Erdős-Ko-Rado in Random Hypergraphs

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Abstract

Let $3 \leq k < n/2$. We prove the analogue of the Erdős-Ko-Rado theorem for the random k-uniform hypergraph $G^k(n,p)$ when $k < (n/2)^{1/3}$; that is, we show that with probability tending to 1 as $n \to \infty$, the maximum size of an intersecting subfamily of $G^k(n,p)$ is the size of a maximum trivial family. The analogue of the Erdős-Ko-Rado theorem does not hold for all p when $k \gg n^{1/3}$.

We give quite precise results for $k < n^{1/2-\varepsilon}$. For larger k we show that the random Erdős-Ko-Rado theorem holds as long as p is not too small and fails to hold for a wide range of smaller values of p. Along the way, we prove that every nontrivial intersecting k-uniform hypergraph can be covered by k^2-k+1 pairs, which is sharp as evidenced by projective planes. This improves upon a result of Sanders [7]. Several open questions remain.

1 Introduction

A *k-graph* with vertex set V is a collection of k-element subsets of V; these subsets are called edges. We say that a k-graph is *intersecting* if every two edges have nonempty

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intersection; it is *trivial* if there is a fixed element that lies in all edges. The Erdős-Ko-Rado theorem [5] is one of the basic results in extremal set theory. It states that every intersecting k-graph on n vertices, for $n > 2k$, has at most $\binom{n-1}{k-1}$ edges, and equality holds if and only if it is trivial. Our goal in this paper is to extend this theorem to the random setting. Analogous investigations have been carried out for Turán's theorem, see for example [6, 8]. Intersecting hypergraphs generated by a random greedy process were studied in [3, 4].

Let $G^k(n,p)$ be the k-graph on vertex set $[n]$ where each edge is included independently with probability p. Technically, $G^k(n,p)$ is of course a probability space whose elements are k-graphs on $[n]$, where a particular k-graph with e edges has probability $p^e(1-p)^{\binom{n}{k}-e}$. Throughout this paper, whenever we say that $G^k(n,p)$ satisfies some property S we mean that S holds with high probability (w.h.p.) in $G^k(n,p)$, or more precisely that

$$\mathbb{P}(G^k(n,p) \text{ satisfies } S) \to 1 \text{ as } n \to \infty.$$

With few exceptions, all asymptotic statements or limits are to be taken as $n \to \infty$, and in particular, $a(n) \ll b(n)$ means that $a(n)/b(n) \to 0$, and $a(n) \sim b(n)$ that $\lim a(n)/b(n), \lim b(n)/a(n) > 0$. Please note that this is somewhat nonstandard use of the notation $\sim$. For simplicity, we will write G_p^k for $G^k(n,p)$.

Definition. Let H be a k-graph. Then $i(H)$ is the maximum number of edges in an intersecting subhypergraph of H. Say that H *satisfies (strong) EKR* if $i(H)$ is achieved only by a trivial subhypergraph. If $i(H)$ is achieved by a trivial subhypergraph and possibly by some nontrivial subhypergraphs as well, then say that H *satisfies weak EKR.*

The motivation for this definition is that the Erdős-Ko-Rado theorem says that for $n > 2k$, the complete k-graph satisfies EKR. In this paper we are interested in the following question:

Given $2 \le k < n/2$, for which functions $p = p(n) \in [0,1]$ does G_p^k satisfy EKR?

The case $k = 2$ is easy to solve by standard results for the random graph $G_p = G_p^2$. A graph is intersecting if and only if it is a triangle or a star. Now if $p \gg 1/n^{5/4}$, then G_p contains a vertex of degree at least 4 (see Bollobás [2] for example), on the other hand, if $p \ll 1/n$, then G_p has no triangle. So for $k = 2$ the strong EKR always holds. We henceforth assume that $k \ge 3$.

A slightly weaker statement than saying that G_p^k satisfies EKR is the statement that $i(G_p^k) = (1+o(1))p\binom{n-1}{k-1}$. In this case we say that G_p^k satisfies EKR asymptotically. Indeed, as long as p is not terribly small, we know that G_p^k has a trivial subhypergraph of size $(1+o(1))p\binom{n-1}{k-1}$, so we can conclude that $i(G_p^k) \geq (1+o(1))p\binom{n-1}{k-1}$. In certain ranges of k and p, we will only prove this asymptotic statement. Our first result essentially settles the case when $k < n^{1/2-\varepsilon}$.

Theorem 1. *Let $p = p(n) \in [0,1]$, $\rho := p\binom{n-1}{k-1}$ and $0 < \varepsilon < 1/3$ be a fixed constant. We have the following results:*

(i) If $k \ll n^{1/4}$ then G_p^k satisfies (strong) EKR.

(ii) If $k \ll n^{1/3}$ then G_p^k satisfies weak EKR. Furthermore, if $n^{1/4} \ll k \ll n^{1/3}$ then G_p^k satisfies strong EKR if $\rho \ll k^{-1}$ or $n^{-1/4} \ll \rho$, and does not satisfy strong EKR when $k^{-1} \ll \rho \ll n^{-1/4}$.

(iii) In the range $n^{1/3} \ll k \leq n^{1/2-\varepsilon}$ we have the following negative result: For every integer $t \geq 3$ if $n^{1/2-1/(2t)} \ll k$, and

$$n^{(t-3)/2} \cdot k^{2-t} \ll \rho \ll n^{-1/t}$$

then G_p^k does not satisfy EKR. If $n^{(1/2)(1-(t-1)/(t+1)(t-2))} \ll k$ and

$$n^{(t-3)/2} \cdot k^{2-t} \ll \rho \ll n^{-1/(t+1)}$$

then G_p^k does not satisfy weak EKR.

(iv) If $k \leq n^{1/2-\varepsilon}$ and $\rho = \Omega(1)$, then G_p^k satisfies EKR.

The main structural obstacles to the EKR property in G_p^k for $k \leq n^{1/2-\varepsilon}$ are collections $\mathcal{F}$ of t sets, where t is a constant, which have pairwise nonempty intersections but $X \cap Y \cap Z = \emptyset$ for distinct $X, Y, Z \in \mathcal{F}$. It turns that for $n^{1/4} \ll k \ll n^{1/2-\varepsilon}$ there are ranges of p for which such structures appear before vertices of degree $t+1$ or even before vertices of degree t. This is the main observation in parts (ii) and (iii) of Theorem 1. The proof of Theorem 1 can be found in Section 3. Part (i) is proved in Sections 3.1, 3.3.2 and 3.3.5, part (ii) is proved in Sections 3.3.3 and 3.3.5, part (iii) is proved in Section 3.3.4, part (iv) is proved in Sections 3.2 and 3.3.5.

Our next result applies for a much larger range of k, but our requirement on p is also much stronger, i.e., p has to be much larger too.

Theorem 2. *Let* $\varepsilon = \varepsilon(n) > 0$. *If* $\log n \ll k < (1-\varepsilon)n/2$ *and* $p \gg (1/\varepsilon)\sqrt{\frac{\log n}{k}}$, *then* G_p^k *satisfies EKR asymptotically, i.e.*

$$i(G_p^k) \leq (1+o(1))p\binom{n-1}{k-1}.$$

The proof of Theorem 2 appears in Section 4. Note that the smallest useful value of ε in Theorem 2 is roughly $\varepsilon = \sqrt{\frac{\log n}{n}}$, and this establishes asymptotic EKR for G_p^k with p a constant for $k \leq \frac{n}{2} - \omega(\sqrt{n \log n})$. The fact that as k gets closer to $n/2$, the requirement on p in order for G_p^k to satisfy EKR (or its other forms) becomes stronger is no coincidence. Indeed, as the next result shows, if k is much larger than $\sqrt{n}$, then G_p^k fails to even satisfy EKR asymptotically for a rather large range of p.

Theorem 3. *Let* $\sqrt{n \log\log n} \ll k < n/2$ *and*

$$\frac{\log n}{\binom{n-1}{k}} \ll p \ll \frac{e^{k^2/2n}}{\binom{n}{k}}.$$

Then G_p^k *is nontrivial and intersecting. In particular,* $i(G_p^k) = (1+o(1))p\binom{n}{k}$ *and* G_p^k *does not satisfy EKR asymptotically.*

The proof of Theorem 3 appears in Section 5.

For $k \gg \sqrt{n}$ there is a wide range of 'intermediate' values of p for which we have not established any results. For these values G_p^k is not intersecting itself and there could be large intersecting subfamilies that are not trivial. It would be interesting to determine whether or not such structures persist over wide ranges of p. A specific case of this general question is the following

Question 4. *Let* $n/2 - \sqrt{n} < k < n/2$ *and* $p = \frac{99}{100}$. *Does* G_p^k *satisfy EKR or weak EKR?*

See Remark 3 in Section 5 below for further discussion of this question.

2 Concentration facts

Our main tools will be some variants of Chernoff's inequality (see the Appendix of [1] for more details).

Theorem 5. *Let $X_1, \ldots, X_m$ be independent $\{0,1\}$ random variables with $P(X_i = 1) = q$ for each i. Let $X = \sum_i X_i$. Then the following inequalities hold, where $a > 0$:*
(1) $\mathbb{P}(X > \mathbb{E}X + a) < \exp(-2a^2/m)$.
(2) Let $r \geq m$. Then $\mathbb{P}(X > 3qr/2) < \exp(-qr/16)$.

We can immediately conclude some properties of G_p^k. Recall that the degree of vertex v in a hypergraph H is denoted $d_H(v)$. More generally, the degree of a set S of vertices is the number of edges containing S and is denoted $d_H(S)$. We omit the standard proof of the following Lemma.

Lemma 6. *Let $k = k(n) \geq 3$. The following hold in G_p^k:*
(1) Let $p \gg \log n / \binom{n-2}{k-2}$. Then $d_{G_p^k}(A) = (1 + o(1))p\binom{n-2}{k-2}$ for every $A \in \binom{[n]}{2}$.
(2) Let $p \gg \log n / \binom{n-3}{k-3}$. Then for every $A, B \in \binom{[n]}{2}$, with $A \cap B = \emptyset$, the number of edges containing A and intersecting B is at most $3p\binom{n-3}{k-3}$.

3 Small k

In this section we give the proof of Theorem 1. We partition the argument into several cases, depending on the value of p and k. This does not correspond to a proper partition of the collection of values of the parameters p, k as some possible values of p, k are covered by more than one of the following subsections.

3.1 $p \gg \frac{\log n}{k\binom{n-2}{k-2}}$ and $k < (n/2)^{1/3}$

Our main tool for this range of p and k is a result about covering nontrivial intersecting families by pairs. Actually, we could use a result due to Sanders [7] for our purposes, but the improvement below is quite simple. Since it is also best possible, as evidenced by projective planes, we believe it may be of independent interest. Recall that for a hypergraph $\mathcal{H}$ and a subset of vertices A we let $d_{\mathcal{H}}(A) := |\{E \in \mathcal{H} : A \subset E\}|$. We say that $\mathcal{H}$ is *covered* by a graph G if for every hyperedge E of $\mathcal{H}$ there is an edge uv of G such that $\{u, v\} \subset E$.

Lemma 7. *For all n, k there is a collection of at most $\binom{n}{k}^3$ graphs on vertex set $[n]$ such that every graph G in this collection has $|E(G)| \leq k^2 - k + 1$ and $\Delta(G) \leq k$ and every nontrivial intersecting k-graph on vertex set $[n]$ is covered by some graph G in this collection.*

Proof. Let $\mathcal{H}$ be a nontrivial intersecting k-graph. Pick two edges A, B of $\mathcal{H}$ with minimum intersection size, say a and let $C := A \cap B$. First consider the case when $a > 1$. Let G be the complete bipartite graph with parts $A - C$ and B, together with (any) one edge e within C. Then G has $k(k-a)+1 < k^2 - k + 1$ edges. Now let us show that each edge D of $\mathcal{H}$ is covered by G. If D has a vertex in $A - C$, then, since it intersects B, it must also have a point in B, so it is covered. Otherwise, by the minimality of a, it must contain all of C and is covered by e. The maximum degree of G is k.

Now suppose that $|C| = a = 1$. Put $C = \{c\}$ and let E be an edge of $\mathcal{H}$ that misses C. Note that E exists because $\mathcal{H}$ is nontrivial. Let G be the union of the complete bipartite graph with parts $A - c$ and $B - c$ and the star with center c and leaves in E. Then G has $(k-1)^2 + k = k^2 - k + 1$ edges, and maximum degree k. Now pick any $D \in \mathcal{H}$ and let us show that it is covered by G. If D does not contain c, then it has a point in both $A - c$ and $B - c$ and is covered by the complete bipartite subgraph of G. Otherwise, D contains c, and since $\mathcal{H}$ is intersecting, D must contain some point of E, and so it is covered by the star in G.

The bound on the number of possible choices for G follows from the fact that G is determined by choosing at most 2 hyperedges and a graph edge (A, B, e), or by choosing 3 hyperedges. □

Proof of Theorem 1 when $p \gg \log n/(k\binom{n-2}{k-2})$.
Let $3 \le k \le (n/2)^{1/3}$ and let $\mathcal{H} \subseteq G_p^k$ be a nontrivial intersecting k-graph. For this $\mathcal{H}$ there is a covering graph G by Lemma 7, where the number of choices for G is less than $\binom{n}{k}^3$. Fix such a graph G. Let m be the number of edges of $\binom{[n]}{k}$ covered by G, which is at most $r := k^2\binom{n-2}{k-2}$. We apply Theorem 5 part (2), where $q = p$ and $X_1, \ldots, X_m$ are the indicator random variables for the m covered hyperedges, and $X = \sum_{i=1}^m X_i$ is the number of these edges. We obtain

$$\mathbb{P}\left(X > 3pr/2\right) < \exp\left(-pr/16\right). \tag{1}$$

So if $\binom{n}{k}^3 \ll \exp(pr/16)$, or $\log n/(k\binom{n-2}{k-2}) \ll p$, then by using the union bound, we conclude that every nontrivial intersecting k-graph in G_p^k has size at most $1.5pr$ w.h.p. Since $k < (n/2)^{1/3}$, this is smaller than the size of a trivial k-graph in G_p^k:

$$1.5pk^2\binom{n-2}{k-2} < \frac{1.5pk^3}{n}\binom{n-1}{k-1} < (1-o(1))p\binom{n-1}{k-1}.$$

We conclude that G_p^k satisfies EKR. □

3.2 $p \gg \frac{k \log n}{\binom{n-3}{k-3}}$ and $1 \ll k < n^{1/2-\varepsilon}$

In this section, we extend the proof in the previous section to $k \ll n^{1/2-\varepsilon}$. Our goal is to prove the following result.

Theorem 8. *Fix $0 < \varepsilon < 1/10$ and $f := \lceil (1/2+\varepsilon)/(2\varepsilon) \rceil + 1$. Suppose that*

$$f \ll k < n^{1/2-\varepsilon} \quad \text{and} \quad p \gg \frac{fk \log n}{\binom{n-3}{k-3}}.$$

Then G_p^k satisfies EKR.

Our first subsection contains some preparation for the proof.

3.2.1 Flowers

The common part of a collection of sets (or edges) $\mathcal{F}$ is $\cup_{A,B \in \mathcal{F}} A \cap B$. We denote this by $C = C(\mathcal{F})$. Sometimes we will call $\mathcal{F}$ a *flower* and C its *core*. The edges of a flower will be called *petals*. Call a flower *degenerate* if one of its petals is contained in the core, and *non-degenerate* otherwise. Let us recall that any statements made about parameters depending on G_p^k should be interpreted as statements that hold w.h.p.

Lemma 9. *Let n, k, p, f be parameters satisfying the conditions of Theorem 8. Then G_p^k satisfies the following property: for every $\mathcal{F} \in \binom{\binom{[n]}{k}}{f}$, the number of edges in G_p^k intersecting every edge of $\mathcal{F}$ and missing $C(\mathcal{F})$ is $o(p\binom{n-1}{k-1})$.*

Proof. For a fixed flower $\mathcal{F}$ with f petals and core C, let $X_{\mathcal{F}}$ be number of edges of G_p^k intersecting each member of $\mathcal{F}$ and avoiding C. As $X_{\mathcal{F}} = 0$ for degenerate $\mathcal{F}$, we shall assume from now that $\mathcal{F}$ is non-degenerate. Fix an arbitrarily small constant $\gamma > 0$ and let $r := \gamma\binom{n-1}{k-1}$. Then

$$\mathbb{E}(X_{\mathcal{F}}) \le pk^f \binom{n-f}{k-f} \le p\frac{k^{2f-1}}{n^{f-1}} \binom{n-1}{k-1} \le pr,$$

for n sufficiently large. By Theorem 5 part (2)

$$\mathbb{P}(X_{\mathcal{F}} > 3pr/2) < \exp(-pr/16).$$

The number of choices for $\mathcal{F}$ is at most $\binom{n}{k}^f \le (en/k)^{kf}$. So if $(en/k)^{kf}\exp(-pr/16) = o(1)$ then the probability that there exists a non-degenerate $\mathcal{F}$ such that the number of edges in G_p^k intersecting every edge of $\mathcal{F}$ and missing $C(\mathcal{F})$ is more than $3pr/2$ is $o(1)$. However this condition holds if $p \gg fk\log n/\binom{n-1}{k-1}$. $\square$

Lemma 10. *Let n, k, p, f be parameters satisfying the conditions of Theorem 8. Then every maximum intersecting family $\mathcal{H} \subset G_p^k$ with $d_{\mathcal{H}}(v) = \Omega\left(p\binom{n-1}{k-1}\right)$ for some vertex v is trivial.*

Proof. Let $\mathcal{H} \subseteq G_p^k$ be an intersecting family of maximum size. Let $\mathcal{F} \subset \mathcal{H}$ be a maximum flower over all flowers with core $\{v\}$. Let q denote the number of its petals. Note that it could be that $\mathcal{F}$ consists of only one hyperedge. The number of such flowers is at most $n\binom{n-1}{k-1}^q$.

Let us first suppose that $q < f$. First we will show the following fact: for every flower with $q < f$ petals and core consisting of v, the number of edges in G_p^k containing v and hitting one of the petals outside v is $o(p\binom{n-1}{k-1})$. For a fixed such flower, the expected number of edges in G_p^k satisfying the condition above is at most

$$pq(k-1)\binom{n-2}{k-2} = pq\frac{(k-1)^2}{n-1}\binom{n-1}{k-1} = o\left(p\binom{n-1}{k-1}\right),$$

since $k < n^{1/2-\varepsilon}$. Noting that we can apply Theorem 5 part (2) to each such flower and that the number of such flowers is at most $n\binom{n-1}{k-1}^f$, the fact follows by the choice of p. Now since $\mathcal{F}$ is a maximum flower, every edge of $\mathcal{H}$ containing v must intersect one of the petals outside v, and so $d_{\mathcal{H}}(v) = o\left(p\binom{n-1}{k-1}\right)$. This contradicts the hypothesis.

We may therefore assume that $q = f$ (if $q \ge f$, simply delete some petals; we will not be using maximality of $\mathcal{F}$ anymore). In order to prove that $\mathcal{H}$ is a trivial family, we shall refine the computation from the proof of Lemma 9. The expected size of a non-trivial intersecting family in G_p^k containing flower $\mathcal{F}$ and some edge E missing v is

$$p\left[k\binom{n-2}{k-2} + (k-1)^f\binom{n-f}{k-f}\right] + f. \tag{2}$$

Explanation: The first term upper estimates the number of edges containing v and intersecting E, and the second term counts the number of edges not containing v but intersecting each of the petals in a different vertex, the edges of $\mathcal{F}$ are not counted.

We may count nontrivial intersecting families for each vertex v and an edge E missing

v. The number of ways to choose (v, E) and f petals is at most

$$n\binom{n-1}{k}\binom{n-1}{k-1}^{f} < n^2\binom{n-1}{k-1}^{f+1}.$$

We apply Theorem 5 part (2) with $r = \gamma\binom{n-1}{k-1}$, where $\gamma > 0$ is small. Noting that $k\binom{n-2}{k-2} + (k-1)^f\binom{n-f}{k-f} = o(\binom{n-1}{k-1})$, to prove that w.h.p. every non-trivial intersecting hypergraph has at most $3r/2$ hyperedges, it suffices to have $p \gg fk\log n/\binom{n-1}{k-1}$, which is guaranteed by the choice of p. We conclude that a maximum intersecting family is trivial. □

3.2.2 Proof of Theorem 8

In this subsection, we will complete the proof of Theorem 8. We will use a classical result in graph theory due to Vizing. It states that every graph with maximum degree Δ has a proper edge-coloring with $\Delta + 1$ colors. Taking a color class of largest size, we also deduce that there is a matching of size at least $e/(\Delta + 1)$, where e is the number of edges of the graph.

Proof of Theorem 8. Let $\mathcal{H} \subseteq G_p^k$ be a non-trivial intersecting k-graph. Note that we may assume (by Lemma 10) that for every vertex v we have $d_{\mathcal{H}}(v) = o\left(p\binom{n-1}{k-1}\right)$. We prove that under this assumption $|\mathcal{H}|$ is smaller than the size of a trivial intersecting k-subhypergraph of G_p^k, therefore $\mathcal{H}$ cannot have maximum size.

Using Lemma 7 we can construct a graph $G = (V, E)$ with vertex set $V \subset [n]$ such that $\Delta(G) \leq k$, $|E| \leq k^2$ and for every $X \in \mathcal{H}$ there is an $e \in E$ such that $e \subset X$. Set $D := 10fp\binom{n-3}{k-3}$. Define

$$E_1 := \{e \in E : d_{\mathcal{H}}(e) \leq D\} \quad \text{and} \quad E_2 := E \setminus E_1.$$

Case 1. There exists a matching $M \subseteq E_2$ such that $|M| \geq f$.
Let $e_1, \ldots, e_f \in E_2$ form a matching. Our aim is to find for every $i \in [f]$ an edge $A_i \in \mathcal{H}$ such that

- $e_i \subset A_i$, for all i,
- $A_i \cap e_j = \emptyset$ for all $i < j$,
- $|A_j \cap A_i| < f$ for $i \neq j$.

For every i, since $d_{\mathcal{H}}(e_i) > D$, for a fixed $j > i$ by Lemma 6 part (2) there are at most $3p\binom{n-3}{k-3}$ edges $A \in \mathcal{H}$ with $e_i \subset A$ and $e_j \cap A \neq \emptyset$. So by the choice of D, for every i at least half of the edges containing e_i are disjoint for every $j > i$ from e_j. In particular, we can choose A_1 with the property that

$$A_1 \cap e_j = \emptyset \quad \text{for } j = 2, \dots, f.$$

Assuming that we have constructed edges $A_1, \dots, A_{i-1}$, we consider the at least $D/2$ edges that contain e_i and are disjoint from e_j for $j > i$. To satisfy our third condition, observe that for a given $j < i$ the number of k-sets in $\binom{[n]}{k}$ intersecting A_j in at least f vertices and containing e_i is at most

$$k^f \binom{n-f-2}{k-f-2} < \frac{k^{2f-1}}{n^{f-1}} \binom{n-3}{k-3} = o\left(\binom{n-3}{k-3}\right).$$

Now suppose $\delta > 0$ is an arbitrarily small constant. Theorem 5 part (2) implies that the probability that there are more than $(3p/2)\delta\binom{n-3}{k-3}$ of these k-sets in G_p^k is at most $\exp[-p\delta\binom{n-3}{k-3}/16]$. As the number of ways $A_1, \dots, A_f$ can be chosen is at most $\binom{n}{k}^f$, we have the third condition w.h.p. if $\binom{n}{k}^f \exp[-p\delta\binom{n-3}{k-3}/16] = o(1)$ which is satisfied with our choice of parameters.

We conclude that the number of such edges is smaller than $D/2f$, implying that there is an A_i satisfying our requirements.

The edges $A_1, \dots, A_f$ form a flower with core C where

$$|C| \leq \binom{f}{2} f.$$

As we assumed that for every vertex v we have $d_{\mathcal{H}}(v) = o(p\binom{n-1}{p-1})$, applying Lemma 9 we have

$$|\mathcal{H}| \leq \sum_{v \in C} d_{\mathcal{H}}(v) + o\left(p\binom{n-1}{k-1}\right) = o\left(p\binom{n-1}{k-1}\right).$$

Case 2. Every matching $M \subseteq E_2$ satisfies $|M| < f$.
It follows from Vizing's Theorem that we have $|E_2| < (k+1)f$. Therefore, using Lemma 6 part (1)

$$\begin{aligned}
|\mathcal{H}| &\leq \sum_{e \in E_1} d_{\mathcal{H}}(e) + \sum_{e \in E_2} d_{\mathcal{H}}(e) \\
&\leq k^2 10 f p \binom{n-3}{k-3} + |E_2| p \binom{n-2}{k-2} (1 + o(1)) \\
&\leq \frac{p}{4} \binom{n-1}{k-1} + \frac{p}{4} \binom{n-1}{k-1}.
\end{aligned}$$

Since this is much smaller than the size of the trivial intersecting families, the proof is complete. □

3.3 Small p

We now focus on the cases of small p. Throughout this section we will assume that $0 < \varepsilon < 1/3$ and $3 \leq k < n^{1/2-\varepsilon}$.

We define ρ by

$$p = \frac{\rho}{\binom{n-1}{k-1}}.$$

In other words ρ is the expected number of edges that contain a particular vertex. We are interested in the case where ρ is relatively small. In the previous sections we assumed $p \gg \log n/(k\binom{n-2}{k-2})$ and $p \gg \frac{fk\log n}{\binom{n-3}{k-3}}$ so in this section we will assume that

$$p \ll k\log^2 n/\binom{n-3}{k-3}.$$

This covers all $p \in [0,1]$ for n sufficiently large. We may therefore assume that

$$\rho \leq \frac{n^2\log^2 n}{k}. \tag{3}$$

We shall use the following identity several times:

$$p\binom{n-2}{k-2} = \rho\frac{k-1}{n-1}.$$

Before we begin our arguments, we will need various facts about random hypergraphs. These are collected and proved in the next subsection.

3.3.1 More concentration facts

In order to estimate certain probabilities in the proof, we will use the Janson inequalities which we now describe.

Setup. Let Ω be a finite universal set. Let R be a random subset of Ω with $\mathbb{P}(r \in R) = p_r$ and these events are mutually independent over $r \in \Omega$ (usually $R = G_p^k$). For a finite index set I and $i \in I$, let $A_i \subset \Omega$ and B_i be the event $A_i \subset R$. For $i, j \in I$, write $i \sim j$ if $i \neq j$ and $A_i \cap A_j \neq \emptyset$. Define

$$\Delta = \sum_{i \sim j} \mathbb{P}(B_i \wedge B_j),$$

where the sum is over all ordered pairs. Put

$$M = \prod_{i \in I} \mathbb{P}(\overline{B_i}) \qquad \text{and} \qquad \mu = \sum_{i \in I} \mathbb{P}(B_i).$$

Theorem 11. *Let B_i, I, Δ, M be as in the setup above and assume that $\mathbb{P}(B_i) \leq \epsilon$ for every $i \in I$. Then*

$$M \leq \mathbb{P}\left(\bigwedge_{i \in I} \overline{B_i}\right) \leq \exp\left\{-\mu + \frac{\Delta}{2(1-\epsilon)}\right\}.$$

If in addition we have $\Delta \geq \mu(1-\epsilon)$, then

$$\mathbb{P}\left(\bigwedge_{i \in I} \overline{B_i}\right) \leq \exp\left\{\frac{-\mu^2(1-\epsilon)}{2\Delta}\right\}.$$

The lower bound in Theorem 11 follows from the FKG inequality; for a proof of the Janson Inequalities and further discussion see Alon and Spencer [1].

We begin with an easy (though tedious) consequence of the Janson Inequalities.

Lemma 12. *Let $d \geq 3$ be a constant. If $\rho \ll n^{-1/d}$ then G_p^k does not have a vertex of degree d and if $\rho \gg n^{-1/d}$ then G_p^k has a vertex of degree d.*

Proof. We apply Theorem 11. Let $\mathcal{A}$ be the set of collection of ordered pairs consisting of a vertex $v \in [n]$ and a collection of d sets in $\binom{[n]}{k}$ that contain v. For each $A_i \in \mathcal{A}$ let B_i be the event $A_i \subseteq G_p^k$. Note that

$$\mathbb{P}(B_i) = \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^d$$

so by the choice of ρ, we conclude that $\mathbb{P}(B_i)$ tends to 0 as n goes to ∞.

We have that the expected number of B_i's

$$\mu = n\binom{\binom{n-1}{k-1}}{d}\left[\frac{\rho}{\binom{n-1}{k-1}}\right]^d \sim n\frac{\rho^d}{d!}.$$

So, if $\rho \ll n^{-1/d}$ then

$$\mathbb{P}\left(\wedge \overline{B_i}\right) \geq \prod_i \left(1 - \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^d\right) \sim e^{-\mu} \to 1$$

proving the first assertion of the theorem. For the second assertion, we estimate Δ.

$$\Delta \leq 2n \sum_{i=1}^{d-1} \binom{\binom{n-1}{k-1}}{2d-i} \binom{2d-i}{d}^2 \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^{2d-i} + n^2 \sum_{i=1}^{d-1} \binom{\binom{n-2}{k-2}}{i} \binom{\binom{n-1}{k-1}}{d-i}^2 \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^{2d-i}$$

$$\sim 2n\left(\frac{\rho^d}{d!}\right)^2 \left[\sum_{i=1}^{d-1} \frac{d!}{\rho^i(2d-i)\cdots(d+1)} \binom{2d-i}{d}^2\right] + \left(n\frac{\rho^d}{d!}\right)^2 \left[\sum_{i=1}^{d-1} \frac{(d!)^2}{i![(d-i)!]^2}\left(\frac{k}{\rho n}\right)^i\right]$$

$$\sim 2\mu^2 \left[\sum_{i=1}^{d-1} \frac{d!}{n\rho^i(2d-i)\cdots(d+1)} \binom{2d-i}{d}^2\right] + \mu^2 \left[\sum_{i=1}^{d-1} \frac{(d!)^2}{i![(d-i)!]^2}\left(\frac{k}{\rho n}\right)^i\right].$$

Note that $n^{-1/d} \ll \rho$ implies that $\mu \to \infty$. If $\Delta < \mu(1-\epsilon)$, then the first part of Theorem 11 implies that

$$\mathbb{P}\left(\wedge \overline{B_i}\right) \leq e^{-\mu + \frac{\Delta}{2(1-\epsilon)}} \to 0.$$

We may therefore suppose that $\Delta \geq \mu(1-\epsilon)$. Since $d \geq 3$, we have $n^{1-1/d} \geq n^{2/3} \gg k$. Therefore $\rho \gg n^{-1/d} \gg k/n$ and $\rho^i n \gg 1$ for $i \leq d-1$. This immediately gives $\Delta = o(\mu^2)$ and then we can apply the second part of Theorem 11:

$$\mathbb{P}\left(\wedge \overline{B_i}\right) \leq e^{-\frac{\mu^2(1-\epsilon)}{2\Delta}} \to 0.$$

This completes the proof of the claim. □

A triplet of hyperedges (A, B, C) is a *triangle* if it is a pairwise intersecting system, but $A \cap B \cap C = \emptyset$. Now we consider nontrivial intersecting families. For integers $1 < j < t$ define a (t, j)-**simplex** to be a collection $\mathcal{A} \subset \binom{[n]}{k}$ such that

- $|\mathcal{A}| = t$,
- the intersection of every j sets from $\mathcal{A}$ is nonempty,
- the intersection of every $j+1$ sets from $\mathcal{A}$ is empty.

So, for example, a triangle is a $(3, 2)$-simplex.

We will employ the following observation several times in what follows. Let $\mathcal{F}$ be a nontrivial intersecting family. Define $s_{\mathcal{F}}$ to be the size of a minimal subfamily M of $\mathcal{F}$ with empty intersection. Then clearly M is a $(j+1, j)$-simplex for $j = s_{\mathcal{F}} - 1$.

Now let $\mathcal{M}'$ be a maximal (t,j)-simplex in $\mathcal{F}$ with maximum possible t. Note that $\mathcal{M}'$ has the following property: Every set in $\mathcal{F}$ intersects one of the $\binom{t}{j}$ intersections of j sets from $\mathcal{M}'$. Indeed, each set $X \in \mathcal{F}$ intersects every intersection of $(j-1)$ sets from $\mathcal{M}$ for the choice of j ensures that every j members of $\mathcal{F}$ have nonempty intersection. Consequently, if $X \in \mathcal{F}$ misses each of the $\binom{t}{j}$ intersections of j sets from $\mathcal{M}'$, then we could add X to $\mathcal{M}'$.

Our main observation about non-trivial intersecting families in G_p^k for p in this range is a lower bound on the threshold function of the appearance of a (t,j)-simplex.

Lemma 13. *Let $1 < j < t$ be fixed constants. If*

$$\rho \ll \frac{n^{\binom{t-1}{j-1}\left(1-\frac{1}{j}\right)-1}}{k^{\binom{t-1}{j-1}-1}}$$

then G_p^k does not have a (t,j)-simplex.

Proof. We compute an upper bound on the expected number (t,j)-simplices. The $\binom{t}{j}$ non-empty intersections in a (t,j)-simplex determine $\binom{t}{j}$ pairwise disjoint sets, since if two of these sets intersect, then we obtain at least $j+1$ members of the (t,j)-simplex with nonempty intersection. We begin our expected computation by picking one point from each of these $\binom{t}{j}$ sets. Note that this specifies exactly $\binom{t-1}{j-1}$ elements of each set in the simplex. We allow the remaining $k-\binom{t-1}{j-1}$ elements of each set in the simplex to be chosen arbitrarily. Thus, the expected number of (t,j)-simplices is at most

$$\begin{aligned}
&n^{\binom{t}{j}} \binom{n}{k-\binom{t-1}{j-1}}^t \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^t \\
&\leq n^{\binom{t}{j}} \left[\frac{n}{(n-k+\binom{t-1}{j-1})\cdot \cdots \cdot (n-k+1)} \cdot \frac{(k-1)!}{(k-\binom{t-1}{k-1})!} \cdot \rho\right]^t \\
&\leq 2n^{\binom{t}{j}} \left[\left(\frac{k}{n}\right)^{\binom{t-1}{j-1}-1} \rho\right]^t \\
&= o(1),
\end{aligned}$$

using $n^{\binom{t}{j}/t} \leq n^{\binom{t-1}{j-1}/j}$. □

Lemma 14. *If $\rho \ll 1/k$ then G_p^k does not contain a triangle. If $\rho \gg 1/k$ then G contains a triangle.*

Proof. We apply Theorem 11 again. The expected number of triangles, μ satisfies

$$\binom{n}{3}\binom{n-3}{k-2}\binom{n-3-k}{k-2}\binom{n-3-2k}{k-2}p^3 \leq \mu \sim \binom{n}{3}\binom{n-3}{k-2}^3\left[\frac{\rho}{\binom{n-1}{k-1}}\right]^3 \sim \rho^3 k^3.$$

This already proves the statement for $\rho \ll 1/k$. For the other part, note that the ratio of the lower and upper bound on μ tends to a constant. We also have

$$\Delta \leq n^4\binom{n-2}{k-2}^3 k^2\binom{n-3}{k-2}^2\left[\frac{\rho}{\binom{n-1}{k-1}}\right]^5 + n^5\binom{n-3}{k-3}^2\binom{n-2}{k-2}^2\left[\frac{\rho}{\binom{n-1}{k-1}}\right]^4 = o(\mu^2).$$

The first term in our bound on Δ comes from pairs of triangles that share 1 hyperedge while the second term comes from pairs of triangles that share 2 hyperedges; note that the first term dominates. By Theorem 11 we are done. □

We will also need the following generalization of Lemma 14.

Lemma 15. *Suppose $t \geq 4$ is fixed and $n^{1/2-1/(2t-4)} \ll k < n^{1/2-\varepsilon}$. If $n^{(t-3)/2}k^{2-t} \ll \rho = o(1)$, then G_p^k contains a $(t,2)$-simplex.*

Proof. Following the calculation in (4) we have

$$\mu \sim n^{\binom{t}{2}}\left[\left(\frac{k}{n}\right)^{t-2}\rho\right]^t$$

where μ is the expected number of $(t,2)$-simplices. The remainder of the proof is very similar to the proof of Lemma 14 and is therefore omitted. □

3.3.2 $\rho = o(1)$ and $k \ll n^{1/4}$

In this subsection we prove Theorem 1 in the range specified above. We define an a-**extended triangle** to be a triangle B_1, B_2, B_3 together with a sets $B_4, \ldots, B_{3+a}$ such that B_{3+j} intersects at least one of $B_1 \cap B_2, B_1 \cap B_2, B_2 \cap B_3$ and B_{3+j} intersects B_1, B_2, B_3 for $j = 1, \ldots, a$.

A careful calculation using Lemma 13 shows that in this case there is no (t,j)-simplex for $t \geq 4$ in G_p^k. Now any nontrivial intersecting subfamily of G_p^k contains a (t,j)-simplex with $t \geq 3$. We choose a (t,j)-simplex with t maximum possible, i.e, with $t = 3$. Hence every nontrivial intersecting subfamily of G_p^k (of size at least three) contains a triangle with the property that every set in the family intersects one of the three intersections determined by the triangle (else we would have a $(4, j)$-simplex for some j).

We define three events and use the union bound to show that they are rare. Let $\mathcal{A}$ be the event that there exist $e, f \in G_p^k$ such that $|e \cap f| > 2$.

$$\mathbb{P}(\mathcal{A}) \leq \binom{n}{3}\binom{n-3}{k-3}^2 p^2 \leq \frac{\rho^2 n^3}{3!}\left(\frac{(k-1)(k-2)}{(n-1)(n-2)}\right)^2 \leq \frac{\rho^2 k^4}{n} = o(1).$$

Let $\mathcal{B}_a$ be the event that there is an a-extended triangle for $a = 1, 2, 3$. As we have already shown that $\mathbb{P}(\mathcal{A}) = o(1)$, we shall assume $\overline{\mathcal{A}}$. Then we obtain

$$\mathbb{P}(\mathcal{B}_a \wedge \overline{\mathcal{A}}) \leq \binom{n}{3}\binom{n-2}{k-2}^3 6^a k^a \binom{n-2}{k-2}^a \left[\frac{\rho}{\binom{n-1}{k-1}}\right]^{3+a} \sim \frac{\rho^{3+a} k^{3+2a}}{n^a}. \tag{4}$$

Explanation: Once the hyperedges in the triangle are chosen, each additional hyperedge must contain one of the at most 6 vertices in the pairwise intersections of the hyperedges in the triangle (note that we use here our assumption that $\mathcal{A}$ does not hold). We have $\mathbb{P}(\mathcal{B}_2) = o(1)$ and if $\rho \leq n^{-1/16}$ then $\mathbb{P}(\mathcal{B}_1) = o(1)$.

Let us first assume that $\rho \leq n^{-1/16}$. We first argue that the cardinality of a nontrivial intersecting family in G_p^k is at most three. Indeed, every such family contains a (t, j)-simplex with t maximum, and the previous observations imply that $t = 3$. Furthermore, if a nontrivial intersecting family with size at least four has a triangle and no $(4, j)$-simplex, then it must contain a 1-extended triangle, and we have shown that there are no 1-extended triangles. Therefore the largest nontrivial intersecting subfamily of G_p^k has size three (i.e. is a triangle). By Lemmas 12, 14 and $k \ll n^{1/4}$, a triangle does not appear until after a vertex of degree 4 as the threshold for the appearance of a vertex of degree 4 is smaller, and so we conclude that G_k^p has the EKR property when $\rho \leq n^{-1/16}$.

Now consider $\rho > n^{-1/16}$. Since $\mathbb{P}(\mathcal{B}_2) = o(1)$, the cardinality of a nontrivial intersecting family in G_p^k is now at most four (achieved by a 1-extended triangle). By Lemma 12 there is a vertex of degree at least 15 in this range hence we again have the EKR property.

3.3.3 $\rho = o(1)$ and $n^{1/4} \ll k \ll n^{1/3}$

Here there is a small range where (strong) EKR does not hold but we will see below that weak EKR does hold. Recall that weak EKR is the property that there exists a maximum intersecting family that is trivial, and so weak EKR allows for the possibility that there is also a maximum intersecting family that is not trivial. So, in this Section we continue to be very careful about the precise sizes of the various intersecting families that might appear. It will be useful to think of ρ as an increasing parameter, with more complicated structures emerging as ρ grows. Note that a triangle appears at around $\rho = k^{-1}$, a vertex of degree 3 appears at around $\rho = n^{-1/3}$, a vertex of degree 4 appears at around $\rho = n^{-1/4}$ and we have $n^{-1/3} \ll k^{-1} \ll n^{-1/4}$. These observations alone show that there is an interval where EKR does not hold. To complete the proof of the $\rho = o(1)$ portion of part (ii) of Theorem 1, it remains to show that nontrivial intersecting families on ℓ hyperedges with $\ell \geq 4$ do not appear until after vertices of degree $\ell + 1$.

Following the arguments in the previous subsection, pairwise intersections have cardinality at most 3. Note that the only (t, j)-simplex that appears (other than a triangle) is the $(4, 2)$-simplex that can appear at $\rho = n^{1/2}/k^2 \gg n^{-1/6}$, but at that time there is a vertex of degree at least 5. Also, w.h.p. no fifth hyperedge can be attached to a $(4, 2)$-simplex, as the expected number of such structures is bounded by

$$\binom{n}{6}\binom{n-3}{k-3}^4\left[\frac{\rho}{\binom{n-1}{k-1}}\right]^4\left\{18k^2\binom{n-3}{k-3}\frac{\rho}{\binom{n-1}{k-1}} + 18\cdot 3\cdot\binom{n-2}{k-2}\frac{\rho}{\binom{n-1}{k-1}}\right\}$$
$$= O\left(\rho^5\left(\frac{k^{12}}{n^4} + \frac{k^9}{n^3}\right)\right).$$

(The second term in this expression is for the case of a fifth set that intersects two of the pairwise intersections in such a way that inclusion of these two points achieves intersection with all 4 sets in the $(4, 2)$-simplex.) Furthermore, by (4) event $\mathcal{B}_1$ does not happen before $\rho = n^{-1/6}$, $\mathcal{B}_2$ does not happen before $\rho = n^{-1/15}$ and $\mathcal{B}_3$ does not happen. So by the time an extended triangle with 4 edges appears there is a vertex with degree at least 5, and when an extended triangle with 5 edges appears there is a vertex with degree at least 14.

3.3.4 $\rho = o(1)$ and $n^{1/3} \ll k \leq n^{1/2-\varepsilon}$

From Lemmas 12 and 15 we know that a vertex of degree t appears at $\rho \gg n^{-1/t}$, and a $(t, 2)$-simplex at $\rho \gg n^{(t-3)/2} \cdot k^{2-t}$. Then we can conclude that for $n^{1/2-1/(2t)} \ll k$, and for

$$n^{(t-3)/2} \cdot k^{2-t} \ll \rho \ll n^{-1/t}$$

EKR does not hold, as a $(t,2)$-simplex appears earlier than a vertex of degree t. (We needed the lower bound on k given in part (iii) of Theorem 1 to make sure that the range of ρ is not empty.) Similar conclusion can be made regarding not having the weak EKR property: a $(t,2)$-simplex appears earlier than a vertex of degree $t+1$ for $n^{(1/2)(1-(t-1)/(t+1)(t-2))} \ll k$ and

$$n^{(t-3)/2} \cdot k^{2-t} \ll \rho \ll n^{-1/(t+1)}.$$

Note that we do not determine the likely size of the largest intersecting family in G_p^k here as we did in Section 3.3.3. Of course, this simplification is reflected in the fact that we make only negative assertions in part (iii) of Theorem 1 for this range of k and ρ. For the values of k and ρ considered here there are many competing non-trivial intersecting structures which appear earlier than a trivial intersecting family with the same size. The number of such structures grows as k approaches $n^{1/2}$. Even though it seems that usually the $(t,2)$-simplex is the earliest to show up when it counts, verifying this using our methods would involve tedious calculations. So, for the sake of brevity, we did not work out more details.

3.3.5 $\rho = \Omega(1)$

In this subsection we show that the cardinality of a nontrivial intersecting family is at most a constant if ρ is less than a large constant, and at most a small multiple of ρ when ρ is larger than this constant. Let us recall that $k < n^{1/2-\varepsilon}$.

We begin by noting that, by Lemma 13 and (3), there is no (t,j)-simplex in G_p^k for $j \geq 6$; $j = 5$ and $t \geq 7$; $j = 4$ and $t \geq 6$; $j = 3$ and $t \geq 7$; or $j = 2$ and $t \geq 3/\varepsilon > 2/\varepsilon + 3$. Thus, we may restrict our attention to small (t,j)-simplices. We define two events and use the union bound to show that they are rare. Let $\mathcal{A}_r$ be the event that there exist $e, f \in G_p^k$ such that $|e \cap f| \geq r$. Then

$$\mathbb{P}(\mathcal{A}_r) \leq \binom{n}{r}\binom{n-r}{k-r}^2 p^2 \leq \frac{\rho^2 n^r}{r!}\left(\frac{(k-1)_{r-1}}{(n-1)_{r-1}}\right)^2 \leq \frac{\left(2\log^4 n\right) k^{2r-4}}{n^{r-6}} = o(1),$$

when $r > 3/\varepsilon$, using $k < n^{1/2-\varepsilon}$. Fix $r = \lceil 3/\varepsilon \rceil$, and write $\mathcal{A} = \mathcal{A}_r$. Now define

$$a = \begin{cases} 5/\varepsilon & \text{if } \rho \leq 500\varepsilon^{-4} \\ \rho\,\varepsilon^3/100 & \text{if } \rho > 500\varepsilon^{-4} \end{cases}$$

and consider the event $\mathcal{B}$ that there is a vertex x and a k-set A such that $x \notin A$ and A is in G_p^k and there are at least a sets in G_p^k that both contain x and intersect A.

$$\mathbb{P}(\mathcal{B}) \leq np\binom{n}{k}\frac{k^a}{a!}\binom{n-2}{k-2}^a p^a \leq \frac{\rho n^2}{k}\left[\frac{ek^2\rho}{an}\right]^a = o(1).$$

Now suppose that we are in the event $\overline{\mathcal{A} \vee \mathcal{B}}$ and consider a non-trivial intersecting family $\mathcal{F}$ in G_p^k. Let $\mathcal{M}$ be a maximal (t,j)-simplex in $\mathcal{F}$. We begin with the case $j = 2$. The cardinality of the union of the pairwise intersections of the sets in the system is at most $\binom{t}{2}(r-1) < (3/\varepsilon)^3$. Each of the sets in $\mathcal{F}$ contains one of these points and intersects some other k-set (depending on the point). Since we have assumed $\overline{\mathcal{A} \vee \mathcal{B}}$, this gives

$$|\mathcal{F}| \leq \begin{cases} 135/\varepsilon^4 & \text{if } \rho \leq 500\varepsilon^{-4} \\ (0.27)\rho & \text{if } \rho > 500\varepsilon^{-4} \end{cases}. \tag{5}$$

Note that if $j \geq 3$ then more restrictive conditions apply to the hyperedges in $\mathcal{F}$ and (5) still holds for ε sufficiently small. If $\Omega(1) = \rho \leq 500\varepsilon^{-4}$ then there is a vertex of unbounded degree (as n tends to infinity). If $\rho > 500\varepsilon^{-4}$ then, by Lemma 12 for $\rho = \Theta(1)$ and the Chernoff bound for $\rho = \omega(1)$, there is a vertex of degree at least $\rho/2$, as the expected degree of a vertex is ρ. In both cases G_p^k has the EKR property.

4 Large k

In this section, we prove Theorem 2. We begin with the case $\log n \ll k < n/3$.

Lemma 16. *Let $3k+2 \leq n$. Write the numbers $[n]$ in some cyclic order. Consider the k-graph K of n edges which are formed by k consecutive elements in the cyclic order. If $\mathcal{H} \subset K$ is intersecting, then there is a non-empty interval on the cycle which is contained in each edge of $\mathcal{H}$.*

Proof. By relabeling if necessary, we may assume that the cyclic order is $\{1, 2, \ldots, n\}$ and that $[k] \in \mathcal{H}$. Each $A \in \mathcal{H}$ contains either 1 or k but not both, since $k < n/3$. If $1 \in A$, then label the largest element of $A \cap [k]$ with $\ell(A)$. If $k \in A$, then label the smallest element in $A \cap [k]$ with $r(A)$. The easy observation is that for every sets $A, B \in \mathcal{H}$, if both $r(A)$ and $\ell(B)$ are defined then $r(A) \leq \ell(B)$, since otherwise $3k+2 \leq n$ implies that $\mathcal{H}$ is not intersecting. Now let $I = [a,b] \subset [k]$, where a is the largest element assigned for some set A as $r(A)$ and b is the smallest element assigned as $\ell(B)$ for some B. Clearly every edge of $\mathcal{H}$ contains an element of I. $\square$

Given a k-graph $\mathcal{H}$ with vertex set $[n]$ and a cyclic permutation σ of $[n]$, define $\mathcal{H}_\sigma \subset \mathcal{H}$ to be the k-graph whose edges occur as consecutive elements of σ.

Proof of Theorem 2 for $\log n \ll k < n/3$. We say that a cyclic permutation σ of S_n is **bad** if $|\mathcal{H}_\sigma| > kp(1+\eta)$ where $\eta = (2/p)\sqrt{\frac{\log n}{k}}$. By Lemma 16, there exists some $x \in [n]$ such that all edges of $\mathcal{H}_\sigma$ contain x. The expected number of edges in G_p^k that are in $\mathcal{H}_\sigma$ and contain x is pk. Therefore, Theorem 5 part (1) and the union bound gives

$$P(\mathcal{H}_\sigma \text{ is bad}) < n\exp(-2\eta^2p^2k) < \frac{1}{n^2}.$$

Let X be the number of bad permutations. We have $E[X] \leq \frac{(n-1)!}{n^2}$. Markov's inequality then implies

$$\mathbb{P}\left(X > \frac{(n-1)!}{n}\right) < \frac{1}{n}.$$

We may therefore assume that there are only few bad permutations. This implies that

$$k!(n-k)!|\mathcal{H}| = \sum_\sigma |\mathcal{H}_\sigma| \leq (n-1)! \cdot kp(1+\eta) + \frac{(n-1)!}{n} \cdot k.$$

Consequently,

$$|\mathcal{H}| \leq \binom{n-1}{k-1} p \left(1 + \frac{2}{p}\sqrt{\frac{\log n}{k}}\right) + \binom{n-1}{k-1}\frac{1}{n} = p\binom{n-1}{k-1}(1+o(1))$$

and the proof is complete for the $k \leq n/3$ case. $\square$

This proof can be extended to $k \leq (1-\epsilon)n/2$ by defining a collection $\mathcal{A}$ of intersecting families in $\binom{[n]}{k}_\sigma$ (the set of k-element subsets of $[n]$ that occur as consecutive elements σ) with the property that $|\mathcal{A}|$ is sufficiently small and every intersecting family is contained in some family in $\mathcal{A}$. That is, we need an extended version of Lemma 16.

Lemma 17. *Let $k < (1-\varepsilon)n/2$, where $0 < \varepsilon < 1/3$. Then the number of maximal intersecting k-uniform hypergraphs whose edges occur as consecutive elements of $[n]$ is at most $(2n)^{2/\varepsilon}$.*

Proof. A maximal intersecting hypergraph $\mathcal{H}$ can be characterized with a sequence $\{\ell, r, 0\}^n$, where the meaning of the i-th coordinate is the following: if it is ℓ then $[i-k+1, i] \in \mathcal{H}$, if it is r then $[i+1, i+k] \in \mathcal{H}$, if it is 0 then none of them in $\mathcal{H}$ (the intervals are understood modulo n). Such a sequence has two important properties: after an ℓ and before an r there must be at least εn 0 digits, otherwise $\mathcal{H}$ is not intersecting, and between two ℓ's if there is a 0 then there must be at least εn of them, otherwise $\mathcal{H}$ is not maximal. This implies that a sequence has the following form: it has some ℓ's then at least εn 0's, then either a run of ℓ's or r's, and so on. At each change we have at most 2 choices for the new digit, and n choices for the length of the run, and the number of runs is at most $2/\varepsilon$. Therefore the number of such sequences is at most $(2n)^{2/\varepsilon}$. □

Now the proof of the theorem for the $n/3 \le k < (1-\varepsilon)n/2$ case is almost the same as before. We note that every intersecting family in $\binom{[n]}{k}_\sigma$ has at most k elements, and we change the value of η to $\eta = \frac{2}{\varepsilon p}\sqrt{\frac{\log n}{k}}$. □

5 When G_p^k fails EKR

In this section we show examples when G_p^k fails EKR, in particular we prove Theorem 3. Actually, the proof of this theorem is a very simple application of the probabilistic method, and shows that G_p^k itself is a nontrivial intersecting family.

Proof of Theorem 3. The probability that there exist edges $X, Y \in G_p^k$ such that $X \cap Y = \emptyset$ is bounded above by $\binom{n}{k}\binom{n-k}{k}p^2$. Using the fact that $(n-k)_k \le (1-k/n)^k (n)_k$, this is at most

$$p^2(1-k/n)^k \frac{(n)_k^2}{k!^2} < p^2 e^{-k^2/n}\binom{n}{k}^2. \tag{6}$$

For each $i \in [n]$, the probability that all edges of G_p^k contain i is at most $(1-p)^{\binom{n-1}{k}}$. Consequently, the probability that G_p^k is trivial is bounded by

$$n(1-p)^{\binom{n-1}{k}} < n \exp\left(-p\binom{n-1}{k}\right). \tag{7}$$

By the choice of p, both (6) and (7) are $o(1)$ and the proof is complete. □

There are several ways one might hope to show that EKR fails for the range of k considered here. We present some of these below.

Remark 1. One might think that the simple deletion method would provide a better construction than the trivial hypergraph for some other p but unfortunately it is not the case: Consider G_p^k, and from each pair of disjoint edges remove one of them. The resulting hypergraph will be intersecting, and the expected number of edges remaining is at least

$$\binom{n}{k}p - \binom{n}{k}\binom{n-k}{k}p^2 \geq \frac{n}{k}\left(1 - p\exp(-k^2/n)\binom{n}{k}\right)p\binom{n-1}{k-1}.$$

This will be larger than the size of the trivial hypergraph but the range of p for which this holds is essentially the same as in Theorem 3.

Remark 2. Construct a graph G whose vertices are the edges of G_p^k, and two vertices (sets) are connected by an edge if they are disjoint. The expected number of vertices of G is $p\binom{n}{k}$. The expected degree of a vertex is $p\binom{n-k}{k}$. Hence there is an independent set of order $\binom{n}{k}/\binom{n-k}{k}$. Using (6), we can check that this is better than a trivial intersecting hypergraph if $p \ll e^{k^2/n}/\binom{n-1}{k-1}$ (and p is not too small).

Remark 3. It is interesting to look at what the following simple minded method gives. First keep all the edges of a maximum sized trivial subhypergraph of G_p^k. We may assume that the common element was u. Then fix an arbitrary vertex v, and include each set A from G_p^k which contains v and not u, if no $u \in B \subset \bar{A}$ is in G_p^k. The expected number of sets chosen is

$$p\binom{n-1}{k-1} + p(1-p)^{\binom{n-k-1}{k-1}}\binom{n-2}{k-1}.$$

For arbitrary p this is somewhat larger than the number of edges of the trivial hypergraph only if $n = 2k$. Even for $n = 2k+1$, a single edge is likely to be added only if $p < 3/4$, and for $n = 2k+2$ if $p \ll 1/k$. This might suggest that for 'large' p, even for large $k < n/2$, the random k-graph G_p^k satisfies EKR.

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