $\ell$-partite hypergraph in intersection pattern $I$ and $I \notin \mathcal{R}$. Applying $\texttt{CliqueDisc}[\ell, s+1]$ to $F$ also tells us about $k$-sets which have more than one vertex in some part of $\mathcal{P}$, but an inclusion/exclusion argument is used to ignore these $k$-sets. In summary, we restrict from $G$ to $G_{\mathcal{P},\mathcal{R}}$ so that we have room to add a complete $\ell$-partite graph of intersection pattern $I$ without interfering with the edges of $G$, and use inclusion/exclusion argument to study only the $k$-sets with exactly one vertex in each part since only for these $k$-sets can we transfer knowledge between $G_{\mathcal{P},\mathcal{R}}$ and $F$. The next definition gives a symbol to these $k$-sets with exactly one vertex in each part which induce exactly $s$ edges of $G$.

**Definition.** *Let $G$ be an $\ell$-uniform hypergraph, let $\mathcal{P} = (P_1, \dots, P_k)$ be an ordered partition of $V(G)$ into $k$ parts, and let $s$ and $k$ be integers where $k > \ell$ and $1 \le s \le \binom{k}{\ell}$. Define*

$$W(G, \mathcal{P}, s) = \left\{ T \in \binom{V(G)}{k} : \forall i, T \cap P_i \neq \emptyset \text{ and } e_G(T) = s \right\},$$

*where $e_G(T) = |E(G[T])|$ is the number of edges of $G$ induced by $T$.*

**Lemma 41.** *Let $k$, $\ell$, and $s$ be integers with $2 \le \ell < k$ and $1 \le s < \binom{k}{\ell}$. Let $\mathcal{H} = \{H_n\}_{n \to \infty}$ be a sequence of $k$-uniform hypergraphs with $|V(H_n)| = n$ and assume $\mathcal{H}$ satisfies $\texttt{CliqueDisc}_p[\ell, s+1]$. Let $G$ be an $\ell$-uniform hypergraph with $V(G) \subseteq V(H_n)$, let $\mathcal{P} = (P_1, \dots, P_k)$ be an ordered partition of $V(G)$ into $k$ parts, and let $\mathcal{R} \subseteq \binom{[k]}{\ell}$ where $|\mathcal{R}| = s$. Then*

$$|W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H_n)| = p\,|W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)| + o(n^k).$$

*Proof.* Throughout this proof, the subscripts $n$ and $p$ are dropped for clarity. Since $s < \binom{k}{\ell}$, pick some $I \in \binom{[k]}{\ell}$ where $I \notin \mathcal{R}$. Define

$$F = G_{\mathcal{P},\mathcal{R}} \cup \left\{ X \in \binom{V(G)}{\ell} : \{i : X \cap P_i \neq \emptyset\} = I \right\}.$$

Now define maps $f_{K_n}, f_H, g_{K_n}, g_H : 2^{[k]} \to \mathbb{N}$ as follows:

$$\begin{aligned} f_{K_n}(A) &= \left| \left\{ T \in \binom{V(G)}{k} : e_F(T) \geq s+1, \{i : T \cap P_i \neq \emptyset\} = A \right\} \right|, \\ f_H(A) &= \left| \{ T \in E(H) \quad : e_F(T) \geq s+1, \{i : T \cap P_i \neq \emptyset\} = A \} \right|, \\ g_{K_n}(A) &= \left| \left\{ T \in \binom{V(G)}{k} : e_F(T) \geq s+1, \{i : T \cap P_i \neq \emptyset\} \subseteq A \right\} \right|, \\ g_H(A) &= \left| \{ T \in E(H) \quad : e_F(T) \geq s+1, \{i : T \cap P_i \neq \emptyset\} \subseteq A \} \right|. \end{aligned}$$

Note that in the above definitions, the set $T$ could have more than one vertex in each part of $\mathcal{P}$.

**Claim 1.** *For all $B \subseteq [k]$, $g_H(B) = p\, g_{K_n}(B) + o(n^k)$.*

*Proof.* This will follow from applying `CliqueDisc`$[\ell, s+1]$ to $F' = F[\cup_{i \in B} P_i]$ as follows. Consider the sets

$$\begin{aligned} \Delta_1 &= \left\{ T \in \binom{V(F')}{k} : e_{F'}(T) \geq s+1 \right\}, \\ \Delta_2 &= \left\{ T \in \binom{V(G)}{k} : e_F(T) \geq s+1, \{i : T \cap P_i \neq \emptyset\} \subseteq B \right\}. \end{aligned}$$

Since $F'$ is $F$ restricted to $\cup_{i \in B} P_i$, $\Delta_1 = \Delta_2$. Next, applying `CliqueDisc`$[\ell, s+1]$ to $F'$ shows that $|\Delta_1 \cap E(H)| = p|\Delta_1| + o(n^k)$. Lastly, by the definitions of $g_H$ and $g_{K_n}$, $g_H(B) = |\Delta_2 \cap E(H)|$ and $g_{K_n}(B) = |\Delta_2|$. Since $\Delta_1 = \Delta_2$, the proof is complete. ☐

**Claim 2.**

$$g_{K_n}(A) = \sum_{B : B \subseteq A} f_{K_n}(B) \qquad g_H(A) = \sum_{B : B \subseteq A} f_H(B)$$

*Proof.* Let $T \in \binom{V(G)}{k}$ with $e_F(T) \geq s+1$ and $\{i : T \cap P_i \neq \emptyset\} \subseteq A$. Define $B = \{i : T \cap P_i \neq \emptyset\}$ so that $B \subseteq A$. Now $T$ will be counted once by $g_{K_n}(A)$ and once by $f_{K_n}(B)$ but will not be counted by any $f_{K_n}(B')$ with $B' \neq B$. A similar argument shows that if $T \in E(H)$, $T$ will be counted once by $g_H(A)$ and once by $f_H(B)$. ☐

**Claim 3.**

$$f_{K_n}(A) = \sum_{B : B \subseteq A} (-1)^{|A|-|B|} g_{K_n}(B) \qquad f_H(A) = \sum_{B : B \subseteq A} (-1)^{|A|-|B|} g_H(B)$$

*Proof.* Apply Claim 2 and Inclusion/Exclusion (Theorem 40) to $f_{K_n}, g_{K_n}$ and $f_H, g_H$. $\square$

**Claim 4.** *For all $A \subseteq [k]$, $f_H(A) = p\, f_{K_n}(A) + o(n^k)$.*

*Proof.* Combine Claims 1 and 3 to obtain

$$f_H(A) = \sum_{B:B\subseteq A} (-1)^{|B|-|A|} g_H(B) = p \sum_{B:B\subseteq A} (-1)^{|B|-|A|} g_{K_n}(B) + o(n^k) = p\, f_{K_n}(A) + o(n^k).$$

$\square$

Claim 4 for $A = [k]$ implies that among the $k$-sets $T$ with exactly one vertex in each part and inducing at least $s+1$ edges of $F$, a $p$-fraction of them are hyperedges of $H$. The remainder of the proof translates this knowledge back to $k$-sets inducing at least $s$ edges of $G_{\mathcal{P},\mathcal{R}}$, using that $F$ was built from $G_{\mathcal{P},\mathcal{R}}$ by adding the complete $\ell$-partite, $\ell$-uniform hypergraph with intersection pattern $I$.

**Claim 5.**

$$f_{K_n}([k]) = |W(F, \mathcal{P}, s+1)| \qquad \textit{and} \qquad f_H([k]) = |W(F, \mathcal{P}, s+1) \cap E(H)|$$

*Proof.* First, we show that $f_{K_n}([k]) \le |W(F, \mathcal{P}, s+1)|$. Let $T \in \binom{V(G)}{k}$ with $e_F(T) \ge s+1$ and $T \cap P_i \neq \emptyset$ for all $i \in [k]$, so that $T$ is counted by $f_{K_n}([k])$. Now consider the set $\mathcal{R}' = \{J : \exists X \in E(F[T]), \{i : X \cap P_i \neq \emptyset\} = J\}$. Since $T$ has exactly one vertex in each $P_i$, $|\mathcal{R}'| \ge s+1$ and also every $J \in \mathcal{R}'$ has size $\ell$. By the definition of $F$, $\mathcal{R}' \subseteq \mathcal{R} \cup \{I\}$ which when combined with $|\mathcal{R}| = s$ shows that $\mathcal{R}' = \mathcal{R} \cup \{I\}$. In particular, $|\mathcal{R}'| = s+1$ so $e_F(T) = s+1$, which implies that $T \in W(F, \mathcal{P}, s+1)$ and thus $f_{K_n}([k]) \le |W(F, \mathcal{P}, s+1)|$.

Next, we prove that $f_{K_n}([k]) \ge |W(F, \mathcal{P}, s+1)|$. Let $T \in W(F, \mathcal{P}, s+1)$. Then $e_F(T) = s+1$ and $|T \cap P_i| = 1$ for all $i \in [k]$ so $\{i : T \cap P_i \neq \emptyset\} = [k]$. Thus $T$ is counted by $f_{K_n}([k])$ so $f_{K_n}([k]) = |W(F, \mathcal{P}, s+1)|$.

A similar argument shows that $f_H([k]) = |W(F, \mathcal{P}, s+1) \cap E(H)|$, since the previous two paragraphs can be applied to sets $T$ which are edges of $H$. $\square$

**Claim 6.**

$$|W(F, \mathcal{P}, s+1)| = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)| \; \textit{and} \; |W(F, \mathcal{P}, s+1) \cap E(H)| = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)|\,.$$

*Proof.* Let $T \in \binom{V(G)}{k}$ with $|T \cap P_i| = 1$ for all $i$. We would like to show that $e_F(T) = s+1$ if and only if $e_{G_{\mathcal{P},\mathcal{R}}}(T) = s$. As in the previous proof, define $\mathcal{R}' = \{J : \exists X \in E(F[T]), \{i : X \cap P_i \neq \emptyset\} = J\}$. Now $e_F(T) = s+1$ if and only if $\mathcal{R}' = \mathcal{R} \cup \{I\}$. Since $F$ is defined as the edges of $G_{\mathcal{P},\mathcal{R}}$ together with all $\ell$-sets with intersection pattern $I$, $\mathcal{R}' = \mathcal{R} \cup \{I\}$ if and only if $e_{G_{\mathcal{P},\mathcal{R}}}(T) = s$. This implies that $|W(F, \mathcal{P}, s+1)| = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)|$. A similar argument where $T$ is restricted to an edge of $H$ shows that $|W(F, \mathcal{P}, s+1) \cap E(H)| = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)|$. $\square$

We can now complete the proof of Lemma 41. Combining Claims 5 and 6 shows that

$$f_{K_n}([k]) = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)|$$
$$f_H([k]) = |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)|\,.$$

Claim 4 then shows that

$$|W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)| = p\,|W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)| + o(n^k),$$

completing the proof of the lemma. □

**Lemma 42.** *Let $k$, $\ell$, and $s$ be integers with $2 \le \ell < k$ and $1 \le s < \binom{k}{\ell}$. Then $\texttt{CliqueDisc}_p[\ell, s+1] \Rightarrow \texttt{CliqueDisc}_p[\ell, s]$.*

*Proof.* Let $\mathcal{H} = \{H_n\}_{n\to\infty}$ be a sequence of $k$-uniform hypergraphs with $|V(H_n)| = n$ and assume $\mathcal{H}$ satisfies $\texttt{CliqueDisc}[\ell, s+1]$. Let $G$ be an $\ell$-uniform hypergraph with $V(G) \subseteq V(H)$ and let $n' = |V(G)|$. Then

$$\left|\left\{T \in \binom{V(G)}{k} : e_G(T) = s\right\}\right| = \frac{1}{k!k^{n'-k}} \sum_{\mathcal{P},\mathcal{R}} |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)|\,. \tag{16}$$

Indeed, let $T = \{t_1, \ldots, t_k\} \subseteq V(H)$ with $e_G(T) = s$. The number of times $T$ is counted in the sum is $k!k^{n'-k}$ since $T$ will be counted on the right hand side of (16) only if $T \cap P_i \neq \emptyset$ for each $i$. There are $k!$ ways of assigning the vertices of $T$ to the parts of $\mathcal{P}$, and $k^{n'-k}$ ways of assigning the other $n' - k$ vertices of $G$ to parts of $\mathcal{P}$. Once such a partition $\mathcal{P}$ is chosen, there is a unique choice for $\mathcal{R}$ since $T$ induces exactly $s$ edges and $T$ has exactly one vertex in each $P_i$. A similar counting argument shows that

$$|\{T \in E(H) : e_G(T) = s\}| = \frac{1}{k!k^{n'-k}} \sum_{\mathcal{P},\mathcal{R}} |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)|\,.$$

Note that the number of terms in the sum is $k!S(n', k)\binom{\binom{k}{\ell}}{s} = \Theta(k^{n'})$, so applying Lemma 41 implies that

$$\begin{aligned}
|\{T \in E(H) : e_G(T) = s\}| &= \frac{1}{k!k^{n'-k}} \sum_{\mathcal{P},\mathcal{R}} |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s) \cap E(H)| \\
&= \frac{p}{k!k^{n'-k}} \sum_{\mathcal{P},\mathcal{R}} |W(G_{\mathcal{P},\mathcal{R}}, \mathcal{P}, s)| + o\left(\frac{n^k k! S(n', k)\binom{\binom{k}{\ell}}{s}}{k!k^{n'-k}}\right) \\
&= p\left|\left\{T \in \binom{V(G)}{k} : e_G(T) = s\right\}\right| + o(n^k). \qquad (17)
\end{aligned}$$

Applying $\texttt{CliqueDisc}[\ell, s+1]$ to $G$ shows that

$$|\{T \in E(H) : e_G(T) \ge s+1\}| = p\left|\left\{T \in \binom{V(G)}{k} : e_G(T) \ge s+1\right\}\right| + o(n^k). \tag{18}$$

Combining (17) and (18) shows that

$$|\{T \in E(H) : e_G(T) \geq s\}| = p \left| \left\{ T \in \binom{V(G)}{k} : e_G(T) \geq s \right\} \right| + o(n^k),$$

implying that `CliqueDisc`$[\ell, s]$ holds. □

**Lemma 43.** *`CliqueDisc`$_p[\ell, 1] \Rightarrow$ `CliqueDisc`$_p[\ell, \binom{k}{\ell}]$*

*Proof.* Let $G$ be an $\ell$-uniform hypergraph with $V(G) \subseteq V(H)$ and denote by $\bar{G}$ the hypergraph with $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \binom{V(G)}{\ell} - E(G)$. Now define

$$A := \left\{ T \in \binom{V(G)}{k} : e_{\bar{G}}(T) \geq 1 \right\}.$$

By the definition of $\bar{G}$,

$$A = \left\{ T \in \binom{V(G)}{k} : e_G(T) < \binom{k}{\ell} \right\}.$$

Thus if we define

$$B := \left\{ T \in \binom{V(G)}{k} : e_G(T) = \binom{k}{\ell} \right\},$$

then $\binom{V(G)}{k} - A = B$. Intersecting this equation with $E(H)$, we also obtain that $E(H) - (A \cap E(H)) = B \cap E(H)$. Apply `CliqueDisc`$[\ell, 1]$ to $\bar{G}$ to imply that $|A \cap E(H)| = p|A| + o(n^k)$ which implies that

$$|B \cap E(H)| = \Big(|E(H)| - |A \cap E(H)|\Big) = |E(H)| - p|A| + o(n^k).$$

Since `CliqueDisc`$[\ell, 1]$ applied to $K_n^{(\ell)}$ shows that $|E(H)| = p\binom{n}{k} + o(n^k)$, we have

$$|B \cap E(H)| = |E(H)| - p|A| + o(n^k) = p\binom{n}{k} - p|A| + o(n^k) = p|B| + o(n^k),$$

which implies that `CliqueDisc`$[\ell, \binom{k}{\ell}]$ holds for $\mathcal{H}$. □

Combining Lemmas 42 and 43 completes the proof of Theorem 2. Note that the proof of Lemma 43 extends to show that `CliqueDisc`$[\ell, s] \Leftrightarrow$ `CliqueDisc`$[\ell, \binom{k}{\ell} - s + 1]$ by defining $A$ as the $k$-sets inducing at least $s$ edges of $\bar{G}$. Also note that the proof of Lemma 43 works even when the definition of `CliqueDisc`$[\ell, s]$ is restricted to spanning graphs as in Chung's [4] original definition, so Lemma 43 provides an alternate contradiction to [4].

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# References


[1] N. Alon, A. Coja-Oghlan, H. Hàn, M. Kang, V. Rödl, and M. Schacht. Quasi-randomness and algorithmic regularity for graphs with general degree distributions. *SIAM J. Comput.*, 39(6):2336–2362, 2010.

[2] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. With an appendix on the life and work of Paul Erdős.

[3] T. Austin and T. Tao. Testability and repair of hereditary hypergraph properties. *Random Structures Algorithms*, 36(4):373–463, 2010.

[4] F. Chung. Quasi-random hypergraphs revisited. *Random Structures Algorithms*, 40(1):39–48, 2012.

[5] F. Chung and R. Graham. Sparse quasi-random graphs. *Combinatorica*, 22(2):217–244, 2002. Special issue: Paul Erdős and his mathematics.

[6] F. Chung and R. Graham. Quasi-random graphs with given degree sequences. *Random Structures Algorithms*, 32(1):1–19, 2008.

[7] F. R. K. Chung. Quasi-random classes of hypergraphs. *Random Structures Algorithms*, 1(4):363–382, 1990.

[8] F. R. K. Chung and R. L. Graham. Quasi-random hypergraphs. *Random Structures Algorithms*, 1(1):105–124, 1990.

[9] F. R. K. Chung and R. L. Graham. Quasi-random set systems. *J. Amer. Math. Soc.*, 4(1):151–196, 1991.

[10] F. R. K. Chung and R. L. Graham. Cohomological aspects of hypergraphs. *Trans. Amer. Math. Soc.*, 334(1):365–388, 1992.

[11] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[12] D. Conlon, H. Hàn, Y. Person, and M. Schacht. Weak quasi-randomness for uniform hypergraphs. *Random Structures Algorithms*, 40(1):1–38, 2012.

[13] P. Erdős and A. Hajnal. On Ramsey like theorems. Problems and results. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 123–140. Inst. Math. Appl., Southend, 1972.

[14] W. T. Gowers. Quasirandomness, counting and regularity for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):143–184, 2006.

[15] W. T. Gowers. Hypergraph regularity and the multidimensional Szemerédi theorem. *Ann. of Math. (2)*, 166(3):897–946, 2007.

[16] W. T. Gowers. Quasirandom groups. *Combin. Probab. Comput.*, 17(3):363–387, 2008.

[17] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[18] S. Janson. Quasi-random graphs and graph limits. *European J. Combin.*, 32(7):1054–1083, 2011.

[19] P. Keevash. A hypergraph regularity method for generalized Turán problems. *Random Structures Algorithms*, 34(1):123–164, 2009.

[20] Y. Kohayakawa, B. Nagle, V. Rödl, and M. Schacht. Weak hypergraph regularity and linear hypergraphs. *J. Combin. Theory Ser. B*, 100(2):151–160, 2010.

[21] Y. Kohayakawa and V. Rödl. Regular pairs in sparse random graphs. I. *Random Structures Algorithms*, 22(4):359–434, 2003.

[22] Y. Kohayakawa, V. Rödl, and P. Sissokho. Embedding graphs with bounded degree in sparse pseudorandom graphs. *Israel J. Math.*, 139:93–137, 2004.

[23] Y. Kohayakawa, V. Rödl, and J. Skokan. Hypergraphs, quasi-randomness, and conditions for regularity. *J. Combin. Theory Ser. A*, 97(2):307–352, 2002.

[24] M. Krivelevich and B. Sudakov. Pseudo-random graphs. In *More sets, graphs and numbers*, volume 15 of *Bolyai Soc. Math. Stud.*, pages 199–262. Springer, Berlin, 2006.

[25] J. Lenz and D. Mubayi. Eigenvalues and linear quasirandom hypergraphs. submitted. `http://arxiv.org/abs/1208.4863`.

[26] J. Lenz and D. Mubayi. Eigenvalues of non-regular linear quasirandom hypergraphs. online at `http://arxiv.org/abs/1309.3584`.

[27] L. Lovász and V. T. Sós. Generalized quasirandom graphs. *J. Combin. Theory Ser. B*, 98(1):146–163, 2008.

[28] J. S. Myers. Graphs without large complete minors are quasi-random. *Combin. Probab. Comput.*, 11(6):571–585, 2002.

[29] J. S. Myers and A. Thomason. The extremal function for noncomplete minors. *Combinatorica*, 25(6):725–753, 2005.

[30] B. Nagle, V. Rödl, and M. Schacht. The counting lemma for regular $k$-uniform hypergraphs. *Random Structures Algorithms*, 28(2):113–179, 2006.

[31] V. Nikiforov. Eigenvalues and extremal degrees of graphs. *Linear Algebra Appl.*, 419(2-3):735–738, 2006.

[32] V. Rödl and J. Skokan. Regularity lemma for $k$-uniform hypergraphs. *Random Structures Algorithms*, 25(1):1–42, 2004.

[33] V. Rödl and J. Skokan. Applications of the regularity lemma for uniform hypergraphs. *Random Structures Algorithms*, 28(2):180–194, 2006.

[34] A. Shapira. Quasi-randomness and the distribution of copies of a fixed graph. *Combinatorica*, 28(6):735–745, 2008.

[35] A. Shapira and R. Yuster. The effect of induced subgraphs on quasi-randomness. *Random Structures Algorithms*, 36(1):90–109, 2010.

[36] M. Simonovits and V. T. Sós. Szemerédi's partition and quasirandomness. *Random Structures Algorithms*, 2(1):1–10, 1991.

[37] M. Simonovits and V. T. Sós. Hereditarily extended properties, quasi-random graphs and not necessarily induced subgraphs. *Combinatorica*, 17(4):577–596, 1997.

[38] M. Simonovits and V. T. Sós. Hereditary extended properties, quasi-random graphs and induced subgraphs. *Combin. Probab. Comput.*, 12(3):319–344, 2003. Combinatorics, probability and computing (Oberwolfach, 2001).

[39] J. Skokan and L. Thoma. Bipartite subgraphs and quasi-randomness. *Graphs Combin.*, 20(2):255–262, 2004.

[40] E. Szemerédi. On sets of integers containing no $k$ elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975.

[41] T. Tao. A variant of the hypergraph removal lemma. *J. Combin. Theory Ser. A*, 113(7):1257–1280, 2006.

[42] A. Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987.

[43] A. Thomason. Random graphs, strongly regular graphs and pseudorandom graphs. In *Surveys in combinatorics 1987 (New Cross, 1987)*, volume 123 of *London Math. Soc. Lecture Note Ser.*, pages 173–195. Cambridge Univ. Press, Cambridge, 1987.

[44] R. Yuster. Quasi-randomness is determined by the distribution of copies of a fixed graph in equicardinal large sets. *Combinatorica*, 30(2):239–246, 2010.

# A Deviation

This section contains the proofs of Lemmas 10 and 11. The ideas behind these two lemmas are the cornerstone of Chung's [7] proofs that Deviation[$\ell$] $\Rightarrow$ Deviation[$\ell-1$] and Deviation[$\ell$] $\Rightarrow$ CliqueDisc[$\ell - 1$]. Since our proof that Deviation[2] $\Rightarrow$ Expand[$\pi$] (which appears in Section 2.3) is based on the these same ideas, we factored out these two lemmas from Chung's [7] proofs. Chung doesn't explicitly state these lemmas, so for completeness we give proofs in this section.

*Proof of Lemma 10.* Let $P, Q \subseteq V(H)^k$ and assume that $Q$ is complete in coordinate $i$ so that there exists a $Q' \subseteq V(H)^{k-1}$ with $Q = \{(x_1, \dots, x_k) : (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \in Q', x_i \in V(H)\}$. By definition,

$$\begin{aligned}\mathrm{dev}_{\ell,P}(H) &= \sum_{\substack{x_1,\dots,x_{k-\ell},y_{1,0},y_{1,1},\dots,y_{\ell,0},y_{\ell,1}\in V(H)\\ \mathcal{O}[x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1}]\subseteq P}} \eta_H(x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1})\\ &= \sum_{\substack{\vec{x}\in V(H)^{k-\ell}\\ \vec{y}\in V(H)^{2\ell}\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}).\end{aligned}$$

where for notational convenience we write $\mathcal{O}[\vec{x};\vec{y}]$ for $\mathcal{O}[x_1;\dots;x_{k-\ell};y_{1,0},y_{1,1};\dots;y_{\ell,0},y_{\ell,1}]$ and similarly for $\eta$. Let $j = i - k + \ell$ and rearrange the sum to obtain

$$\mathrm{dev}_{\ell,P}(H) = \sum_{\vec{x}\in V(H)^{k-\ell}} \sum_{\substack{y_{1,0},y_{1,1},\dots,y_{j-1,0},y_{j-1,1}\in V(H)\\ y_{j+1,0},y_{j+1,1},\dots,y_{\ell,0},y_{\ell,1}\in V(H)}} \sum_{\substack{y_{j,0},y_{j,1}\in V(H)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}). \tag{19}$$

Now fix $\vec{x}$ and $y_{1,0}, y_{1,1}, \dots, y_{j-1,0}, y_{j-1,1}, y_{j+1,0}, y_{j+1,1}, \dots, y_{\ell,0}, y_{\ell,1}$ and consider the sum over $y_{j,0}, y_{j,1}$ in the above expression. Call a vertex $z$ *even* if

$$|\tilde{\mathcal{O}}[\vec{x}; y_{1,0}, y_{1,1}; \dots; y_{j-1,0}, y_{j-1,1}; z; y_{j+1,0}, y_{j+1,1}; \dots; y_{\ell,0}, y_{\ell,1}] \cap E(H)|$$

is even and *odd* otherwise. In other words, $z$ is even if the squashed octahedron formed using $z$ in the $i$th part is even. Define $N = \{z : \mathcal{O}[\vec{x}; y_{1,0}, y_{1,1}; \dots; y_{j-1,0}, y_{j-1,1}; z; y_{j+1,0}, y_{j+1,1}; \dots; y_{\ell,0}, y_{\ell,1}] \subseteq P\}$. Now expand the sum over $y_{j,0}$ and $y_{j,1}$ by cases depending on if the vertices are even or odd.

$$\begin{aligned}\sum_{\substack{y_{j,0},y_{j,1}\in V(H)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}) &= \sum_{y_{j,0},y_{j,1}\in N} \eta(\vec{x};\vec{y})\\ &= \sum_{\substack{y_{j,0}\text{ even in }N\\ y_{j,1}\text{ even in }N}} \eta(\vec{x};\vec{y}) + \sum_{\substack{y_{j,0}\text{ even in }N\\ y_{j,1}\text{ odd in }N}} \eta(\vec{x};\vec{y})\\ &\quad+ \sum_{\substack{y_{j,0}\text{ odd in }N\\ y_{j,1}\text{ even in }N}} \eta(\vec{x};\vec{y}) + \sum_{\substack{y_{j,0}\text{ odd in }N\\ y_{j,1}\text{ odd in }N}} \eta(\vec{x};\vec{y}).\end{aligned} \tag{20}$$

If $y_{j,0}$ and $y_{j,1}$ are both even or both odd then $\eta(\vec{x};\vec{y}) = +1$ and if exactly one of $y_{j,0}, y_{j,1}$ is even then $\eta(\vec{x};\vec{y}) = -1$. Let $\Gamma_0$ be the number of even vertices in $N$ and $\Gamma_1$ the number of odd vertices in $N$. Then continuing the above equation we have

$$\sum_{\substack{y_{j,0},y_{j,1}\in V(H)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}) = \Gamma_0^2 - \Gamma_0\Gamma_1 - \Gamma_1\Gamma_0 + \Gamma_1^2 = (\Gamma_0 - \Gamma_1)^2 \geq 0. \tag{21}$$

In particular, this implies that the above sum is always non-negative. Now return to (19). Since the innermost sum is always non-negative for any choice of $\vec{x}$ and $y_{1,0}, y_{1,1}, \ldots, y_{j-1,0}, y_{j-1,1}, y_{j+1,0}, y_{j+1,1}, \ldots, y_{\ell,0}, y_{\ell,1}$, the middle sum in (19) can be restricted to $Q'$ and this restriction cannot make the value of the sum go up. More precisely,

$$\begin{aligned}\mathrm{dev}_{\ell,P}(H) &= \sum_{\vec{x}\in V(H)^{k-\ell}} \sum_{\substack{y_{1,0},y_{1,1},\ldots,y_{j-1,0},y_{j-1,1}\in V(H)\\ y_{j+1,0},y_{j+1,1},\ldots,y_{\ell,0},y_{\ell,1}\in V(H)}} \sum_{\substack{y_{j,0},y_{j,1}\in V(H)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}) \\ &\geq \sum_{\vec{x}\in V(H)^{k-\ell}} \sum_{\substack{y_{1,0},y_{1,1},\ldots,y_{j-1,0},y_{j-1,1}\in V(H)\\ y_{j+1,0},y_{j+1,1},\ldots,y_{\ell,0},y_{\ell,1}\in V(H)\\ \mathcal{O}[\vec{x};y_{1,0},y_{1,1};\ldots;y_{j-1,0},y_{j-1,1};y_{j+1,0},y_{j+1,1};\ldots;y_{\ell,0},y_{\ell,1}]\subseteq Q'}} \sum_{\substack{y_{j,0},y_{j,1}\in V(H)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}).\end{aligned}$$

Notice that the octahedron in the middle sum skips the $j$th coordinate of the $y$s which corresponds to the $i$th coordinate of the octahedron. This matches with the fact that $Q' \subseteq V(H)^{k-1}$. By definition of $Q'$, $\mathcal{O}[\vec{x}; y_{1,0}, y_{1,1}; \ldots; y_{j-1,0}, y_{j-1,1}; y_{j+1,0}, y_{j+1,1}; \ldots; y_{\ell,0}, y_{\ell,1}] \subseteq Q'$ if and only if $\mathcal{O}[\vec{x};\vec{y}] \subseteq Q$. Thus the above sum simplifies to

$$\mathrm{dev}_{\ell,P}(H) \geq \sum_{\vec{x}\in V(H)^{k-\ell}} \sum_{\substack{\vec{y}\in V(H)^{2\ell}\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P\cap Q}} \eta(\vec{x};\vec{y}) = \mathrm{dev}_{\ell,P\cap Q}(H).$$

□

Next we prove Lemma 11, which is a consequence of the Cauchy-Schwartz inequality.

**Theorem 44.** *(Cauchy-Schwartz Inequality) If $n$ is a positive integer and $\alpha_i, \beta_i \in \mathbb{R}$ for $1 \leq i \leq n$, then*

$$\left(\sum_{i=1}^{n} \alpha_i\beta_i\right)^2 \leq \left(\sum_{i=1}^{n} \alpha_i^2\right)\left(\sum_{i=1}^{n} \beta_i^2\right).$$

*Proof of Lemma 11.* Let $\mathcal{H}$ be a sequence of hypergraphs where $\mathrm{dev}_{\ell,P}(H_n) = o(n^{k+\ell})$. Using the same notation as the proof of Lemma 10, we have

$$\begin{aligned}\mathrm{dev}_{\ell,P}(H_n) &= \sum_{\vec{x}\in V(H_n)^{k-\ell}} \sum_{\substack{\vec{y}\in V(H_n)^{2\ell}\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}) \\ &= \sum_{\vec{x}\in V(H_n)^{k-\ell}} \sum_{y_{2,0},y_{2,1},\ldots,y_{\ell,0},y_{\ell,1}\in V(H_n)} \sum_{\substack{y_{1,0},y_{1,1}\in V(H_n)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}).\end{aligned}$$

For $j = 1$, the equations (20) and (21) show that

$$\sum_{\substack{y_{1,0},y_{1,1}\in V(H_n)\\ \mathcal{O}[\vec{x};\vec{y}]\subseteq P}} \eta(\vec{x};\vec{y}) = (\Gamma_0 - \Gamma_1)^2,$$

where $\Gamma_0$ is the number of even vertices in $N = \{z : \mathcal{O}[\vec{x}; z; y_{2,0}, y_{2,1}; \ldots ; y_{\ell,0}, y_{\ell,1}] \subseteq P\}$ and $\Gamma_1$ is the number of odd vertices in $N$. But by the definition of an even and odd vertex,

$$\Gamma_0 - \Gamma_1 = \sum_{z\in N} \eta(\vec{x}; z; y_{2,0}, y_{2,1}; \ldots ; y_{\ell,0}, y_{\ell,1}).$$

Thus

$$\operatorname{dev}_{\ell,P}(H_n) = \sum_{\vec{x}\in V(H_n)^{k-\ell}} \; \sum_{y_{2,0},y_{2,1},\ldots,y_{\ell,0},y_{\ell,1}\in V(H_n)} \left( \sum_{z\in N} \eta(\vec{x}; z; y_{2,0}, y_{2,1}; \ldots ; y_{\ell,0}, y_{\ell,1}) \right)^2 .$$

Now apply Cauchy-Schwartz with $\alpha_i = 1$ and $\beta_i = \sum_z \eta(\cdots)$ to obtain

$$\operatorname{dev}_{\ell,P}(H_n) \geq \frac{1}{n^{k+\ell-2}} \left( \sum_{\vec{x}\in V(H_n)^{k-\ell}} \; \sum_{y_{2,0},y_{2,1},\ldots,y_{\ell,0},y_{\ell,1}\in V(H_n)} \; \sum_{z\in N} \eta(\vec{x}; z; y_{2,0}, y_{2,1}; \ldots ; y_{\ell,0}, y_{\ell,1}) \right)^2 .$$

The expression inside the square is $\operatorname{dev}_{\ell-1,P}(H_n)$, since the sum is over $\vec{x}$ and $z$ which sums over $k - \ell + 1$ parts of the squashed octahedron with one vertex and a sum over $\ell - 1$ parts of the squashed octahedron with two vertices. The restriction of $z \in N$ translates to $\mathcal{O}[\vec{x}; z; y_{2,0}, y_{2,1}; \ldots ; y_{\ell,0}, y_{\ell,1}] \subseteq P$, exactly the restriction in $\operatorname{dev}_{\ell-1,P}(H_n)$. Thus

$$\begin{aligned}
\frac{1}{n^{k+\ell-2}} \left(\operatorname{dev}_{\ell-1,P}(H_n)\right)^2 \leq \operatorname{dev}_{\ell,P}(H_n) &= o(n^{k+\ell}) \\
\left(\operatorname{dev}_{\ell-1,P}(H_n)\right)^2 &= o(n^{2k+2\ell-2}) \\
\operatorname{dev}_{\ell-1,P}(H_n) &= o(n^{k+\ell-1}).
\end{aligned}$$

□

For completeness, we give the two proofs of Chung [7] which were the original motivation for Lemmas 10 and 11.

**Lemma 45.** ***(Chung [7])*** *For $2 \leq \ell \leq k$,* `Deviation`*[$\ell$] $\Rightarrow$* `Deviation`*[$\ell - 1$].*

*Proof.* Apply Lemma 10 with $P = V(H)^k$. □

**Lemma 46.** ***(Chung [7])*** *For $2 \leq \ell \leq k$,* `Deviation`*[$\ell$] $\Rightarrow$* `CliqueDisc`$_{1/2}$*[$\ell - 1$].*

*Proof.* Let $G$ be an $(\ell-1)$-uniform hypergraph. For $k-\ell+1 \le i \le k$, define

$$P_i = \Bigg\{(x_1,\dots,x_k) \in V(H)^k : |\{x_1,\dots,x_{i-1},x_{i+1},\dots,x_k\}| = k-1, \\ \binom{\{x_1,\dots,x_{i-1},x_{i+1},\dots,x_k\}}{\ell-1} \subseteq E(G)\Bigg\}.$$

That is, $P_i$ is the collection of tuples where the vertices besides the $i$th coordinate are distinct and form a $(k-1)$-clique in $G$. Note that $P_i$ is complete in coordinate $i$. We claim that $\cap P_i$ is the collection of $k$-tuples of distinct vertices which form a $k$-clique in $G$. Indeed, let $x_1,\dots,x_k$ be distinct vertices forming a $k$-clique of $G$. Then $(x_1,\dots,x_k) \in P_i$ for every $i$ since all $(\ell-1)$-subsets of $\{x_1,\dots,x_k\}$ are edges of $G$. In the other direction, let $(x_1,\dots,x_k) \in \cap P_i$ and let $R$ be any $(\ell-1)$-subset of $\{x_1,\dots,x_k\}$. Since $|R| = \ell-1$ and $i$ ranges from $k-\ell+1$ to $k$, there is some $i$ such that $x_i \notin R$. But now $(x_1,\dots,x_k) \in P_i$ implies that $R \subseteq E(G)$ showing that $G[\{x_1,\dots,x_k\}]$ is a clique. Therefore, Lemma 10 and the fact that $\texttt{Deviation}[\ell]$ holds imply that

$$\mathrm{dev}_{\ell,\cap P_i}(H_n) \le \mathrm{dev}_\ell(H_n) = o(n^{k+\ell}).$$

Now a repeated application of Lemma 11 implies that

$$\mathrm{dev}_{0,\cap P_i}(H_n) = o(n^k).$$

Expanding the definition of $\mathrm{dev}_{0,\cap P_i}(H_n)$, we have

$$\begin{aligned}
\mathrm{dev}_{0,\cap P_i}(H_n) &= \sum_{(x_1,\dots,x_k)\in \cap P_i} \eta(x_1;\dots;x_k) \\
&= k! \sum_{\substack{\{x_1,\dots,x_k\}\subseteq V(H_n) \\ G[\{x_1,\dots,x_k\}] \text{ is a clique}}} \eta(x_1;\dots;x_k) \\
&= k!\Big(|\mathcal{K}_k(G)\cap E(H)| - \big|\mathcal{K}_k(G)\cap E(\bar{H})\big|\Big).
\end{aligned}$$

Thus $\mathrm{dev}_{0,\cap P_i}(H_n) = o(n^k)$ implies that $|\mathcal{K}_k(G)\cap E(H)| = \big|\mathcal{K}_k(G)\cap E(\bar{H})\big| + o(n^k)$ which implies that $\texttt{CliqueDisc}_{1/2}[\ell-1]$ holds for $\mathcal{H}$. □