# Extremal problems for $t$-partite and $t$-colorable hypergraphs

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**Abstract**

Fix integers $t \geq r \geq 2$ and an $r$-uniform hypergraph $F$. We prove that the maximum number of edges in a $t$-partite $r$-uniform hypergraph on $n$ vertices that contains no copy of $F$ is $c_{t,F}\binom{n}{r} + o(n^r)$, where $c_{t,F}$ can be determined by a finite computation.

We explicitly define a sequence $F_1, F_2, \ldots$ of $r$-uniform hypergraphs, and prove that the maximum number of edges in a $t$-chromatic $r$-uniform hypergraph on $n$ vertices containing no copy of $F_i$ is $\alpha_{t,r,i}\binom{n}{r} + o(n^r)$, where $\alpha_{t,r,i}$ can be determined by a finite computation for each $i \geq 1$. In several cases, $\alpha_{t,r,i}$ is irrational. The main tool used in the proofs is the Lagrangian of a hypergraph.

## 1 Introduction

An *$r$-uniform hypergraph* or *$r$-graph* is a pair $G = (V, E)$ of *vertices*, $V$, and *edges* $E \subseteq \binom{V}{r}$, in particular a 2-graph is a graph. We denote an edge $\{v_1, v_2, \ldots, v_r\}$ by $v_1 v_2 \cdots v_r$. Given $r$-graphs $F$ and $G$ we say that $G$ is *$F$-free* if $G$ does not contain a copy of $F$. The maximum number of edges in an $F$-free $r$-graph of order $n$ is $\mathrm{ex}(n, F)$. For $r = 2$ and $F = K_s$ ($s \geq 3$)

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this number was determined by Turán [T41] (earlier Mantel [M07] found $\mathrm{ex}(n, K_3)$). However in general (even for $r = 2$) the problem of determining the exact value of $\mathrm{ex}(n, F)$ is beyond current methods. The corresponding asymptotic problem is to determine the *Turán density* of $F$, defined by $\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{r}}$ (this always exists by a simple averaging argument due Katona et al. [KNS64]). For 2-graphs the Turán density is determined by the chromatic number of the forbidden subgraph $F$. The explicit relationship is given by the following fundamental result.

**Theorem 1** (Erdős–Stone–Simonovits [ES46],[ES66]). *If $F$ is a 2-graph then $\pi(F) = 1 - \frac{1}{\chi(F)-1}$.*

When $r \geq 3$, determining the Turán density is difficult, and there are only a few exact results. Here we consider some closely related hypergraph extremal problems. Call a hypergraph $H$ $t$-partite if its vertex set can be partitioned into $t$ classes, such that every edge has at most one vertex in each class. Call $H$ $t$-colorable, if its vertex set can be partitioned into $t$ classes so that no edge is entirely contained within a class.

**Definition 2.** *Fix $t, r \geq 2$ and an $r$-graph $F$. Let $ex^*_t(n, F)$ ($ex_t(n, F)$) denote the maximum number of edges in a $t$-partite ($t$-colorable) $r$-graph on $n$ vertices that contains no copy of $F$. The $t$-partite Turán density of $F$ is $\pi^*_t(F) = \lim_{n\to\infty} ex^*_t(n, F)/\binom{n}{r}$ and the $t$-chromatic Turán density of $F$ is $\pi_t(F) = \lim_{n\to\infty} ex_t(n, F)/\binom{n}{r}$.*

Note that it is easy to show that these limits exist. In this paper, we determine $\pi^*_t(F)$ for all $r$-graphs $F$ and determine $\pi_t(F)$ for an infinite family of $r$-graphs (previously no nontrivial value of $\pi_t(F)$ was known). In many cases our examples yield irrational values of $\pi_t(F)$. For the usual Turán density, $\pi(F)$ has not been proved to be irrational for any $F$, although there are several conjectures stating irrational values.

In order to describe our results, we need the concept of $G$-colourings which we introduce now. If $F$ and $G$ are hypergraphs (not necessarily uniform) then $F$ is *$G$-colourable* if there exists $c : V(F) \to V(G)$ such that $c(e) \in E(G)$ whenever $e \in E(F)$. In other words, $F$ is $G$-colourable if there is a homomorphism from $F$ to $G$.

Let $K^{(r)}_t$ denote the complete $r$-graph of order $t$. Then an $r$-graph $F$ is *$t$-partite* if $F$ is $K^{(r)}_t$-colourable, and $F$ is *$t$-colourable* if it is $H^{(r)}_t$-colourable where $H^{(r)}_t$ is the (in general non-uniform) hypergraph consisting of all subsets $A \subseteq \{1, 2, \ldots, t\}$ satisfying $2 \leq |A| \leq r$). The *chromatic number* of $F$ is $\chi(F) = \min\{t \geq 1 : F \text{ is } t\text{-colourable}\}$. Note that while a

2-graph is $t$-colourable iff it is $t$-partite this is no longer true for $r \geq 3$, for example $K_4^{(3)}$ is 2-colourable but not 2-partite or 3-partite.

Let $\mathcal{G}_t^{(r)}$ denote the collection of all $t$-vertex $r$-graphs. A tool which has proved very useful in extremal graph theory and which we will use later is the Lagrangian of an $r$-graph. Let

$$\mathbb{S}_t = \{\vec{x} \in \mathbb{R}^t : \sum_{i=1}^{t} x_i = 1, x_i \geq 0 \text{ for } 1 \leq i \leq t\}.$$

If $G \in \mathcal{G}_t^{(r)}$ and $\vec{x} \in \mathbb{S}_t$ then we define

$$\lambda(G, \vec{x}) = \sum_{v_1 v_2 \cdots v_r \in E(G)} x_{v_1} x_{v_2} \cdots x_{v_t}.$$

The *Lagrangian* of $G$ is $\max_{\vec{x} \in \mathbb{S}_t} \lambda(G, \vec{x})$. The first application of the Lagrangian to extremal graph theory was due to Motzkin and Strauss who gave a new proof of Turán's theorem. We are now ready to state our main result.

**Theorem 3.** *If $F$ is an $r$-graph and $t \geq r \geq 2$ then*

$$\pi_t^*(F) = \max\{r!\lambda(G) : G \in \mathcal{G}_t^{(r)} \text{ and } F \text{ is not } G\text{-colourable}\}.$$

As an example of Theorem 3, suppose that $t = 4, r = 3$, and $F = K_4^{(3)}$. Let $H$ denote the unique 3-graph with four vertices and three edges. Now $F$ is $F$-colorable, but it is not $H$-colorable, and the lagrangian $\lambda(H)$ of $H$ is $4/81$, achieved by assigning the degree three vertex a weight of $1/3$ and the other three vertices a weight of $2/9$. Consequently, Theorem 3 says that the maximum number of edges in an $n$-vertex 4-partite 3-graph containing no copy of $K_4^{(3)}$ is $(8/27)\binom{n}{3} + o(n^3)$. This is clearly achievable, by the 4-partite 3-graph with part sizes $n/3, 2n/9, 2n/9, 2n/9$, with all possible triples between three parts that include the largest (of size $n/3$), and no triples between the three small parts.

Chromatic Turán densities were previously considered in [T07] where they were used to give an improved upper bound on $\pi(H)$, where $H$ is defined in the previous paragraph. However no non-trivial chromatic Turán densities have previously been determined. For each $r \geq t \geq 2$ we are able to give an infinite sequence of $r$-graphs whose $t$-chromatic Turán densities are determined exactly.

For $l \geq t \geq 2$ and $r \geq 2$ define

$$\beta_{r,t,l} := \max\{\lambda(G) : G \text{ is a } t\text{-colorable } r\text{-graph on } l \text{ vertices}\}.$$

It seems obvious that $\beta_{r,t,l}$ is achieved by the $t$-chromatic $r$-graph of order $l$ with all color classes of size $\lfloor l/t \rfloor$ or $\lceil l/t \rceil$ and all edges present except those within the classes. Note that if $t|l$ then this would give

$$\beta_{r,t,l} = \left( \binom{l}{r} - t\binom{l/t}{r} \right) \frac{1}{l^r}.$$

However, we are only able to prove this for $r = 2, 3$. If the above statement is true, then $\beta_{r,t,l}$ can be computed by calculating the maximum of an explicit polynomial in one variable over the unit interval. In any case it can be obtained by a finite computation (for fixed $r, t, l$). Let $\alpha_{r,t,l} = r!\beta_{r,t,l}$.

**Theorem 4.** *Fix $l \geq r \geq 2$. Let $L^{(r)}_{l+1}$ be the $r$-graph obtained from the complete graph $K_{l+1}$ by enlarging each edge with a set of $r-2$ new vertices. If $t \geq 2$ then*

$$\pi_t(L^{(r)}_{l+1}) = \alpha_{r,t,l}$$

*where $\alpha_{r,t,l}$ is defined above.*

The remainder of the paper is arranged as follows. In the next section we prove Theorem 3 and in the last section we prove Theorem 4 and the statements about computing $\beta_{r,t,l}$, for $r = 2, 3$.

# 2 Proof of Theorem 3

If $G \in \mathcal{G}^{(r)}_t$ and $\vec{x} = (x_1, \ldots, x_t) \in \mathbb{Z}^t_+$ then the *$\vec{x}$-blow-up* of $G$ is the $r$-graph $G(\vec{x})$ constructed from $G$ by replacing each vertex $v$ by a class of vertices of size $x_v$ and taking all edges between any $r$ classes corresponding to an edge of $G$. More precisely we have $V(G(\vec{x})) = X_1 \dot{\cup} \cdots \dot{\cup} X_t$, $|X_i| = x_i$ and

$$E(G(\vec{x})) = \{\{v_{i_1}v_{i_2}\cdots v_{i_r}\} : v_{i_j} \in X_{i_j}, \{i_1 i_2 \cdots i_r\} \in E(G)\}.$$

If $\vec{x} = (s, s, \ldots, s)$ and $G = K^{(r)}_t$ then $G(\vec{x})$ is the complete $t$-partite $r$-graph with class size $s$, denoted by $K^{(r)}_t(s)$. Note that if $F$ and $G$ are both $r$-graphs then $F$ is $G$-colourable iff there exists $\vec{x} \in \mathbb{Z}^t_+$ such that $F \subseteq G(\vec{x})$.

An $r$-graph $G$ is said to be *covering* if each pair of vertices in $V(G)$ is contained in a common edge. If $W \subset V$ and $G$ is an $r$-graph with vertex $V$ then $G[W]$ is the induced subgraph of $G$ formed by deleting all vertices not in $W$ and removing all edges containing these vertices.

**Lemma 5** (Frankl and Rödl [FR84]). *If $G$ is an $r$-graph of order $n$ then there exists $\vec{y} \in \mathbb{S}_n$ with $\lambda(G) = \lambda(G, \vec{y})$, such that if $P = \{v \in V(G) : y_v > 0\}$ then $G[P]$ is covering.*

Supersaturation for ordinary Turán densities was shown by Erdős [E71]. The proof for $G$-chromatic Turán densities is essentially identical but for completeness we give it. We require the following classical result.

**Theorem 6** (Erdős [E64]). *If $r \geq 2$ and $t \geq 1$ then $ex(n, K_r^{(r)}(t)) = O(n^{r-\lambda_{r,t}})$, with $\lambda_{r,t} > 0$.*

**Lemma 7** (Supersaturation). *Fix $t \geq r \geq 2$. If $G$ is an $r$-graph, $\mathcal{H}$ is a finite family of $r$-graphs, $s \geq 1$ and $\vec{s} = (s, s, \ldots, s)$ then $\pi_t^*(\mathcal{H}(\vec{s})) = \pi_t^*(\mathcal{H})$ (where $\mathcal{H}(\vec{s}) = \{H(\vec{s}) : H \in \mathcal{H}\}$).*

*Proof:* Let $p = \max\{|V(H)| : H \in \mathcal{H}\}$. By adding isolated vertices if necessary we may suppose that every $H \in \mathcal{H}$ has exactly $p$ vertices.

First we claim that if $F$ is an $n$-vertex $r$-graph with density at least $\alpha + 2\epsilon$, where $\alpha, \epsilon > 0$, and $r \leq m \leq n$ then at least $\epsilon\binom{n}{m}$ of the $m$-vertex induced subgraphs of $F$ have density at least $\alpha + \epsilon$. To see this note that if it fails to hold then

$$\binom{n-r}{m-r}(\alpha + 2\epsilon)\binom{n}{r} \leq \sum_{W \in \binom{V(F)}{m}} e(F[W]) < \epsilon\binom{n}{m}\binom{m}{r} + (1-\epsilon)\binom{n}{m}(\alpha+\epsilon)\binom{m}{r},$$

which is impossible.

Let $\epsilon > 0$ and suppose that $F$ is an $n$-vertex $r$-graph with density at least $\pi_t^*(\mathcal{H}) + 2\epsilon$. We need to show that if $n$ sufficiently large then $F$ contains a copy of $\mathcal{H}(\vec{s})$. Let $m \geq m(\epsilon)$ be sufficiently large that any $t$-partite $m$-vertex $r$-graph with density at least $\pi_t^*(\mathcal{H}) + \epsilon$ contains a copy of some $H \in \mathcal{H}$. We say that $W \in \binom{V(F)}{m}$ is *good* if $F[W]$ contains a copy of some $H \in \mathcal{H}$. By the claim at least $\epsilon\binom{n}{m}$ $m$-sets are good, so if $\delta = \epsilon/|\mathcal{H}|$ then at least $\delta\binom{n}{m}$ $m$-sets contain a fixed $H^* \in \mathcal{H}$.

Thus the number of $p$-sets $U \subset V(F)$ such that $F[U] \simeq H^*$ is at least

$$\frac{\delta\binom{n}{m}}{\binom{n-p}{m-p}} = \frac{\delta\binom{n}{p}}{\binom{m}{p}}. \tag{1}$$

Let $J$ be the $p$-graph with vertex set $V(F)$ and edge set consisting of those $p$-sets $U \subset V(F)$ such that $F[U] \simeq H^*$. Now, by Theorem 6, $\mathrm{ex}_t^*(n, K_r^{(r)}(t)) \leq \mathrm{ex}(n, K_r^{(r)}(t)) = O(n^{r-\lambda_{r,t}})$, where $\lambda_{r,t} > 0$. Hence (1) implies that for any $t \geq r$ if $n$ is sufficiently large then $K_p^{(p)}(t) \subset J$.

Finally consider a colouring of the edges of $K_p^{(p)}(t)$ with $p!$ different colours, where the colour of the edge is given by the order in which the vertices of $H^*$ are embedded in it. By Ramsey's theorem if $t$ is sufficiently large then there is a copy of $K_p^{(p)}(s)$ with all edges the same colour. This yields a copy of $H^*(\vec{s})$ in $F$ as required. □

**Proof of Theorem 3.** Let $\alpha_{r,t} = \max\{r!\lambda(G) : G \in \mathcal{G}_t^{(r)}$ and $F$ is not $G$-colourable$\}$. (This is well-defined since $|\mathcal{G}_t^{(r)}| \leq \binom{t}{r}$ is finite.)

If $G \in \mathcal{G}_t^{(r)}$ and $F$ is not $G$-colourable then for any $\vec{x} \in \mathbb{Z}_+^t$ we have $F \not\subseteq G(\vec{x})$. Let $\vec{y} \in \mathbb{S}_t$ satisfy $\lambda(G, \vec{y}) = \lambda(G)$. For $n \geq 1$ let $\vec{x}_n = (\lfloor y_1 n \rfloor, \ldots, \lfloor y_t n \rfloor) \in \mathbb{Z}_+^t$. If $G_n = G(\vec{x}_n)$ then

$$\lim_{n\to\infty} \frac{e(G_n)}{\binom{n}{r}} = r!\lambda(G).$$

Moreover since each $G_n$ is $F$-free, $t$-partite and of order at most $n$ we have $\pi_t^*(F) \geq r!\lambda(G)$. Hence $\pi_t^*(F) \geq \alpha_{r,t}$.

Let $\mathcal{H}(F) = \{H \in \mathcal{G}_t^{(r)} : F$ is $H$-colourable$\}$.

It is sufficient to show that

$$\pi_t^*(\mathcal{H}(F)) \leq \alpha_{r,t}. \tag{2}$$

Indeed, if we assume that (2) holds, then let $s \geq 1$ be minimal such that every $H \in \mathcal{H}(F)$ satisfies $F \subseteq H(\vec{s})$, where $\vec{s} = (s, s, \ldots, s)$. (Note that $s$ exists since $F$ is $H$-colourable for every $H \in \mathcal{H}(F)$). Now by supersaturation (Lemma 7) if $\epsilon > 0$, then any $t$-partite $r$-graph $G_n$ with $n \geq n_0(s, \epsilon)$ vertices and density at least $\alpha_{r,t} + \epsilon$ will contain a copy of $H(\vec{s})$ for some $H \in \mathcal{H}(F)$. In particular $G_n$ contains $F$ and so $\pi_t^*(F) \leq \alpha_{r,t}$.

Let $\pi_t^*(\mathcal{H}(F)) = \gamma$ and $\epsilon > 0$. If $n$ is sufficiently large there exists an $\mathcal{H}(F)$-free, $t$-partite $r$-graph $G_n$ of order $n$ satisfying

$$\frac{r!e(G_n)}{n^r} \geq \gamma - \epsilon.$$

Taking $\vec{y} = (1/n, 1/n, \ldots, 1/n) \in \mathbb{S}_n$ we have

$$r!\lambda(G_n) \geq r!\lambda(G_n, \vec{y}) = \frac{r!e(G_n)}{n^r} \geq \gamma - \epsilon.$$

Now Lemma 5 implies that there exists $\vec{z} \in \mathbb{S}_n$ satisfying

- $\lambda(G_n) = \lambda(G_n, \vec{z})$ and
- $G_n[P]$ is covering where $P = \{v \in V(G) : z_v > 0\}$.

Since $G_n$ is $t$-partite, we conclude that $G_n[P]$ has at most $t$ vertices. Moreover, $G_n$ is $\mathcal{H}(F)$-free and so $G_n[P] \notin \mathcal{H}(F)$. Thus $F$ is not $G_n[P]$-colorable, and we have $\gamma - \epsilon \leq r!\lambda(G_n[P]) \leq \alpha_{r,t}$. Thus $\pi_t^*(\mathcal{H}(F)) \leq \alpha_{r,t} + \epsilon$ for all $\epsilon > 0$. Hence (2) holds and the proof is complete. □

# 3 Infinitely many chromatic Turán densities

For $l, r \geq 2$ let $\mathcal{K}_l^{(r)}$ be the family of $r$-graphs with at most $\binom{l}{2}$ edges that contain a set $S$, called the *core*, of $l$ vertices, with each pair of vertices from $S$ contained in an edge. Note that $L_{l+1}^{(r)} \in \mathcal{K}_{l+1}^{(r)}$. We need the following Lemma that was proved in [M06]. For completeness, we repeat the proof below.

**Lemma 8.** *If $K \in \mathcal{K}_{l+1}^{(r)}$, $s = \binom{l+1}{2} + 1$ and $\vec{s} = (s, s, \ldots, s)$ then $L_{l+1}^{(r)} \subseteq K(\vec{s})$.*

*Proof.* We first show that $L_{l+1}^{(r)} \subset L(\binom{l+1}{2} + 1)$ for every $L \in \mathcal{K}_{l+1}^{(r)}$. Pick $L \in \mathcal{K}_{l+1}^{(r)}$, and let $L' = L(\binom{l+1}{2} + 1)$. For each vertex $v \in V(L)$, suppose that the clones of $v$ are $v = v^1, v^2, \ldots, v^{\binom{l+1}{2}+1}$. In particular, identify the first clone of $v$ with $v$.

Let $S = \{w_1, \ldots, w_{l+1}\} \subset V(L)$ be the core of $L$. For every $1 \leq i < j \leq l+1$, let $E_{ij} \in L$ with $E_{ij} \supset \{w_i, w_j\}$. Replace each vertex $z$ of $E_{ij} - \{w_i, w_j\}$ by $z^q$ where $q > 1$, to obtain an edge $E'_{ij} \in L'$. Continue this procedure for every $i, j$, making sure that whenever we encounter a new edge it intersects the previously encountered edges only in $L$. Since the number of clones is $\binom{l+1}{2} + 1$, this procedure can be carried out successfully and results in a copy of $L_{l+1}^{(r)}$ with core $S$. Therefore $L_{l+1}^{(r)} \subset L' = L(\binom{l+1}{2} + 1)$. Consequently, Lemma 7 implies that $\pi(L_{l+1}^{(r)}) \leq \pi(\mathcal{K}_{l+1}^{(r)})$. □

**Proof of Theorem 4.** Let $l \geq r \geq 2$ and $t \geq 2$. We will prove that

$$\pi_t(\mathcal{K}_{l+1}^{(r)}) = \alpha_{r,t,l}. \tag{3}$$

The theorem will then follow immediately from Lemmas 7 and 8. Let

$$\mathcal{B}_{r,t,l} = \{G : G \text{ is a } t\text{-colourable } \mathcal{K}_{l+1}^{(r)}\text{-free } r\text{-graph}\}.$$

**Claim.** $\max\{\lambda(G) : G \in \mathcal{B}_{r,t,l}\} = \beta_{r,t,l} = \alpha_{r,t,l}/r!$.

**Proof of Claim.** If $G \in \mathcal{B}_{r,t,l}$ has order $n$ then Lemma 5 implies that there is $\vec{y} \in \mathbb{S}_n$ such that $\lambda(G) = \lambda(G, \vec{y})$ with $G[P]$ covering, where $P = \{v \in V(G) : y_v > 0\}$. Since $G$ is

$\mathcal{K}_{l+1}^{(r)}$-free, we conclude that $|P| = p \le l$. Hence there is $H \in \mathcal{B}_{r,t,l}$ such that $\lambda(H) = \lambda(G)$ and $H$ has order at most $l$. Consequently, $\max\{\lambda(G) : G \in \mathcal{B}_{r,t,l}\} \le \beta_{r,t,l}$. For the other inequality, we just observe that an $l$ vertex $r$-graph must be $\mathcal{K}_{l+1}^{(r)}$-free. □

Now we can quickly complete the proof of the theorem by proving (3). For the upper bound, observe that if $G \in \mathcal{B}_{r,t,l}$ has order $n$ then by the Claim

$$\frac{e(G)}{n^r} \le \lambda(G) \le \frac{\alpha_{r,t,l}}{r!}$$

and so $\pi_t(\mathcal{K}_{l+1}^{(r)}) \le \alpha_{r,t,l}$. For the lower bound, suppose that $G \in \mathcal{B}_{r,t,l}$ has order $p$ and satisfies $\lambda(G) = \beta_{r,t,l}$. Then there exists $\vec{y} \in \mathbb{S}_p$ such that $\lambda(G, \vec{y}) = \lambda(G) = \beta_{r,t,l}$. For $n \ge p$ define $\vec{y}_n = (\lfloor y_1 n \rfloor, \ldots, \lfloor y_p n \rfloor)$. Now $\{G(\vec{y}_n)\}_{n=p}^{\infty}$ is a sequence of $t$-colourable $\mathcal{K}_{l+1}^{(r)}$-free $r$-graphs and hence

$$\pi_t(\mathcal{K}_{l+1}^{(r)}) \ge \lim_{n\to\infty} \frac{e(G_n)}{\binom{n}{r}} = r!\lambda(G) = \alpha_{r,t,l}.$$

□

Now we prove that $\beta_{r,t,l}$ can be computed by only considering maximum $t$-colorable $r$-graphs with almost equal part sizes when $r = 2, 3$. The case $r = 2$ follows trivially from Lemma 5 so we consider the case $r = 3$.

**Theorem 9.** *Fix $l \ge t \ge 2$. Then $\beta_{3,t,l}$ is achieved by the $t$-chromatic 3-graph of order $l$ with all color classes of size $\lfloor l/t \rfloor$ or $\lceil l/t \rceil$ and all edges present except those within the classes.*

**Remark:** Note that if $t|l$ then this implies that $\beta_{3,t,l} = (\binom{l}{3} - t\binom{l/t}{3})\frac{1}{l^3}$.

*Proof.* Let $G$ be a $t$-chromatic 3-graph of order $l$. We may suppose (by adding edges as required) that $V(G) = V_1 \cup V_2 \cup \cdots \cup V_t$ and that all edges not contained in any $V_i$ are present. We may also suppose that $|V_1| \ge |V_2| \ge \cdots \ge |V_t|$. Let $\vec{x} \in \mathbb{S}_p$ satisfy $\lambda(G, \vec{x}) = \lambda(G)$.

If $v, w \in V_i$ and $x_v > x_w$ then for a suitable choice of $\delta > 0$ we can increase $\lambda(G, \vec{x})$ by increasing $x_w$ by $\delta$ and decreasing $x_v$ by $\delta$. Hence we may suppose that there are $x_1, \ldots, x_t$ such that all vertices in $V_i$ receive weight $x_i$.

Let $l = bt + c, 0 \le c < t$. To complete the proof we need to show that all of the $V_i$ have order $b$ or $b + 1$. Suppose, for a contradiction, that there exist $V_i$ and $V_j$ with $a_i = |V_i|$, $a_j = |V_j|$ and $a_i \ge a_j + 2$. Moving a vertex $v$ from $V_i$ to $V_j$ and inserting all new allowable edges (i.e. those which contain $v$ and 2 vertices from $V_i \backslash \{v\}$) while deleting any edges which now lie in $V_j$ we cannot increase $\lambda(G, \vec{x})$. This implies that

$$\binom{a_j}{2} x_i x_j^2 \geq \binom{a_i - 1}{2} x_i^3, \tag{4}$$

and so in particular $x_i < x_j$. Let $\tilde{G}$ denote this new $t$-colourable 3-graph.

We give a new weighting $\vec{y}$ for $\tilde{G}$ by setting

$$y_v = \begin{cases} a_i x_i/(a_i - 1), & v \in V_i, \\ a_j x_j/(a_j + 1), & v \in V_j, \\ x_k, & v \in V_k \text{ and } k \neq i, j. \end{cases}$$

It is easy to check that $\vec{y} \in \mathbb{S}_l$ is a legal weighting for $\tilde{G}$. We will derive a contradiction by showing that $\lambda(\tilde{G}) \geq \lambda(\tilde{G}, \vec{y}) > \lambda(G, \vec{x}) = \lambda(G)$.

If $w = a_i x_i + a_j x_j = (a_i - 1) y_i + (a_j + 1) y_j$ then

$$\begin{aligned} \lambda(\tilde{G}, \vec{y}) - \lambda(G, \vec{x}) &= (1 - w) \left( \binom{a_i - 1}{2} y_i^2 + \binom{a_j + 1}{2} y_j^2 + (a_i - 1)(a_j + 1) y_i y_j \right. \\ &\quad \left. - \binom{a_i}{2} x_i^2 - \binom{a_j}{2} x_j^2 - a_i a_j x_i x_j \right) + \binom{a_i - 1}{2} (a_j + 1) y_i^2 y_j + \\ &\quad \binom{a_j + 1}{2} (a_i - 1) y_i y_j^2 - \binom{a_i}{2} a_j x_i^2 x_j - \binom{a_j}{2} a_i x_i x_j^2 \\ &= \frac{(1 - w)}{2} \left( \frac{a_j x_j^2}{a_j + 1} - \frac{a_i x_i^2}{a_i - 1} \right) + \frac{a_i a_j x_i x_j}{2} \left( \frac{x_j}{a_j + 1} - \frac{x_i}{a_i - 1} \right). \end{aligned}$$

Using (4) it is easy to check that this is strictly positive. □

**Corollary 10.** *The $t$-chromatic Turán density can take irrational values.*

*Proof.* We consider $\beta_{3,2,2k}$ for $k \geq 3$. In fact, we focus of $\beta_{3,2,6}$, the maximum density of a 2-chromatic 3-graph that contains no copy of $\mathcal{K}_6^{(3)}$. By the previous Theorem, this is 6 times the lagrangian of the 3-graph with vertex set $\{a, a', a'', b, b'\}$ and all edges present except $\{a, a', a''\}$. Assigning weight $x$ to the $a$'s and weight $y$ to the $b$'s, we must maximize $6(6x^2y + 3xy^2)$ subject to $3x + 2y = 1$ and $0 \leq x \leq 1/3$. A short calculation shows that the choice of $x$ that maximizes this expression is $(\sqrt{13} - 2)/9$, and this results in an irrational value for the lagrangian. Similar computations hold for larger $k$ as well. □

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