# Rainbow Turán Problems

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**Abstract**

For a fixed graph $H$, we define the rainbow Turán number $\mathrm{ex}^*(n, H)$ to be the maximum number of edges in a graph on $n$ vertices that has a proper edge-colouring with no rainbow $H$. Recall that the (ordinary) Turán number $\mathrm{ex}(n, H)$ is the maximum number of edges in a graph on $n$ vertices that does not contain a copy of $H$. For any non-bipartite $H$ we show that $\mathrm{ex}^*(n, H) = (1+o(1))\mathrm{ex}(n, H)$, and if $H$ is colour-critical we show that $\mathrm{ex}^*(n, H) = \mathrm{ex}(n, H)$. When $H$ is the complete bipartite graph $K_{s,t}$ with $s \le t$ we show $\mathrm{ex}^*(n, K_{s,t}) = O(n^{2-1/s})$, which matches the known bounds for $\mathrm{ex}(n, K_{s,t})$ up to a constant. We also study the rainbow Turán problem for even cycles, and in particular prove the bound $\mathrm{ex}^*(n, C_6) = O(n^{4/3})$, which is of the correct order of magnitude.

## 1 Introduction

In this paper, we address the following question. For a fixed graph $H$, determine the maximum number of edges in a properly edge-coloured graph on $n$ vertices which does not contain a rainbow $H$, i.e. a copy of $H$ all of whose edges have different colours. This maximum is denoted $\mathrm{ex}^*(n, H)$, and we refer to it as the rainbow Turán number of $H$.

There are two main motivations for our study of rainbow Turán numbers. One is the possibility of applying purely combinatorial methods to certain extremal problems in additive number theory. Call a subset $A$ of an abelian group $G$ a $B_k^*$-set if does not contain disjoint $k$-subsets $B, C$ with the same sum. Given a set $A$ consider the following edge-coloured bipartite graph. The two parts $X, Y$ are both copies of $G$, we join $x \in X$ to $y \in Y$ if $x - y \in A$, and then the edge $xy$ is assigned the colour $x - y$. This is a properly coloured graph, and if $A$ is a $B_k^*$-set then it does not contain a rainbow $C_{2k}$, the cycle of length $2k$.

Another motivation is that it seems to be a natural meeting point of two areas of extremal graph theory. Firstly, there is the classical Turán problem, which has a rich history in combinatorics. This asks for the maximum number of edges in a graph on $n$ vertices that contains no copy of some fixed

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graph $H$. The maximum here is denoted $\mathrm{ex}(n, H)$, and is known as the Turán number. Next there is the literature on extremal problems for edge-colourings (not necessarily proper). An example is the Canonical Ramsey Theorem, proved by Erdős and Rado [12], a special case of which shows that any proper colouring of $K_n$ produces a rainbow $K_m$, provided $n$ is large relative to $m$. Motivated by this and work in [10] and [22], Alon, Jiang, Miller and Pritikin [2] introduced the problem of finding a rainbow copy of a graph $H$ in a colouring of $K_n$ in which each colour appears at most $m$ times at each vertex. The rainbow Turán problem is a natural Turán-type extension of this problem.

We will discuss these motivations in greater detail before the statement of our relevant results, which we divide into subsections according to the nature of the forbidden fixed graph $H$.

### 1.1 Colour-critical graphs

We start this subsection with some more background information on the Turán problem. Its systematic study originated with Turán, who considered forbidding $K_r$, the complete graph on $r$ vertices. The *Turán graph* $T_{r-1}(n)$ is the complete $(r-1)$-partite graph with part sizes as equal as possible; we write $t_{r-1}(n)$ for the number of edges in $T_{r-1}(n)$. Then Turán's Theorem [35] states that $\mathrm{ex}(n, K_r) = t_{r-1}(n)$, and $T_{r-1}(n)$ is the unique extremal $K_r$-free graph. Erdös and Stone [14] showed that the behaviour of the Turán number of a general graph $H$ is determined by its chromatic number. They proved that if $\chi(H) = r$ then $\mathrm{ex}(n, H) = t_{r-1}(n) + o(n^2)$, which gives asymptotics except when $H$ is bipartite.

Clearly the rainbow Turán number for any $H$ satisfies $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$. Examples when equality holds include the cases when $H$ is a star or a triangle, as then any proper edge-colouring of $H$ is rainbow, and so $\mathrm{ex}^*(n, H) = \mathrm{ex}(n, H)$. We can describe a general class of graphs in which equality holds as follows. Say that $H$ is colour-critical if it contains an edge $e$ so that $\chi(H \backslash e) = \chi(H) - 1$, where $\chi(H)$ denotes the chromatic number of $H$. If $H$ is colour-critical and $\chi(H) = r$ then a result of Simonovits [33] shows that $\mathrm{ex}(n, H) = t_{r-1}(n)$ for sufficiently large $n$. Our first result determines $\mathrm{ex}^*(n, H)$ asymptotically for any non-bipartite $H$ and exactly for colour-critical graphs, a class that includes for example all complete graphs and all cycles of odd length.

**Theorem 1.1** *The rainbow Turán number $ex^*(n, H)$ satisfies $ex(n, H) \leq ex^*(n, H) \leq ex(n, H) + o(n^2)$. Furthermore, if $H$ is colour-critical then $ex^*(n, H) = ex(n, H)$ for $n$ sufficiently large.*

### 1.2 Bipartite graphs

For bipartite graphs, even the order of magnitude of Turán numbers is not well understood. In the case of complete bipartite graphs, Kövari, Sós and P. Turán [24] showed $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$, where the implied constant depends only on $s$ and $t$. The best known bound on the constant is due to Füredi [15]. For $t > (s-1)!$ there is a lower bound of the same order of magnitude given by a construction of [4] (modifying that of [23]). Generalising the upper bound for $K_{s,t}$, Alon, Krivelevich and Sudakov [3] showed $\mathrm{ex}(n, H) = O(n^{2-1/s})$ whenever $H$ is a bipartite graph in which the vertices of one part all have degree at most $s$. We prove the following rainbow version of this result.

**Theorem 1.2** *Let $H$ be a bipartite graph in which the vertices of one part all have degree at most $s$. Then $ex^*(n, H) = O(n^{2-1/s})$.*

It seems difficult to determine whether $\text{ex}^*(n, K_{s,t}) \sim \text{ex}(n, K_{s,t})$, even in the simplest case $s = t = 2$. This leads us to our next topic - the rainbow Turán problem for even cycles.

### 1.3 Even cycles and $B_k$-sets

The case of even cycles is of particular interest, not only in the context of rainbow Turán numbers, but in its relation to the problem of $B_k$-sets in combinatorial number theory. A $B_k$-set in an abelian group is a subset $A$ of the group with the property that all the $k$-element subsets of $A$ have distinct sums. We use the existence of dense $B_k$-sets, constructed by Bose and Chowla [8], to give the lower bound for $\text{ex}^*(n, C_{2k})$ in the following theorem.

**Theorem 1.3** *For all $k \geq 2$, there exists an absolute constant $c > 0$ such that $ex^*(n, C_{2k}) \geq cn^{1+1/k}$. Furthermore, if $G$ is a properly edge-coloured graph without any cycles of length less than $2k$ and without any rainbow cycle of length $2k$, then $G$ has $O(n^{1+1/k})$ edges.*

It is a little surprising that one can find a lower bound $\text{ex}^*(n, C_{2k}) \geq cn^{1+1/k}$ when all the known constructions of $C_{2k}$-free graphs have much fewer edges. Lower bounds on $\text{ex}(n, C_{2k})$ of order $n^{1+1/k}$ are only known for $k$ equal to 2, 3 or 5. Here an upper bound $\text{ex}(n, C_{2k}) < c(k)n^{1+1/k}$ was obtained by Bondy and Simonovits [7]. The best known bound on the constant $c(k)$ is due to Verstraëte [36]. We conjecture that the same bound holds for $\text{ex}^*(n, C_{2k})$.

**Conjecture 1.4** *For all $k \geq 2$, $ex^*(n, C_{2k}) = O(n^{1+1/k})$.*

Although we cannot prove this conjecture in general, we prove it in the case $k = 3$. (The case $k = 2$ is covered by Theorem 1.2, which gives the bound $\text{ex}^*(n, K_{2,2}) = O(n^{3/2})$.)

**Theorem 1.5** *There exists absolute constants $c_2 \geq c_1 > 1$ such that*

$$c_1 \, ex(n, C_6) \leq ex^*(n, C_6) \leq c_2 \, ex(n, C_6).$$

*In particular, $ex^*(n, C_6) = \Theta(n^{4/3})$.*

A $B_k^*$-set in an abelian group $G$ is a set $A \subset G$ with the property that no pair of disjoint $k$-element subsets of $A$ have the same sum. Note that a $B_k$-set is, in particular, a $B_k^*$-set. By considering the bipartite Cayley graph of a $B_3^*$-set $A \subset G$ – consisting of edges $\{g, g + a\}$ for $g \in G$ and $a \in A$ – and noting that this graph contains no rainbow cycle of length six, we deduce that $|A| = O(|G|^{1/3})$ by applying Theorem 1.6. This gives the correct order of magnitude for the maximum size of a $B_3^*$-set, by the constructions of Bose and Chowla [8]. We will give a more detailed discussion of these reductions in Section 3.

A graph on $n$ vertices without any cycle at all has at most $n - 1$ edges, but how many edges can there be in a properly coloured graph without a rainbow cycle? We will give constructions of such graphs with $\Omega(n \log n)$ edges, but we cannot improve the upper bound of $O(n^{4/3})$ given by Theorem 1.5. In fact we can construct graphs with $\Omega(n \log n)$ edges with no cycle that uses more than half as many colours as edges, so it is natural to relax our problem and ask how many edges are sufficient to find such a cycle. Here we prove the following result.

**Theorem 1.6** *Let $G$ be a graph on $n$ vertices so that for all $k$, any cycle of length $2k$ uses at most $k$ different colours. Then the number of edges of $G$ satisfies $e(G) < n \log_2(n+3) - 2n$. Furthermore, when $n$ is a power of 2 then there is an example of such a graph with $\frac{1}{2} n \log_2 n$ edges.*

The rest of this paper is organised as follows. In the next section we prove Theorems 1.1 and 1.2 by reductions to appropriate ordinary Turán problems. Section 3 contains the proofs of the theorems on cycles and $B_k^*$-sets. In Section 4 we make some brief observations about the rainbow Turán problem for hypergraphs, and the final section contains some concluding remarks and open problems.

## 2 Reductions to Turán problems

In this section we show how for certain graphs $H$ the rainbow Turán problem for a graph $H$ can be reduced to the ordinary Turán problem for some larger graph $H'$. We will use this method to prove Theorems 1.1 and 1.2. The reductions are based on a simple greedy algorithm, which we formulate as follows.

**Lemma 2.1** *Suppose $G$ is a properly edge-coloured graph and $X$ is a subset of its vertices for which the induced graph $G_X$ is rainbow. If $Y$ is any set of vertices disjoint from $X$ with $|Y| > (|X|-2)e(G_X)$ then there is a vertex $y \in Y$ so that $X \cup \{y\}$ induces a rainbow subgraph of $G$.*

**Proof.** Let $C$ be the set of colours that appear on the edges of $G_X$. By assumption $|C| = e(G_X)$. For each $x \in X$ let $d_x$ denote the degree of $x$ in $G_X$. There are at most $|C|$ vertices $y$ such that $xy$ has a colour in $C$, so at most $|C| - d_x$ such vertices in $Y$. Therefore the number of vertices in $Y$ that are joined to any vertex in $X$ by an edge with a colour in $C$ is at most $\sum_{x \in X}(|C| - d_x) = |X||C| - 2e(G_X) = (|X|-2)e(G_X)$. Since $|Y|$ is larger than this we can choose $y \in Y$ so that no colour in $C$ appears on the edges from $y$ to $X$. Since $G$ is properly edge-coloured these edges all have different colours, so $X \cup \{y\}$ induces a rainbow subgraph. $\square$

From this we deduce the following lemma, which provides the reduction for Theorem 1.1. First we need some notation. We write $K_r(t)$ for the complete $r$-partite graph with $t$ vertices in each class and $K_r(t)^+$ for the graph obtained from $K_r(t)$ by adding an edge to one of the classes.

**Lemma 2.2**
(1) *Any proper colouring of $K_r(r^3t^3)$ contains a rainbow $K_r(t)$ and*
(2) *Any proper colouring of $K_r(r^3t^3)^+$ contains a rainbow $K_r(t)^+$.*

**Proof.** It suffices to prove the second statement. For then given any properly coloured $K_r(r^3t^3)$, we can add an edge inside a class with some new colour and find a rainbow $K_r(t)^+$. This must use the added edge, which we delete to get a rainbow $K_r(t)$ in the original graph. Consider then a properly coloured $K_r(r^3t^3)^+$ with parts $X_1, \cdots, X_r$ in which the extra edge joins $a$ and $b$ in $X_1$. We need to find $Y_1, \cdots, Y_r$ spanning a rainbow subgraph, where $Y_i \subset X_i$ has size $t$ for each $i$ and $\{a, b\} \subset Y_1$. This can be achieved by selecting each $Y_i$ in turn by the greedy algorithm, starting with $Y_1$ which can be any $t$ vertices of $X_1$ including $a$ and $b$. At any stage we have selected at most $tr$ vertices and they span at most $t^2\binom{r}{2}$ edges. We need to choose the next vertex to belong to some $X_i$ and not be one of the previously chosen vertices, so that the subgraph spanned is again rainbow. This is possible by Lemma 2.1 as $|X_i| = r^3t^3 > tr + (tr-2)t^2\binom{r}{2}$. □

**Proof of Theorem 1.1.** Suppose $H$ is a graph on $t$ vertices with chromatic number $r$. Then $H$ is a subgraph of $K_r(t)$. By the Erdős-Stone Theorem [14], any graph $G$ on $n$ vertices with $\text{ex}(n, K_r)+o(n^2)$ edges will contain a copy of $K_r(r^3t^3)$. Then a proper colouring of $G$ will yield a rainbow $K_r(t)$ by Lemma 2.2, which contains a rainbow $H$. Therefore $\text{ex}^*(n, H) < \text{ex}(n, K_r) + o(n^2) = \text{ex}(n, H) + o(n^2)$. Now suppose in addition that $H$ is colour-critical. Then $H$ is a subgraph of $K_{r-1}(t)^+$. Note that $K_{r-1}((r-1)^3t^3)^+$ is colour-critical. Then by the result of Simonovits mentioned earlier any graph $G$ on $n$ vertices with more than $\text{ex}(n, K_r)$ edges will contain a copy of $K_{r-1}((r-1)^3t^3)^+$, for $n$ sufficiently large. A proper colouring of $G$ will yield a rainbow $K_{r-1}(t)^+$ by Lemma 2.2, which contains a rainbow $H$. Therefore $\text{ex}^*(n, H) = \text{ex}(n, H) = \text{ex}(n, K_r)$. □

**Example.** We remark that $\text{ex}^*(n, H)$ is not equal to $\text{ex}(n, H)$ for a general non-bipartite graph. Consider for example a graph $H$ that consists of $t$ triangles that all share exactly one common vertex (a '$t$-fan'). Erdős, Füredi, Gould and Gunderson [11] showed that $\text{ex}(n, H)$ is equal to $\lfloor n^2/4 \rfloor + t^2 - t$ for $t$ odd and $\lfloor n^2/4 \rfloor + t^2 - 3t/2$ for $t$ even, when $n \geq 50t^2$. On the other hand, we have $\text{ex}^*(n, H) \geq \lfloor n^2/4 \rfloor + (t-1)\lfloor n/2 \rfloor$, as shown by the following construction. Start with any proper colouring of the Turán graph $T_2(n)$. Now add $t-1$ new matchings to the graph, each with its own new colour. Any rainbow subgraph of this construction uses at most $t-1$ edges that did not come from the Turán graph. On the other hand it is impossible to obtain a bipartite graph by deleting $t-1$ edges from a $t$-fan, so the construction does not have a rainbow $t$-fan. Therefore for any constant $C$ there is a non-bipartite graph $H$ with $\text{ex}^*(n, H) - \text{ex}(n, H) > Cn$ for large $n$.

Next we prove Theorem 1.2, which states that if $H = (X, Y)$ is any bipartite graph in which the vertices of $X$ all have degree at most $s$, then $\text{ex}^*(n, H) = O(n^{2-1/s})$. Given a graph $G$, call a subset $A$ of vertices $(s, b)$-*common* if every $s$ vertices in $A$ have at least $b$ common neighbours. We use the following consequence of Lemma 2.1 of Alon, Krivelevich and Sudakov [3]: for any $a, b, s$ there is a constant $c$ such that any graph on $n$ vertices with at least $cn^{2-1/s}$ edges contains an $(s, b)$-common set of size $a$. When $H$ has $h$ vertices we choose $a = h$ and $b = h^3$.

**Proof of Theorem 1.2.** Let $H = (X, Y)$ and $h, c$ be as defined in the previous paragraph. Suppose $G$ is a properly coloured graph on $n$ vertices with at least $cn^{2-1/s}$ edges. By definition there is an $(s, h^3)$-common set $A$ of size $h$. Choose any set $Y'$ of $|Y|$ vertices in $A$ to represent the part $Y$ of $H$. We select vertices of $G$ to represent $X$ by a greedy algorithm. Suppose we have come to some $x$ in $X$. Let $Y_x \subset Y$ be its neighbours in $H$ and let $Y'_x$ be their representatives in $Y'$. We have already chosen less than $h$ vertices, so there are less than $\binom{h}{2}$ forbidden colours, and so less than $h\binom{h}{2} < h^3$ vertices that cannot be used as representative for $x$. By definition $|Y'_x| \leq s$ so there are at least $h^3$ common neighbours of $Y'_x$ in $G$. We can choose any of these that is not forbidden as a representative of $x$. □

There are certain complete bipartite graphs for which the Turán numbers are known asymptotically, not just to order of magnitude. Füredi showed in [15] that $\text{ex}(n, K_{3,3}) = (1 + o(1))\frac{1}{2}n^{5/3}$ and in [16] that $\text{ex}(n, K_{2,t}) = (1 + o(1))\frac{1}{2}\sqrt{t-1}\, n^{3/2}$. In the following lemma we analyse our greedy procedure more carefully to get the best constants achievable by our reduction method.

**Lemma 2.3** *Suppose $s, t > 1$ and $t' > (s(s-1)+1)(t-1)$. Then any proper colouring of $K_{s,t'}$ contains a rainbow $K_{s,t}$.*

**Proof.** Let $K_{s,t'}$ be properly coloured, and denote the parts by $A, B$ where $|A| = s$, $|B| = t'$. Suppose we have chosen $v_1, \cdots, v_p$ in $B$ so that $A \cup \{v_1, \cdots, v_p\}$ spans a rainbow subgraph $H$, for some $p < t$. Let $C$ be the colours appearing on edges of $H$, so that $|C| = ps$. For each $a$ in $A$ there are at most $p(s-1)$ vertices $b$ in $B \backslash \{v_1, \cdots, v_p\}$ for which $ab$ has a colour in $C$. Then there are at most $p + |A| \cdot p(s-1) = p(1 + s(s-1)) < t'$ unavailable vertices in $B$, so we can choose $v_{p+1}$. $\square$

It follows from this lemma and the result of Füredi previously mentioned that $\text{ex}^*(n, K_{2,t}) < (1+o(1))\frac{1}{2}\sqrt{3(t-1)}n^{3/2}$. It is natural to ask whether this constant may be improved. For instance, in the simplest case of quadrilaterals ($C_4 = K_{2,2}$) we only know that the inferior and superior limits of $\text{ex}^*(n, C_4)/n^{3/2}$ lie between $1/2$ and $\sqrt{3}/2$.

**Construction.** Here we give a construction that shows that the bound for $t'$ in Lemma 2.3 cannot be improved in general, which suggests that an improvement in the constant for $\text{ex}^*(n, K_{s,t})$ will not come from a reduction to an ordinary Turán problem. A $(v, k, \lambda)$ *difference set* in a group $G$ of order $v$ is a subset $S$ of size $k$ so that for every non-identity element $g \in G$ there are exactly $\lambda$ pairs $(s, t) \in S \times S$ so that $g = st^{-1}$. A well-known result of Singer [34] shows in particular that when $q$ is a prime power there is a $(q^2+q+1, q+1, 1)$ difference set in the cyclic group of order $q^2+q+1$. Let $v = q^2+q+1$ and let $D$ be such a set in the additive group of $\mathbb{Z}/v\mathbb{Z}$.

Consider the following coloured complete bipartite graph. The parts are $D$ and $\mathbb{Z}/v\mathbb{Z}$. For $d \in D$ and $x \in \mathbb{Z}/v\mathbb{Z}$ we colour the edge $(d, x)$ with $d + x \in \mathbb{Z}/v\mathbb{Z}$. This is clearly a proper colouring. Also, for any $x, x' \in \mathbb{Z}/v\mathbb{Z}$ we see that $D \cup \{x, x'\}$ does not span a rainbow subgraph. For by definition of $D$ it contains $d, d'$ such that $d - d' = x' - x$, i.e. $d + x = d' + x'$, so $(d, x)$ and $(d', x')$ have the same colour. Our construction consists of $t-1$ copies of this bipartite graph with all copies of $D$ identified. To be formal, consider the coloured bipartite graph with parts $D$ and $\mathbb{Z}/v\mathbb{Z} \times \{1, \cdots, t-1\}$ (Cartesian product), where for $d \in D$, $x \in \mathbb{Z}/v\mathbb{Z}$ and $i \in \{1, \cdots, t-1\}$ the edge $(d, (x, i))$ has colour $(d + x, i)$. This is properly coloured. Also, for any choice of $t$ points in $\mathbb{Z}/v\mathbb{Z} \times \{1, \cdots, t-1\}$ there are some two of the form $(x, i)$ and $(x', i)$ and so there are $d, d'$ in $D$ such that $(d, (x, i))$ and $(d', (x', i))$ have the same colour. Setting $s = q + 1$ we have a proper colouring of $K_{s,(s(s-1)+1)(t-1)}$ with no rainbow $K_{s,t}$.

**Example.** For bipartite graphs $H$, it is not necessarily the case that $\text{ex}^*(n, H)$ is asymptotically equal to $\text{ex}(n, H)$. The path with 3 edges $P_3$ is a counterexample. It is well-known and easy to show that $\text{ex}(n, P_3)$ is equal to $n$ or $n-1$ (according to whether $n$ is divisible by 3 or not). On the other hand, we note that a properly 3-edge-coloured $K_4$ does not contain a rainbow $P_3$. Taking a disjoint union of such $K_4$'s we obtain a lower bound $\text{ex}^*(n, P_3) \geq 6\lfloor n/4 \rfloor = (1+o(1))\frac{3}{2}\text{ex}(n, P_3)$. (As a slight improvement, if $n = 4p + q$ with $0 \leq q \leq 3$ then clearly $\text{ex}^*(n, P_3) \geq 6p + \binom{q}{2}$. It is not hard to show an upper bound $\text{ex}^*(n, P_3) \leq 3n/2$; the key observation is that if $G$ has no rainbow $P_3$ and $d(x) \geq 4$ then there are no edges incident to the neighbours of $x$ other than those incident to $x$.)

For general $k$ we have $\text{ex}(n, P_k) \leq \frac{k-1}{2}n$, and equality is achieved when $n$ is divisible by $k$ with a disjoint union of cliques $K_k$. It is not so clear what the answer should be for the rainbow problem. A natural conjecture is that the optimal construction should be a disjoint union of cliques of size $c(k)$, where $c(k)$ is chosen as large as possible so that the cliques can be properly coloured with no rainbow $P_k$. It is not hard to see that $k \le c(k) \le 2(k-1)$, where the upper bound is our usual greedy argument. It seems interesting to determine $c(k)$ exactly.

# 3 Even cycles and $B_k^*$-sets

In this section we discuss the rainbow Turán problem for even cycles and its relationship with a certain extremal problem in combinatorial number theory. For a subset $A$ of an abelian group $G$, we define the coloured bipartite Cayley graph as follows. The two parts $X, Y$ are both copies of $G$, we join $x \in X$ to $y \in Y$ if $y - x \in A$, and then the edge $xy$ is assigned the colour $y - x$. Note that this is a properly coloured graph. Suppose that $x_1 y_1 \cdots x_k y_k$ is a rainbow cycle of length $2k$. Let $B = \{y_1 - x_1, \cdots, y_k - x_k\}$ and $C = \{y_1 - x_2, \cdots, y_{k-1} - x_k, y_k - x_1\}$. Then $B, C$ are disjoint $k$-subsets of $A$ with the same sum. We say $A$ is a $B_k^*$-set if no such subsets exist. Thus a bound on the number edges in a graph with no rainbow $C_{2k}$ gives a bound on the size of a $B_k^*$-set.

A related and more commonly studied condition is the following. We call $A$ a $B_k$-set if any element $g \in G$ has at most one representation of the form $g = a_1 + \cdots + a_k$ with $a_i \in A$ for $1 \le i \le k$, where we do not count permutations of the summands as being a different representation. There are $\binom{|A|+k-1}{k}$ different representations, so if $|G| = n$ we have $|A| < (k!n)^{1/k}$. When $G = \mathbb{Z}/n\mathbb{Z}$ Bose and Chowla [8] constructed $B_k$ sets of size $(1 + o(1))n^{1/k}$, showing that $n^{1/k}$ is the correct order of magnitude. Note that a $B_k$-set is in particular a $B_k^*$-set, so there are $B_k^*$-sets in $\mathbb{Z}/n\mathbb{Z}$ of size $(1 + o(1))n^{1/k}$. An upper bound of the same order of magnitude was obtained by Ruzsa [31], who showed that a $B_k^*$-set in the integers $\{1, \cdots, n\}$ has at most $(1 + o(1))k^{2-1/k}n^{1/k}$ elements. One of the outstanding problems in combinatorial number theory is to close the gap between the upper and lower bounds for such sets.

## 3.1 Rainbow Turán for even cycles

In this subsection we prove Theorem 1.3, which concerns the rainbow Turán number $\mathrm{ex}^*(n, C_{2k})$. The lower bound $(1+o(1))(n/2)^{1+1/k}$ follows from the bipartite Cayley graph construction described above, applied to a Bose-Chowla $B_k$-set in $\mathbb{Z}/n\mathbb{Z}$. Now we will show a corresponding upper bound under the additional assumption of there being no strictly shorter cycles in the underlying graph.

**Proof of Theorem 1.3.** Let $G$ be a graph on $n$ vertices with no cycle of length less than $2k$. Suppose $G$ has a proper colouring with no rainbow $C_{2k}$. Let $d = 2e(G)/n$ be the average degree. We will show that $d < 2(k^2 n)^{1/k}$ for large $n$. Note that we can assume that $G$ has minimum degree at least $d/2$, as deleting a vertex of degree less than $d/2$ does not decrease the average degree. We start by showing that the number of rainbow paths of length $k$ satisfies

$$R_k \ge 2^{-k+1} nd(d-1) \prod_{i=1}^{k-2} (d - 4i). \tag{1}$$

This follows by induction on $k$. First of all, by Cauchy-Schwartz there are at least $n\binom{d}{2}$ (rainbow) paths of length 2. For $t \ge 2$ each rainbow path of length $t+1$ contains 2 rainbow paths of length $t$. Also, given a rainbow path of length $t$, each of its endpoints is incident to at least $d/2$ edges, of which at most $t-1$ have endpoints on the path and at most $t-1$ others have a colour that appears on the path, so it can be extended to a rainbow path of length $t+1$ in at least $2(d/2-2(t-1))$ ways. Therefore $R_{t+1} \geq (d/2-2(t-1))R_t$, which proves the claim.

Given a pair of vertices $a, b$ let $p_{ab}$ denote the number of rainbow paths of length $k$ with endpoints $a$ and $b$. Since $G$ has girth at least $2k$ any two such paths $ax_1\cdots x_{k-1}b$ and $ax'_1\cdots x'_{k-1}b$ are internally disjoint, i.e. $ax_1\cdots x_{k-1}bx'_{k-1}\cdots x'_1$ is a $2k$-cycle. By assumption, there are two edges of the same colour on this cycle. Say that a path $y_1\cdots y_t$ is *special* if there is some $i > 1$ such that $y_1y_2$ and $y_iy_{i+1}$ have the same colour, and no other pair of edges have the same colour. We claim that the cycle $ax_1\cdots x_{k-1}bx'_{k-1}\cdots x'_1$ contains a special path of length $k+1$.

To see this, we start with the shortest path that contains two edges with the same colour. One of each must appear on the two rainbow paths joining $a$ and $b$, so suppose the path is $x_i\cdots x_1ax'_1\cdots x'_j$ for some $i, j$. Here $x_{i-1}x_i$ and $x'_{j-1}x'_j$ have the same colour (using the shorthand $x_0 = x'_0 = a$) and no other pair of edges have the same colour. The length of the path is $i+j$. Note that $x_{i-1}x_i$ and $x'_{j-1}x'_j$ belong to the path $x_{i-1}\cdots x_{k-1}bx'_{k-1}\cdots x'_{j-1}$ of length $2(k+1)-(i+j)$, and as we chose the shortest path it has length $i+j \leq k+1$. Now consider the path $x_t\cdots x_1ax'_1\cdots x'_{k+1-t}$ where $t \geq 1$ is chosen as small as possible so that the colour of $x_{t-1}x_t$ is repeated on the path. This path exists by the preceding discussion, and there are no other repetitions of colours, as this would contradict the minimality of $t$. Therefore we have found a special path of length $k+1$.

Note that each special path of length $k+1$ contains a rainbow path of length $k$ (obtained by deleting the end-edge whose colour is repeated) and each rainbow path of length $k$ can be extended to at most $2(k-1)$ special paths of length $k+1$. This shows that there are at most $2(k-1)R_k$ special paths of length $k+1$. Also, since $G$ has girth at least $2k$ there is at most one path of length $k-1$ between any two points, so each special path comes from at most one $C_{2k}$. Each such $C_{2k}$ can be written as the union of two rainbow paths of length $k$ in at most $k$ ways. We conclude that

$$\frac{1}{k}\sum_{a,b}\binom{p_{ab}}{2} \leq 2(k-1)R_k = 2(k-1)\sum_{a,b} p_{ab}.$$

This can be rewritten as $\sum_{a,b} p_{a,b}^2 \leq (4k(k-1)+1)\sum_{a,b} p_{ab}$. By the Cauchy-Schwartz inequality

$$\sum_{a,b} p_{ab}^2 \geq \binom{n}{2}^{-1}\left(\sum_{a,b} p_{ab}\right)^2$$

and so we see that $\sum_{a,b} p_{ab} \leq (4k(k-1)+1)\binom{n}{2}$. Now by equation (1) we have

$$2^{-k+1}nd(d-1)\prod_{i=1}^{k-2}(d-4i) \leq R_k = \sum_{a,b} p_{ab} \leq (4k(k-1)+1)\binom{n}{2},$$

which implies $d < 2(k^2n)^{1/k} + 4k$. □

**Remark.** Rödl and Tuza [29] proved the existence of graphs of arbitrarily large girth for which any proper edge-colouring produces a rainbow cycle. They show that random graphs have this property, but it would be interesting to give explicit examples of such graphs. The preceding theorem would provide this if we knew the existence of certain conjectured constructions of graphs without short even cycles. For example, there are constructions of bipartite graphs with no cycles of length at most 10 with $\Omega(n^{6/5})$ edges, and our theorem implies that any proper edge-colouring of such a graph produces a rainbow $C_{12}$. The best known general constructions are given by Lazebnik, Ustimenko and Woldar [26] who construct graphs with no cycle of length less than $2k$ with $\Omega(n^{1+1/\ell})$ edges, where $\ell$ is approximately $3k/2$. It is conjectured that this can be improved to $\Omega(n^{1+1/(k-1)})$, but even $k^{2/k}n^{1+1/k}$ would allow us to apply Theorem 1.3.

### 3.2 Rainbow Turán for $C_6$

In this subsection we discuss the rainbow Turán problem for the six-cycle (or hexagon). For the ordinary Turán problem the best known bounds are due to Füredi, Naor and Verstraëte [18]. They show that $(1+o(1))\alpha n^{4/3} \le \text{ex}(n, C_6) \le (1+o(1))\beta n^{4/3}$, where $\alpha = 3(\sqrt{5}-2)(\sqrt{5}-1)^{-4/3} \sim 0.534$ and $\beta \sim 0.627$ is the real root of $16\beta^3 - 4\beta^2 + \beta - 3 = 0$. We will prove Theorem 1.5, which states that there are constants $c_2 \ge c_1 > 1$ such that $c_1\text{ex}(n, C_6) \le \text{ex}^*(n, C_6) \le c_2\text{ex}(n, C_6)$. We will not attempt to optimise these constants. First we need a lemma.

**Lemma 3.1** *Let $G$ be a bipartite graph on $n$ vertices with average degree $d = 2e(G)/n$. Suppose $G$ does not contain $K_{2,t}$ and has a proper edge-colouring with no rainbow $C_6$. Then $d < ((11t-12)n)^{1/3} + 4$.*

**Proof of Lemma 3.1.** We can assume that $G$ has minimum degree at least $d/2$, as deleting a vertex of degree less than $d/2$ does not decrease the average degree. As in the proof of Theorem 1.3 we see that the number of rainbow paths of length 3 satisfies

$$R_3 \ge \frac{1}{4}nd(d-1)(d-4). \tag{2}$$

Given a pair of vertices $a, b$ we write $p_{ab}$ for the number of rainbow paths of length 3 that have endpoints $a$ and $b$. We claim that there are at least $\binom{p_{ab}}{2} - (5t-6)p_{ab}$ pairs of such paths $(axyb, ax'y'b)$ for which $axyby'x'$ is a 6-cycle and $x'y'$ has the same colour as $xy$. To see this, fix any rainbow path $axyb$. Since $G$ does not contain $K_{2,t}$ there are at most $t-2$ other paths of the form $axy'b$ and at most $t-2$ other paths of the form $ax'yb$. This shows that there are at most $(t-2)p_{ab}$ (unordered) pairs $(axyb, ax'y'b)$ for which $axyby'x'$ does not form a 6-cycle. Now consider $ax'y'b$ for which $axyby'x'$ is a 6-cycle. By assumption this is not rainbow so there are two edges with the same colour. There are at most 2 vertices $x' \ne x$ such that $ax'$ has a colour from the path $axyb$, so at most $2(t-1)$ paths $ax'y'b$ where $ax'$ has a colour from the path $axyb$. Similarly there are at most $2(t-1)$ paths $ax'y'b$ where $y'b$ has a colour from the path $axyb$. Therefore there are at most $4(t-1)p_{ab}$ pairs $(axyb, ax'y'b)$ such that $axyby'x'$ is a 6-cycle and one of $ax, yb$ has a colour from $ax'y'b$ or one of $ax', y'b$ has a colour from $axyb$. Since there is no rainbow $C_6$, for any 6-cycle $axyby'x'$ not covered by the above exceptions the edges $xy$ and $x'y'$ have the same colour. It follows that there are at least $\binom{p_{ab}}{2} - (t-2)p_{ab} - 4(t-1)p_{ab} = \binom{p_{ab}}{2} - (5t-6)p_{ab}$ pairs $(axyb, ax'y'b)$ for which $axyby'x'$ is a 6-cycle and $x'y'$ has the same colour as $xy$.

Call a path *special* if its first and last edges have the same colour and no other pair of edges have the same colour. (Note that this is slightly different to the definition used in the proof of Theorem 1.3.) A special path of length $k$ contains 2 rainbow paths of length $k-1$, and each rainbow path of length $k-1$ is contained in at most 2 special paths of length $k$, so the number of rainbow paths of length $k-1$ is an upper bound on the number of special paths of length $k$. To each pair $(axyb, ax'y'b)$ for which $axyby'x'$ is a 6-cycle and $x'y'$ has the same colour as $xy$ we can associate the two special paths of length four $xyby'x'$ and $yxax'y'$. Also, each special path of length four belongs to at most $t-1$ 6-cycles (as there is no $K_{2,t}$), and it is counted by exactly one partition of any such 6-cycle into two rainbow paths of length 3. It follows that there are at least $\frac{2}{t-1}\sum_{a,b}\left(\binom{p_{ab}}{2}-(5t-6)p_{ab}\right)$ special paths of length 4. As noted above, the number of special paths of length 4 is at most the number of rainbow paths of length 3, which equals $\sum_{a,b} p_{ab}$ by definition. We conclude that

$$\frac{2}{t-1}\sum_{a,b}\left(\binom{p_{ab}}{2}-(5t-6)p_{ab}\right)\leq\sum_{a,b}p_{ab}.$$

This may be re-written as $\sum_{a,b} p_{ab}^2 \leq (11t-12)\sum_{a,b} p_{ab}$. By the Cauchy-Schwartz inequality

$$\sum_{a,b}p_{ab}^2\geq\binom{n}{2}^{-1}\left(\sum_{a,b}p_{ab}\right)^2$$

and so we see that $\sum_{a,b} p_{ab} \leq (11t-12)\binom{n}{2}$. Recalling that $R_3=\sum_{a,b} p_{ab}$ and equation (2) we get $\frac{1}{2}nd(d-1)(d-4)\leq(11t-12)\binom{n}{2}$, which gives $d<(1+o(1))((11t-12)n)^{1/3}+4$. This completes the proof. □

**Proof of Theorem 1.5.** We start with the upper bound. Let $G$ be a graph on $n$ vertices that is properly coloured with no rainbow $C_6$. It contains a bipartite subgraph $G'$ with $e(G')\geq e(G)/2$. We say that a subgraph $K_{2,t}$ of $G'$ is *maximal* if it is not contained in $K_{2,t'}$ for any $t'>2$. We claim that if $G'$ contains a maximal $K_{2,s}$ and a maximal $K_{2,t}$ with $s,t\geq 9$ then they must be edge-disjoint.

For suppose that $(A_1,B_1)$ is a maximal $K_{2,s}$ with $|A_1|=2$, $|B_1|=s\geq 9$, $(A_2,B_2)$ is a maximal $K_{2,t}$ with $|A_2|=2$, $|B_2|=t\geq 9$, and $xy$ is a common edge. Consider first the case when $x\in A_1\cap A_2$ and $y\in B_1\cap B_2$. By maximality we have $A_1=\{x,z_1\}$ and $A_2=\{x,z_2\}$ with $z_1\neq z_2$. Let $c_1$ be the colour of $yz_1$ and $c_2$ of $yz_2$. There are at most 4 vertices $b_1$ in $B_1$ such that $xb_1$ or $z_1b_1$ has has colour $c_1$ or $c_2$, so we can choose $b_1\in B_1$ so that $xb_1$ has colour $c_3$, $z_1b_1$ has colour $c_4$ and $c_1,\cdots,c_4$ are all different. Now there are at most 8 vertices $b_2$ in $B_2$ so that $xb_2$ or $z_2b_2$ has a colour among $c_1,\cdots,c_4$. Choosing any other $b_2$ we obtain a rainbow 6-cycle $xb_1z_1yz_2b_2$. Now consider the case when $x\in A_1\cap B_2$ and $y\in B_1\cap A_2$. Write $A_1=\{x,z_1\}$ and $A_2=\{y,z_2\}$. Let $c_1$ be the colour of $yz_1$ and $c_2$ of $xz_2$. There are at most 4 vertices $b_1$ in $B_1$ such that $xb_1$ or $z_1b_1$ has has colour $c_1$ or $c_2$, so we can choose $b_1\in B_1$ so that $xb_1$ has colour $c_3$, $z_1b_1$ has colour $c_4$ and $c_1,\cdots,c_4$ are all different. Now there are at most 8 vertices $b_2$ in $B_2$ so that $yb_2$ or $z_2b_2$ has a colour among $c_1,\cdots,c_4$. Choosing any other $b_2$ we obtain a rainbow 6-cycle $xb_1z_1yb_2z_2$. It follows that any edge of $G'$ belongs to at most one maximal $K_{2,t}$ with $t\geq 9$.

Suppose that $(A,B)$ is a maximal $K_{2,t}$ with $A=\{a_1,a_2\}$ and $|B|=t\geq 9$. Delete from $G'$ all edges joining $a_1$ to $B$. Repeat this process as long as there is any (maximal) $K_{2,t}$ with $t\geq 9$. Note that we have considered mutually disjoint sets of edges and deleted half of each, so we have deleted at most half of the edges of $G'$. The remaining graph $G''$ contains no $K_{2,9}$. By Lemma 3.1 it has average degree $d''<(1+o(1))(87n)^{1/3}$. Therefore $e(G)\leq 2e(G')\leq 4e(G'')=2d''n<9n^{4/3}$ for large $n$.

There is a lower bound $\mathrm{ex}^*(n, C_6) = \Omega(n^{4/3})$ from the bipartite Cayley graph construction, but here we will give a better construction to show $\mathrm{ex}^*(n, C_6) > c\,\mathrm{ex}(n, C_6)$ with $c > 1$. Suppose $n$ is even and consider a graph $G_0$ on $n/2$ vertices with no cycle of length at most 6. Let $G$ be the two-point blowup of $G_0$, i.e. for each vertex $v \in G_0$ there are two vertices $v_0, v_1$ in $G$, and for each edge $uv \in G_0$ we have all four edges $u_i v_j$, $0 \le i, j \le 1$ in $G$. Choose an arbitrary proper edge-colouring $c_0$ of $G_0$. We define an edge-colouring $c$ of $G$ by the rule $c(u_i v_j) = (c_0(uv), i + j \bmod 2)$. By this we mean that the colour of an edge is an ordered pair: the first element is the colour of the edge in $G_0$ it came from, and the second element is chosen to be 0 or 1 in a way that ensures that the resulting edge-colouring is proper.

We claim that $G$ has no rainbow $C_6$. For suppose $C = a_\alpha b_\beta c_\gamma d_\delta e_\epsilon f_\zeta$ is a 6-cycle in $G$. Then $abcdef$ is a closed walk in $G_0$, which has no cycle of length at most 6, and it is easy to see that it must consist of a path of length 3 traversed in both directions. Without loss of generality $a$ and $d$ are the endpoints of this path. Then $e = c$ and $f = b$, so $\epsilon = \gamma + 1$ and $\zeta = \beta + 1$, which gives $\epsilon + \zeta = \gamma + \beta \pmod 2$. Thus the edges $b_\beta c_\gamma$ and $e_\epsilon f_\zeta$ have the same colour, so $C$ is not rainbow.

We can choose the graph $G_0$ to have $e(G_0) = (n/4)^{4/3} + O(n)$ (see remark 3 following this proof.) Then $\mathrm{ex}^*(n, C_6) \ge e(G) = 4e(G_0) = 4^{-1/3} n^{4/3} + O(n) \ge (1 + \lambda + o(1))\mathrm{ex}(n, C_6)$, where using the upper bound for $\mathrm{ex}(n, C_6)$ quoted at the beginning of this subsection one may calculate that $\lambda \ge 4^{-1/3}\beta^{-1} - 1 > 1/250$. This completes the proof of the theorem. □

**Remarks.** (1) We have made no attempt to optimise the constants in our arguments, but it seems interesting that we have a purely combinatorial argument that gives the correct order of magnitude.
(2) It follows from the proof that an edge-coloured bipartite graph on $2n$ vertices with no rainbow $C_6$ has at most $\frac{1}{2}9(2n)^{4/3} < 12n^{4/3}$ edges (for large $n$). Applying this to a bipartite Cayley graph we see that in any abelian group of order $n$, a $B_3^*$-set can have at most $12n^{1/3}$ elements.
(3) The construction given in the above proof may be generalised as follows. Write $z(n, H)$ for the maximum number of edges in an $H$-free bipartite graph with $n$ vertices in each part, and let $z^*(n, H)$ denote the rainbow analogue of this definition. Applying the construction when $G_0$ is a bipartite graph with no cycles of length at most $2k$ we see that $z^*(2n, C_{2k}) \ge 4z(n, C_4, \cdots, C_{2k})$ for any $k$. It is shown in [18] that $z(n, C_6) \le 2^{1/3} n^{4/3} + O(n)$, and Hoory [21] and Lam [25] independently showed that $z(n, C_4, C_6) = n^{4/3} + O(n)$. Therefore $z^*(n, C_6) \ge 4z(n/2, C_4, C_6) \ge 4(n/2)^{4/3} + O(n) \ge (1 + o(1))2^{1/3} z(n, C_6)$.

### 3.3 Excluding all cycles

Now we will consider the problem of excluding any rainbow cycle. Note that the ordinary Turán problem is easy in this case: an acyclic graph on $n$ vertices has at most $n - 1$ edges, with equality for a tree. On the other hand, we can construct graphs with order $n \log n$ edges that can be properly coloured with no rainbow cycle.

One construction is the $m$-cube, a bipartite graph in which the vertices are all subsets of $\{1, \cdots, m\}$ and for any $A \subset \{1, \cdots, m\}$ and $i \in A$ there is an edge between $A$ and $A \backslash \{i\}$ of colour $i$. There are no rainbow cycles in this graph, and in fact every cycle of length $2k$ uses at most $k$ different colours. Indeed, if a cycle contains an edge $(A, A\backslash\{i\})$ of colour $i$ then the path continuing along the cycle from

$A\backslash\{i\}$ must again use at least one edge of colour $i$ in order to reach $A$, which contains the element $i$. The $m$-cube has $n = 2^m$ vertices and $m2^{m-1} = \frac{1}{2}n\log_2 n$ edges.

An improvement in the constant is obtained by taking a product of graphs $K_{3,3}$. The building block is the bipartite graph $K_{3,3}$ with parts $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$ in which the edge $x_iy_j$ has colour $i - j$ mod 3. Clearly this contains no rainbow cycles. The vertices of our graph are sequences of length $m$ in which each term is one of $x_0, x_1, x_2, y_0, y_1, y_2$. Two sequences $z = (z_1, \cdots, z_m)$ and $z' = (z'_1, \cdots, z'_m)$ are adjacent if there is some $s$ such that $z_t = z'_t$ for $t \neq s$ and $z_sz'_s$ is an edge of the building block $K_{3,3}$. We colour such an edge $zz'$ with the pair $(s, c)$, where $c$ is the colour of $z_sz'_s$. Consider any cycle $z^1, \cdots, z^k$. For any $1 \leq s \leq m$ the terms $z^1_s, \cdots, z^k_s, z^1_s$ form a sequence of vertices in $K_{3,3}$ in which each term is either adjacent or equal to the one preceding it. There is at least one $s$ for which these terms are not all equal. Then there is a closed walk in $K_{3,3}$ whose edges appear as adjacent members of $z^1_s, \cdots, z^k_s$, so some colour is repeated. It follows that there are no rainbow cycles. This graph has $n = 6^m$ vertices and $6^m \cdot 3m/2 = \frac{3}{2\log_2 6}n\log_2 n > 0.58n\log_2 n$ edges.

We do not have a good upper bound for the problem of finding a rainbow cycle, but we can determine the order of magnitude for finding a cycle with more than half as many colours as edges.

**Proof of Theorem 1.6.** A construction with $\frac{1}{2}n\log_2 n$ edges when $n$ is a power of 2 was described above. For the upper bound let $G$ be a graph on $n$ vertices so that any cycle of length $2k$ uses at most $k$ different colours for any $k$. Let $d = 2e(G)/n$ be the average degree. By deleting vertices of small degree we can assume that the minimum degree is at least $d/2$. We claim that the number of rainbow paths of length $k$ satisfies

$$R_k > 2n\prod_{i=0}^{k-1}(d/2 - i). \tag{3}$$

The proof is a slight improvement on that given for equation (1) in Theorem 1.3. As before we have $R_2 \geq n\binom{d}{2}$ which is larger than $2n(d/2)(d/2-1)$. For $t \geq 2$ each rainbow path of length $t+1$ contains 2 rainbow paths of length $t$. Also, given a rainbow path of length $t$, each of its endpoints is incident to at least $d/2$ edges, of which only one has an endpoint on the path (otherwise there would be a rainbow cycle) and at most $t - 1$ others have a colour that appears on the path, so it can be extended to a rainbow path of length $t + 1$ in at least $2(d/2 - t)$ ways. Therefore $R_{t+1} \geq (d/2 - t)R_t$, which proves the claim.

Note that if a pair of vertices is joined by two rainbow paths then they have the same length. For the symmetric difference of the paths is a disjoint union of cycles, and if one of them is longer then it will contribute more than half of the edges of one of these cycles. Since the path is rainbow this cycle will have more than half as many colours as edges, which is a contradiction. The same argument shows that in fact the two paths use exactly the same set of colours.

For each $k$ let $H_k$ be the graph consisting of all pairs $ab$ for which there is a rainbow path of length $k$ from $a$ to $b$ in $G$. Consider such a path using colours $c_1, \cdots, c_k$. We showed above that any other such path uses a permutation of these colours. Since $G$ is properly coloured it is uniquely determined by the permutation, so there are at most $k!$ such paths. Therefore $e(H_k) \geq R_k/k!$, which gives $\sum_{k\geq 1} R_k/k! \leq \sum_{k\geq 1} e(H_k) \leq \binom{n}{2}$, since for $k \neq k'$ the graphs $H_k$ and $H_{k'}$ are edge-disjoint.

Recalling that $R_k > 2n\prod_{i=0}^{k-1}(d/2-i)$. we see that

$$2n(2^{d/2}-1) = 2n\sum_{k\geq 1}\binom{d/2}{k} < \binom{n}{2},$$

which gives $d < 2\log_2(n+3)-4$, i.e. $e(G) < n\log_2(n+3)-2n$. $\square$

**Remark.** A properly edge-coloured $d$-regular graph on $n$ vertices has at least $nd(d-1)\cdots(d-k+1) = nk!\binom{d}{k}$ rainbow walks of length $k$. It is natural to conjecture that a graph with average degree $d$ should have at least this many rainbow walks of length $k$. Under the assumptions of the above theorem all rainbow walks are in fact paths, so if this conjecture is true we would have $R_k \geq \frac{1}{2}nk!\binom{d}{k}$ and so $\binom{n}{2} \geq \sum_{k\geq 1} R_k/k! \geq \frac{1}{2}n\sum_{k\geq 1}\binom{d}{k} = n(2^d-1)$, i.e. $d \leq \log_2 n$. This would show that the lower bound is tight, not just asymptotically but exactly when $n$ is a power of 2.

We will mention a couple of results to give some intuition as to why this conjecture might be true. Firstly, there is an inequality of Blakley and Roy [6] that implies that a graph with average degree $d$ has at least $nd^k$ walks of length $k$. Secondly, there is the following result of Alon, Hoory and Linial that may be found within the proof in [1]. Consider a walk of length $k$ using the edges $e_1,\cdots,e_k$ in succession. It is a non-returning walk if we never have $e_i = e_{i+1}$. It is shown in [1] that a graph with $n$ vertices and average degree $d$ has at least $nd(d-1)^{k-1}$ non-returning walks of length $k$. A possible generalisation of these results is to count walks of length $k$ in which there are $a_i$ forbidden edges at the $i^{th}$ step (possibly depending on the walk so far). One might think that a graph with $n$ vertices and average degree $d$ has at least $n\prod_{i=1}^{k}(d-a_i)$ such walks. This would include our conjecture on rainbow walks as a special case.

We conclude this subsection with an argument very similar to the previous proof that gives a girth result for this weaker condition on cycle colourings, under a weaker assumption than the type used in Theorem 1.3.

**Theorem 3.2** *Suppose $k > 1$ and let $G$ be a graph on $n$ vertices so that any cycle of length $2t$ uses at most $t$ different colours for any $t \leq k$. Then $e(G) < (1+o(1))(k!)^{1/k}n^{1+1/k}$.*

**Proof of Theorem 3.2.** Let $G$ be a graph on $n$ vertices so that any cycle of length $2t$ uses at most $t$ different colours for any $t \leq k$. Let $d = 2e(G)/n$ be the average degree. By deleting vertices of small degree we can assume that the minimum degree is at least $d/2$. As in the previous proof we have $R_k > 2n\prod_{i=0}^{k-1}(d/2-i)$. Defining $H_k$ as before we have $R_k/k! \leq e(H_k) \leq \binom{n}{2}$. Therefore $d < (1+o(1))2(k!n/4)^{1/k}$, so $e(G) < (1+o(1))k!^{1/k}n^{1+1/k}$. $\square$

## 3.4 Combinatorial number theory, coloured graphs and hypergraphs

We have seen that the extremal problem for $B_k^*$-sets in abelian groups can be reduced to the rainbow Turán problem for cycles, via bipartite Cayley graphs. A thorough analysis of $B_k$-sets and other extremal problems in combinatorial number theory is given in Graham [20] and in Ruzsa [31]. In these papers, upper bounds for the size of $B_k$-sets and $B_k^*$-sets in $\{1,2,\ldots,n\}$ are given using Fourier analysis – more precisely, Parseval's Identity. In the context of this paper, a natural question arises: which other problems in combinatorial number theory can be reduced to extremal problems for properly coloured graphs?

Perhaps the most remarkable example of such a reduction is the Ruzsa-Szemerédi Theorem [32] (otherwise known as the $(6,3)$-Theorem when phrased in terms of hypergraphs): in any properly $n$-edge-coloured $n$-vertex graph of positive density, there is a path of length three whose first and last edges have the same colour. Ruzsa and Szemerédi proved, using a reduction to bipartite Cayley-type graphs, that this implies that every set of integers of positive density contains a three-term arithmetic progression. The first proof of this result was given by Roth [30], using the Hardy-Littlewood circle method. Furthermore, a construction of Behrend [5] gives an $n$-vertex $n$-edge-coloured graph with roughly $\exp(-c\sqrt{\log n}) \cdot n^2$ edges containing no path of length three whose first and last edges have the same colour, so the Ruzsa-Szemerédi theorem is essentially best possible. It is also known that Szemerédi's Theorem on $k$-term arithmetic progressions can be reduced to an extremal problem for hypergraphs. This approach was recently investigated at length by Frankl and Rödl [19]. As in the Ruzsa-Szemerédi theorem, these extremal problems involve graphs which are not rainbow coloured, and the extremal problems exhibit a very different behaviour to those discussed in this paper.

The theorems which we mentioned above can be phrased in terms of simple hypergraphs (a hypergraph is simple if each pair of edges have at most one point in common). Starting with a simple 3-uniform hypergraph $\mathcal{H}$, we can form a graph $G$ consisting of all pairs of points which are contained in a triple of $\mathcal{H}$, and assigning a pair $e \in G$ the colour $x$ if $e \cup \{x\} \in \mathcal{H}$. Since $\mathcal{H}$ is simple, $G$ is a properly $n$-edge coloured graph, whose subgraphs correspond to certain hypergraph configurations in $\mathcal{H}$. As an example, suppose we forbid in $\mathcal{H}$ a set of $k$ triples whose union has at most $\ell$ points. Then it is not hard to see that $G$ cannot contain any $k$-vertex subgraph with at most $\ell - k$ colours on its edges. In the case $k = 3$ and $\ell = 6$, one of the forbidden subgraphs is a path of length three with two colours on its edges, and we recover the $(6,3)$-Theorem of Ruzsa and Szemerédi. A wealth of old problems and results in this vein may be found in the paper of Brown, Erdős and Sós [9]. Finally, we mention one of the outstanding problems in that paper is the following: how many edges can a properly $n$-edge-coloured $n$-vertex graph have if it is not to contain a cycle of length four with at most three colours on its edges?

# 4 Hypergraphs

In this section we briefly consider the rainbow Turán problem for hypergraphs. We use the same notation as for graphs: if $\mathcal{F}$ is a fixed $r$-uniform hypergraph then $\mathrm{ex}(n,\mathcal{F})$ denotes the number of edges in the largest $\mathcal{F}$-free $r$-uniform hypergraph on $n$ vertices, and $\mathrm{ex}^*(n,\mathcal{F})$ is the maximum number of edges in a properly edge-coloured $r$-uniform hypergraph on $n$ vertices with no rainbow $\mathcal{F}$. Since the ordinary Turán theory for hypergraphs is not well understood, we will not attempt a systematic analysis, but will restrict ourselves to two interesting observations.

Firstly, we note that the arguments of Theorem 1.1 can be extended to show that $\mathrm{ex}^*(n,\mathcal{F}) \leq \mathrm{ex}(n,\mathcal{F}) + o(n^r)$ for an $r$-uniform hypergraph $\mathcal{F}$. The following three statements provide a sketch of the proof.

1. Suppose $\mathcal{H}$ is a properly edge-coloured $r$-uniform hypergraph and $X$ is a subset of its vertices for

which the induced hypergraph $\mathcal{H}_X$ is rainbow. If $Y$ is any set of vertices disjoint from $X$ with $|Y| > \binom{|X|}{r-1}e(G_X)$ then there is a vertex $y \in Y$ so that $X \cup \{y\}$ induces a rainbow subhypergraph of $\mathcal{H}$.

2. Let $\mathcal{F}$ be an $r$-uniform hypergraph with $f$ vertices. Write $\mathcal{F}(t)$ for the $t$-point blowup of $\mathcal{F}$ (i.e. making $t$ copies of each vertex and the corresponding $t^r$ copies of each edge). Then any proper edge-colouring of $\mathcal{F}\left(\binom{f}{r}^2\right)$ contains a rainbow $\mathcal{F}$.

3. (Erdős-Simonovits supersaturation [13]) For an $r$-uniform hypergraph $\mathcal{F}$ and a constant $t$ we have $\mathrm{ex}(n, \mathcal{F}(t)) \leq \mathrm{ex}(n, \mathcal{F}) + o(n^r)$.

Secondly, we describe a hypergraph analogue of a phenomenon which we have observed for bipartite graphs, namely an 3-partite 3-uniform hypergraph for which the rainbow and ordinary Turán numbers have the same order of magnitude but appear not to be asymptotically equal. We define an 3-uniform hypergraph $K^{(3)}_{s,t}$ as follows. There are three parts $X, Y, Z$ with $|X| = |Y| = t$ and $|Z| = s$. The elements of $X$ and $Y$ are paired as $\{(x_i, y_i) : 1 \leq i \leq t\}$ and the edges of the hypergraph consist of all triples $\{(x_i, y_i, z) : 1 \leq i \leq t, z \in Z\}$. Mubayi and Verstraëte [28] showed that $\mathrm{ex}(n, K^{(3)}_{2,2}) < 3\binom{n}{2} + 6n$ (improving the bound $3.5\binom{n}{2}$ obtained by Füredi [17]) and that $\mathrm{ex}(n, K^{(3)}_{2,t}) < t^4\binom{n}{2}$ for $t > 2$. A simple construction shows that $\mathrm{ex}(n, K^{(3)}_{2,2}) \geq \binom{n-1}{2} + \lfloor (n-1)/3 \rfloor$, and Füredi conjectured that in fact equality holds.

It is easy to see that a proper edge-colouring of $K^{(3)}_{2,4}$ contains a rainbow copy of $K^{(3)}_{2,2}$, so we have $\mathrm{ex}^*(n, K^{(3)}_{2,2}) \leq \mathrm{ex}(n, K^{(3)}_{2,4}) < 4^4\binom{n}{2} = O(\mathrm{ex}(n, K^{(3)}_{2,2}))$. For a lower bound, consider the following construction. Consider a Steiner $S(n, 6, 2)$ system, i.e. a partition of the edges of the complete graph $K_n$ into copies of $K_6$, with vertex sets $A_1, \cdots, A_m$ (where $m = \frac{1}{15}\binom{n}{2}$). Let $\mathcal{H}$ be the 3-uniform hypergraph with edge set consisting of all triples that belong to some $A_i$, $1 \leq i \leq m$. Then $e(\mathcal{H}) = 20m = \frac{4}{3}\binom{n}{2}$. Colour the edges so that edges $e$ and $f$ have the same colour if and only if $e \cup f = A_i$ for some $i$. Note that $\mathcal{H}$ has no rainbow $K^{(3)}_{2,2}$. For suppose $X, Y, Z$ span $K^{(3)}_{2,2}$ (using the notation above). Both elements of $Z$ form an edge with $(x_1, y_1)$, so they belong to the same $A_i$ as $x_1$ and $y_1$. Similarly they belong to the same clique as $x_2$ and $y_2$, i.e. $X \cup Y \cup Z = A_i$ for some $i$. Then $x_1y_1z_1$ and $x_2y_2z_2$ have the same colour, so the $K^{(3)}_{2,2}$ is not rainbow. We conclude that $\mathrm{ex}^*(n, K^{(3)}_{2,2}) \geq \frac{4}{3}\binom{n}{2}$, so if Füredi's conjecture is true this is not asymptotically equal to the ordinary Turán number.

## 5 Concluding remarks

We have seen that for non-bipartite $H$ the rainbow Turán number $\mathrm{ex}^*(n, H)$ is asymptotically equal to the ordinary Turán number $\mathrm{ex}(n, H)$. For bipartite graphs we have seen some evidence that these quantities may have the same order of magnitude. They are not asymptotically equal in general, as we saw with the examples of a path of length 3 and a six-cycle. It seems plausible that other bipartite graphs, such as even cycles and complete bipartite graphs, should also exhibit this phenomenon. Perhaps it will be helpful for intuition in rainbow Turán problems to prove some natural structural properties. For example, is it true that a properly edge-coloured graph $G$ with no rainbow $H$ has a proper edge-colouring using the minimum possible number of colours (i.e. $\chi'(G)$) which also has no rainbow $H$?

There is a natural generalisation of the rainbow Turán problem along the lines of [2]. For any $m \geq 1$ and graph $H$ we write $\mathrm{ex}^*_m(n, H)$ for the maximum number of edges in a graph on $n$ vertices that has an edge-colouring with no rainbow $H$ in which each colour appears at most $m$ times at each vertex. The following example shows that $\mathrm{ex}^*_m(n, K_r)$ is not equal to $\mathrm{ex}(n, K_r)$ for $m \geq 2$, even in the case $m = 2$ and $r = 3$. Take a balanced complete bipartite graph with parts $A, B$ and add a matching to $A$. Colour the edges so that edges $e_1$ and $e_2$ have the same colour if and only if there is a vertex $b \in B$ and an edge $a_1a_2$ in $A$ so that $e_1 = ba_1$ and $e_2 = ba_2$. Clearly there is no rainbow triangle and each colour appears at most twice at each vertex, so $\mathrm{ex}^*_2(n, K_3) \geq \mathrm{ex}(n, K_3) + n/2$. On the other hand, even under the fairly weak assumption of $m = o(n)$, we can show that $\mathrm{ex}^*_m(n, K_r)$ and $\mathrm{ex}(n, K_r)$ are asymptotically equal. We claim that for every $\epsilon > 0$ there is $\delta > 0$ so that $\mathrm{ex}^*_m(n, K_r) < \mathrm{ex}(n, K_r) + \epsilon n^2$ for $m < \delta n$. To see this, we use the following well-known fact, which follows from an inequality of Moon and Moser [27]: there is a function $f_r : \mathbb{R}^+ \to \mathbb{R}^+$ so that any graph on $n$ vertices with at least $\mathrm{ex}(n, K_r) + \epsilon n^2$ edges contains at least $f_r(\epsilon)n^r$ copies of $K_r$. On the other hand, it is not hard to see that there is a constant $C_r$ so that, in any graph on $n$ vertices in which each colour appears at most $\delta n$ times at each vertex, there are fewer than $C_r\delta n^r$ copies of $K_r$ that are not rainbow. Therefore the claim holds with $\delta < f_r(\epsilon)/C_r$.

Finally, for the convenience of the reader we conclude by collecting what we consider to be the main open problems from this paper.

**Problems.**

1. Does a properly edge-coloured graph on $n$ vertices with $n^{1+\epsilon}$ edges contain a rainbow cycle? How about with $Cn \log n$ edges?
2. Is $\mathrm{ex}^*(n, C_{2k}) = O(n^{1+1/k})$?
3. Is $\mathrm{ex}^*(n, C_4) \sim \frac{1}{2}n^{3/2}$?
4. Does a properly edge-coloured graph on $n$ vertices with average degree $d \geq k$ have at least $nk!\binom{d}{k}$ rainbow walks of length $k$?
5. Find the largest $c(k)$ such that a clique with $c(k)$ vertices can be properly coloured with no rainbow path of length $k$.

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