An intersection theorem for four sets

Dhruv Mubayi *

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Abstract

Fix integers $n, r \geq 4$ and let $\mathcal{F}$ denote a family of r-sets of an n-element set. Suppose that for every four distinct $A, B, C, D \in \mathcal{F}$ with $|A \cup B \cup C \cup D| \leq 2r$, we have $A \cap B \cap C \cap D \neq \emptyset$. We prove that for n sufficiently large, $|\mathcal{F}| \leq \binom{n-1}{r-1}$, with equality only if $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. This is closely related to a problem of Katona and a result of Frankl and Füredi [10], who proved a similar statement for three sets. It has been conjectured by the author [18] that the same result holds for d sets (instead of just four), where $d \leq r$, and for all $n \geq dr/(d-1)$.

This exact result is obtained by first proving a stability result, namely that if $|\mathcal{F}|$ is close to $\binom{n-1}{r-1}$ then $\mathcal{F}$ is close to satisfying $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. The stability theorem is analogous to, and motivated by the fundamental result of Erdős and Simonovits for graphs.

1 Introduction.

Throughout this paper, X is an n-element set. For any nonnegative integer r, we write $\binom{X}{r}$ for the family of all r-element subsets of X. In this paper we initiate a new approach to solving classical intersection type problems in extremal set theory. The approach, which we call the stability method, proves an exact extremal result by first proving an approximate result that gives structural information on the near extremal families.

Questions about stability in extremal combinatorics grew from the seminal work of Erdős and Simonovits [23] on graph stability in the 60's. The notion of stability for properties of set systems was explicitly formulated by the author recently [17]. Several motivations were provided: First,

*Department of Mathematics, Statistics, & Computer Science, University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045; email: mubayi@math.uic.edu. Research partially supported by National Science Foundation Grant DMS-0400812, and an Alfred P. Sloan Research Fellowship.

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and perhaps the most obvious, is that proving a stability theorem tells us more about a problem than just the extremal result. Second, stability results allow one to prove exact results for certain extremal problems. This approach was first used by Simonovits [23] to determine the exact extremal function for k-critical graphs, and more recently has found resurgence in solving several hypergraph Turán problems (see [8, 9, 12, 15, 16, 21, 22]). Third, many proof techniques in extremal set theory (Katona's circle method, shifting, Linear Algebra methods) seem not to give stability analogues, and therefore new methods need to be developed. Consequently, the search for stability results can also yield new proof techniques in extremal set theory. Finally, stability results can be used to accurately enumerate discrete structures. This was shown recently by Balogh-Bollobás-Simonovits [1], who proved (among more general statements) that the number of labeled n vertex graphs containing no copy of K_{l+1} is $2^{(1-1/l)\binom{n}{2}+O(n^{2-\gamma})}$, where $\gamma > 0$. This improved previous results of Erdős-Kleitman-Rothschild [4] and others.

In the following definitions, we consider set-systems whose underlying set of elements are not labeled, so technically speaking, a set-system refers to an isomorphism class of set-systems whose underlying sets are labeled. A *Property* $\mathcal{P}$ ($\mathcal{P}^r$) is an infinite family of set-systems (comprising r-sets). The property $\mathcal{P}$ is monotone if whenever $\mathcal{G} \in \mathcal{P}$ and $\mathcal{G}'$ is obtained from $\mathcal{G}$ by deleting vertices and edges, then $\mathcal{G}' \in \mathcal{P}$. One can characterize monotone properties by properties not containing forbidden subsystems. In fact, the forbidden family for $\mathcal{P}$ is the collection $\mathcal{F}$ of set-systems not contained in any member of $\mathcal{P}$. The property $\mathcal{P}_n$ ($\mathcal{P}_n^r$) is the subfamily of $\mathcal{P}$ ($\mathcal{P}^r$) consisting of those set systems on n elements. The classical extremal problem in this regard is to determine

$$\mathrm{ex}(n, \mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \in \mathcal{P}_n\}, \qquad \text{or} \qquad \mathrm{ex}_r(n, \mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \in \mathcal{P}_n^r\}$$

where $\mathcal{F}$ is the forbidden family for a monotone property $\mathcal{P}$.

In what follows, we write $\mathcal{G}_n$ for a set system whose underlying set has size n. The formulation below applies as well to $\mathcal{P}^r$ even though we write it only for $\mathcal{P}$.

Definition. *Let $t > 0$ be an integer, $\mathcal{P}$ be a monotone property of set systems, and $\mathcal{F}$ be a forbidden family for $\mathcal{P}$. The property $\mathcal{P}$ is t-stable if there exists $m_0 = m_0(\mathcal{F})$ and set systems $\mathcal{H}_m^1, \ldots, \mathcal{H}_m^t$ for every $m > m_0$ such that the following holds: for every $\delta > 0$, there exists $\epsilon > 0$ and $n_0 = n_0(\epsilon)$ such that for all $n > n_0$, if $\mathcal{G}_n \in \mathcal{P}_n$ with*

$$|\mathcal{G}_n| > (1-\epsilon)\mathrm{ex}(n, \mathcal{F}),$$

then $\mathcal{G}_n$ can be transformed to some $\mathcal{H}_n^i$ by adding and removing at most $\delta|\mathcal{G}_n|$ sets. Say that $\mathcal{P}$ is stable if it is 1-stable.

In [17] a stability theorem was proved for a nontrivial problem in extremal set theory, indeed this was one of the first such results. Since we need this result in our proof, we describe it next. A *triangle* is a family of three sets A, B, C that have pairwise nonempty intersections, and $A \cap B \cap C = \emptyset$. An old problem of Erdős was to determine the maximum size of a family $\mathcal{F} \subset \binom{X}{r}$ that contains no triangle. Extending previous results of Chvátal, Frankl, and Füredi, the author and Verstraëte [19] proved that this maximum is $\binom{n-1}{r-1}$ for all $n \geq 3r/2$ and $r \geq 3$. Recently, this was extended by the author [17] to prove a stability version.

Theorem 1. ([17]) *Fix $r \geq 3$. For every $\delta > 0$, there exist $\epsilon > 0$ and $n_0 = n_0(\epsilon, r)$ such that the following holds for all $n > n_0$: if $\mathcal{G} \subset \binom{X}{r}$ contains no triangle, $|\mathcal{G}| > (1-\epsilon)\binom{n-1}{r-1}$, then there exists an $S \subset X$ with $|S| = n-1$ such that $|\mathcal{G} \cap \binom{S}{r}| \leq \delta\binom{n-1}{r-1}$.*

Here we continue this project, and prove a stability result for a generalization of a problem of Katona and a theorem of Frankl and Füredi. Moreover, our approach then yields an exact extremal result. Although the general technique of obtaining an exact result after obtaining structural information is not new (for example, the delta system method initiated by M. Deza is another example), to the authors knowledge, the stability approach in this paper has not been previously used to prove intersection theorems in extremal set theory. In particular, sunflowers are not employed in our proof. An earlier paper of Frankl and Füredi [11] also proves an exact result for an intersection theorem in several steps (building on the Deza technique), one of which obtains a structure theorem. The reader may wish to compare and contrast our approach to that in [11].

A *star* is a family of sets for which there is an element that is contained in all the sets. The seminal result of Erdős-Ko-Rado [5] states that an intersecting family $\mathcal{F} \subset \binom{X}{r}$ of maximum size is a star for $n > 2r$. Motivated by a possible generalization of the Erdős-Ko-Rado theorem to more than two sets, Katona defined the following.

Definition. Let $r \leq s \leq 3r$. Then $f(n, r, s)$ denotes the maximum size of a family $\mathcal{F} \subset \binom{[n]}{r}$ so that whenever $A, B, C \in \mathcal{F}$ satisfy $|A \cup B \cup C| \leq s$, we have $A \cap B \cap C \neq \emptyset$.

Katona asked for the determination of $f(n, r, s)$. Frankl and Füredi [10] proved that for every $2r \leq s \leq 3r$, $f(n, r, s) = \binom{n-1}{r-1}$ as long as $n \geq r^2 + 3r$, and observed that $f(n, r, 2r-1) = \Omega(n^r)$ for fixed r. Note that the lower bound $f(n, r, s) \geq \binom{n-1}{r-1}$ is valid for all $s \in \{2r, \ldots, 3r\}$ by simply letting $\mathcal{F}$ be a maximum sized star. Moreover, by definition $f(n, r, s+1) \leq f(n, r, s)$, hence Frankl and Füredi's first result follows by proving the upper bound just for $s = 2r$. They conjectured that $f(n, r, 2r) = \binom{n-1}{r-1}$ for all $r \geq 3$ and $n \geq 3r/2$, with equality only for a star. The threshold $3r/2$ follows from the fact that for smaller n, three sets $A, B, C \in \mathcal{F}$ whose intersection is empty cannot exist (so in particular, we can have $|\mathcal{F}| = \binom{n}{r}$).

Frankl and Füredi [10] proved their conjecture for $r = 3$, and commented (without proof) that their approach also works for $r = 4, 5$ and more generally for $r > k^2/\log k$. Recently the author [18] gave a short proof of their conjecture using different arguments.

Neither of the two approaches above provides a stability result for Katona's problem, so we provide a third approach. Although we couldn't show that our method gives an exact result for all $n \geq 3r/2$, it proves a stability result for a more general situation which includes Katona's problem as a special case. More precisely, in Katona's problem, the forbidden configuration is a family of three sets with certain properties, while in our results it is a family of d sets with similar properties. Then we prove an exact result (Theorem 3) using the stability theorem. For convenience, we make the following

Definition. Fix $r \geq 2$. A family $\mathcal{G}$ of r-sets is a $K(d)$-family if, whenever distinct sets $A_1, \dots, A_d \in \mathcal{G}$ satisfy $|\bigcup_i A_i| \leq 2r$, we have $\bigcap_i A_i \neq \emptyset$.

A $K(2)$-family of r-sets is simply an intersecting family, and hence its maximum size is given by the Erdős-Ko-Rado theorem. Also, $f(n, r, 2r)$ is just the maximum size of a $K(3)$-family of r-sets on X. Note that there exist families of size $\lfloor n/r \rfloor^r = \Omega(n^r)$ such that every $d \geq 3$ sets have empty intersection provided their union is at most $2r - 1$ (simply partition $[n]$ into r almost equal parts, and take all r-sets with exactly one point in each part). This is the reason for the threshold $2r$ in our definitions. As our theorems below will show, changing the threshold $2r$ to any other $s \geq 2r$ will not alter our results, similar to the situation regarding $f(n, r, s)$ described above (since the lower bound $\binom{n-1}{r-1}$ on the families we consider holds for all $s \geq 2r$). The stability result below shows that if $\mathcal{F}$ is a $K(d)$-family for some $2 \leq d \leq r$, then $\mathcal{F}$ is stable.

Theorem 2. (Stability) *Fix $2 \leq d \leq r$. For every $\delta > 0$, there exists $\epsilon > 0$ and n_0 such that the following holds for all $n > n_0$: Suppose that $\mathcal{G} \subset \binom{X}{r}$ is a $K(d)$-family. If $|\mathcal{G}| \geq (1 - \epsilon)\binom{n-1}{r-1}$, then there exists an $(n - 1)$-set $S \subset X$ with $|\mathcal{G} \cap \binom{S}{r}| < \delta\binom{n-1}{r-1}$. In particular, $|\mathcal{G}| < (1 + \delta)\binom{n-1}{r-1}$.*

It is possible that Theorem 2 holds even when $d > r$. In fact, it is an interesting open problem to determine the largest $d = d(r)$ for which Theorem 2 holds.

Using Theorem 2, we prove a result similar to those of Frankl-Füredi [10] and the author [18] for $K(4)$-families. As mentioned before, our proof technique for Theorem 3 is one of the main new contributions in this work.

Theorem 3. *Let $r \geq 4$ and let n be sufficiently large. Suppose that $\mathcal{G} \subset \binom{[n]}{r}$ is a $K(4)$-family. Then $|\mathcal{G}| \leq \binom{n-1}{r-1}$, with equality only if $\mathcal{F}$ is a star.*

Theorem 3 is also related to the following old problem of Erdős. Let $f_r(n)$ be the maximum size of a family of r-sets of an n element set containing no two pairs of disjoint r-sets with the same union. Since all the forbidden configurations in this question are forbidden configurations in a $K(4)$-family (the converse is not true), an upper bound for this problem yields an upper bound for the $K(4)$-problem. Answering a question of Erdős, Füredi [7] proved that $f_r(n) \leq \frac{7}{2}\binom{n}{r-1}$.

The author and Verstraëte [20] slightly improved Füredi's result by showing that $f_r(n) < 3\binom{n}{r-1}$. Füredi further conjectured that $f_r(n) = \binom{n-1}{r-1} + \lfloor \frac{n-1}{r} \rfloor$ for all $r \geq 4$ and sufficiently large n. Theorem 3 can be viewed as a solution to a relaxation of this problem (ignoring the $(n-1)/r$ term).

The following more general conjecture, posed in [18], remains open. While a complete proof may be out of reach at present, we certainly believe that the stability approach with Theorem 2 should yield a proof for large n.

Conjecture 4. **([18])** *Let $r \geq d \geq 3$ and $n \geq dr/(d-1)$. Suppose that $\mathcal{G} \subset \binom{[n]}{r}$ is a $K(d)$-family. Then $|\mathcal{G}| \leq \binom{n-1}{r-1}$, with equality only if $\mathcal{G}$ is a star.*

It is possible that Conjecture 4 holds even for $d > r$. However, it cannot hold for $d \geq 2^r$, since in this case we can take an r-partite r-graph $\mathcal{G}$ containing no copy of $K(2, \ldots, 2)$, the complete r-partite r-graph with two points in each part. It is known that such $\mathcal{G}$ exists with $|\mathcal{G}| > \Omega(n^{r-1+\gamma})$ for $\gamma = 1 - r/2^{r-1} > 0$ (see [3, 6] and also [13] for slight improvements), and it is easy to verify that $\mathcal{G}$ is a $K(2^r)$-family. It would be interesting to determine the largest $d = d(r)$ so that every $K(d)$-family $\mathcal{G} \subset \binom{X}{r}$ satisfies $|\mathcal{G}| = O(n^{r-1})$.

2 Notation

For $\mathcal{A} \subset \binom{X}{r}$, let $V(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$. For $Y \subset X$, we define $\mathcal{A} - Y = \mathcal{A} \cap \binom{X-Y}{r}$. When $Y = \{y\}$, we write $\mathcal{A} - y$ instead of $\mathcal{A} - \{y\}$. The *trace* of $Y \subset V(\mathcal{A})$ in $\mathcal{A}$ is defined by $\mathrm{tr}(Y) = \mathrm{tr}_{\mathcal{A}}(Y) = \{A \subset X - Y : A \cup Y \in \mathcal{A}\}$. The *degree* of $Y \subset V(\mathcal{A})$ in $\mathcal{A}$ is $\deg(Y) = \deg_{\mathcal{A}}(Y) = |\mathrm{tr}_{\mathcal{A}}(Y)|$. When $Y = \{y\}$, we write $\mathrm{tr}(y)$ and $\deg(y)$. Let $\mathcal{A} \subset \binom{X}{r}$ and $x \in X$. Then we define

$$\mathcal{S}_x = \{Y \in \mathrm{tr}(x) : \deg(Y) = 1\} \qquad \text{and} \qquad \mathcal{L}_x = \mathrm{tr}(x) - \mathcal{S}_x.$$

The *sum of families* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t$, denoted $\sum_i \mathcal{A}_i$, is the family of all sets in each $\mathcal{A}_i$. Note that $\sum \mathcal{A}_i$ may have repeated sets, even if none of the $\mathcal{A}_i$ have repeated sets. The trace of $\mathcal{A}$ is $\mathrm{tr}(\mathcal{A}) = \sum_{x \in X} \mathrm{tr}(x)$. Write $\mathcal{S} = \sum_{x \in X} \mathcal{S}_x$ and $\mathcal{L} = \sum_{x \in X} \mathcal{L}_x = \mathrm{tr}(\mathcal{A}) - \mathcal{S}$. Note that if $A \in \mathcal{L}_x$, then there exists $y \neq x$ such that $A \in \mathcal{L}_y$. The shadow $\partial\mathcal{G}$ of a set system $\mathcal{G} \subset \binom{X}{r}$ is $\partial\mathcal{G} = \{S \in \binom{X}{r-1} :$ there exists $T \in \mathcal{G}$ with $S \subset T\}$.

Throughout the paper, the Greek letters ϵ, δ etc. are real numbers and m, n, r, s, t etc. are integers.

3 Stability

In this section we prove the stability result for those set systems which are $K(d)$-families for some $2 \leq d \leq r$.

Proof of Theorem 2. Fix $r \geq 2$. We proceed by induction on r, handling the cases $r = 2$ and $r = 3$ separately. When $r = 2$, a $K(2)$-family is a graph containing no matching of size two, and in this case it is trivial to observe that such a family with at least four edges must be a star. So for any $\delta > 0$ (even $\delta = 0$), we can let, for example, $\epsilon = \epsilon_2 = 1/8$ and $n_0 = 4$. Indeed, then any graph $\mathcal{F}$ on $n > n_0$ vertices with at least $\lceil (1-\epsilon)\binom{n-1}{r-1} \rceil = \lceil (7/8)(n-1) \rceil \geq 4$ edges must be a star.

More generally, a $K(2)$-family is just an intersecting family. It is well-known (see, e.g., Theorem 2, page 48 of [2]) that an intersecting family of size $\Omega(n^{r-2})$ is already a star (indeed, this also follows from the Hilton-Milner theorem on nontrivial intersecting families), so a $K(2)$-family is certainly stable. Consequently, we may assume that $3 \leq d \leq r$.

When $r = 3$, a $K(3)$-family contains no triangle, since a triangle A, B, C satisfies $A \cap B \cap C = \emptyset$, and $|A \cup B \cup C| \leq |A| + (|B| - 1) + (|C| - 2) = 6 = 2r$. Hence Theorem 1 implies Theorem 2 for $r = 3$. We may therefore assume that $r \geq 4$.

Now suppose we are given $\delta = \delta_r$ as in the theorem. First set

$$\delta_{r-1} = \min\left\{ \frac{1}{2}\left(\frac{2}{5}\delta_r\right)^{r-2}, \frac{\delta_r}{72(r-1)} \right\}. \tag{1}$$

Now choose ϵ_{r-1} and $n_0(\epsilon_{r-1}, r-1)$ that satisfy the conclusion of the theorem for $r - 1$. Such choices exist by the induction hypothesis, and we may also assume that $\epsilon_{r-1} < \delta_{r-1}$. Next let

$$\epsilon_r = \frac{\epsilon_{r-1}}{2}. \tag{2}$$

Finally, choose $n_0 = n_0(\epsilon_r, r) > n_0(\epsilon_{r-1}, r-1) + 1$ so that for all $n > n_0$,

$$\frac{r(1-\epsilon_r)\binom{n-1}{r-1} - \binom{n}{r-1}}{n} > (1 - 2\epsilon_r)\binom{n-2}{r-2}, \tag{3}$$

and

$$\binom{\frac{1}{2}\delta_r(n-2)}{r-2} > \left(\frac{2}{5}\delta_r\right)^{r-2}\binom{n-2}{r-2}. \tag{4}$$

Note that a short calculation shows that for sufficiently large n, both (3) and (4) do indeed hold, hence n_0 is well-defined.

Having fixed all constants, we now begin the argument for the induction step. As argued above, we may assume that $d > 2$, so fix $3 \leq d \leq r$. Let $\mathcal{G} \subset \binom{X}{r}$ be a $K(d)$-family with $|X| = n > n_0$ and $|\mathcal{G}| > (1-\epsilon_r)\binom{n-1}{r-1}$. Our strategy is to obtain the $(n-1)$-set S in the conclusion of the theorem in three steps:

1) Find a vertex w with $|\mathcal{L}_w|$ very large.

2) Study the structure of $|\mathcal{L}_w|$, in particular, show that it contains a large star with center x.

3) Set $S = X - \{x\}$ and show that S satisfies the requirements of the theorem, because $\mathcal{G}$ is a $K(d)$-family.

Step 1.

We begin with the following equation which is an easy double counting exercise.

$$r|\mathcal{G}| = \sum_{x\in X} \deg(x) = \sum_{x\in X}(|\mathcal{S}_x| + |\mathcal{L}_x|) = \sum_{x\in X}|\mathcal{S}_x| + \sum_{x\in X}|\mathcal{L}_x|.$$

Since $\sum_x |\mathcal{S}_x| = |\mathcal{S}| \le \binom{n}{r-1}$, there exists $w \in X$ for which

$$\begin{aligned} |\mathcal{L}_w| \ge \frac{r|\mathcal{G}| - \binom{n}{r-1}}{n} &> \frac{r(1-\epsilon_r)\binom{n-1}{r-1} - \binom{n}{r-1}}{n} \\ &> (1-2\epsilon_r)\binom{n-2}{r-2} \ge (1-\epsilon_{r-1})\binom{n-2}{r-2}, \end{aligned} \tag{5}$$

where the inequalities follow from (3) and (2). This concludes Step 1.

Step 2

Now consider the family $\mathcal{L}_w \subset \binom{X-\{w\}}{r-1}$. We next show that $\mathcal{L}_w$ is a $K(d-1)$-family. Suppose, for a contradiction, that $\mathcal{L}_w$ is not a $K(d-1)$-family of $(r-1)$-sets. Then $\mathcal{L}_w$ contains distinct sets $A_1, \ldots, A_{d-1}$ with $|\bigcup_i A_i| \le 2(r-1)$ and $\bigcap_i A_i = \emptyset$. By definition of $\mathcal{L}_w$, there exists $y \ne w$ such that $A_1 \cup \{y\} \in \mathcal{G}$. Now define $B_i = A_i \cup \{w\}$ for $i = 1, \ldots, d-1$ and $B_d = A_1 \cup \{y\}$. Because $|\bigcup_i A_i| \le 2(r-1)$, we have $|\bigcup_{i=1}^d B_i| = |\bigcup_{i=1}^{d-1} A_i| + |\{y, w\}| \le 2r$. If there is an element $v \in \bigcap_i B_i$, then $v \ne w$, since $w \notin B_d$, and $v \ne y$, since $y \notin B_1$. Thus $v \in \bigcap_i A_i = \emptyset$ which is impossible. Consequently, $\bigcap_i B_i = \emptyset$, contradicting the fact that $\mathcal{G}$ is a $K(d)$-family. We conclude that $\mathcal{L}_w$ is indeed a $K(d-1)$-family.

By (5), we have $|\mathcal{L}_w| > (1-\epsilon_{r-1})\binom{(n-1)-1}{(r-1)-1}$. Because $n_0 > n_0(\epsilon_{r-1}, r-1)+1$, and $2 \le d-1 \le r-1$, the induction hypothesis applied to $\mathcal{L}_w$ provides a vertex $x \in X - \{w\}$ so that $|\mathcal{L}_w \cap \binom{X-\{w,x\}}{r-1}| < \delta_{r-1}\binom{n-2}{r-2}$. Since $\epsilon_{r-1} < \delta_{r-1}$, we conclude that

$$\deg_{\mathcal{L}_w}(x) > (1-\epsilon_{r-1}-\delta_{r-1})\binom{n-2}{r-2} > (1-2\delta_{r-1})\binom{n-2}{r-2}. \tag{6}$$

This concludes Step 2.

Step 3

The rest of the proof is devoted to proving that $\mathcal{G} - x = \mathcal{G} \cap \binom{X-\{x\}}{r}$ satisfies

$$|\mathcal{G} - x| \le \delta_r \binom{n-1}{r-1}. \tag{7}$$

Partition $\mathcal{G} - x$ into $\mathcal{G}_1 \cup \mathcal{G}_2$, where

$$\mathcal{G}_1 = \{S \in \mathcal{G} - x : w \in S\} \qquad \text{and} \qquad \mathcal{G}_2 = \{S \in \mathcal{G} - x : w \notin S\}.$$

We will separately bound the size of each of these families. Let $\mathcal{G}_w = \mathrm{tr}_{\mathcal{G}_1}(w)$. In other words,

$$\mathcal{G}_w = \left\{ S \in \binom{X - \{w, x\}}{r-1} : S \cup \{w\} \in \mathcal{G} \right\}.$$

Note also that $\deg_{\mathcal{G}_1}(w) = |\mathcal{G}_1|$.

Claim 1. $|\mathcal{G}_1| \leq \frac{\delta_r}{2}\binom{n-1}{r-1}$.

Proof. Suppose, for a contradiction, that $|\mathcal{G}_1| > \frac{\delta_r}{2}\binom{n-1}{r-1}$. Then

$$\sum_{T \in \binom{X-\{w,x\}}{r-2}} \deg_{\mathcal{G}_w}(T) = |\mathcal{G}_1|\binom{r-1}{r-2} > \frac{\delta_r}{2}(r-1)\binom{n-1}{r-1}.$$

Consequently, there exists $T_0 \in \binom{X-\{w,x\}}{r-2}$ for which

$$\deg_{\mathcal{G}_w}(T_0) > \frac{\frac{\delta_r}{2}(r-1)\binom{n-1}{r-1}}{\binom{n-2}{r-2}} = \frac{\delta_r}{2}(n-1) > \frac{\delta_r}{2}(n-2).$$

We will now obtain a contradiction to (6). First we show that there is no $E \in \mathcal{L}_w$ satisfying $x \in E$ and $E - \{x\} \subset \mathrm{tr}_{\mathcal{G}_w}(T_0)$. Suppose to the contrary that such an E exists, say $E = \{x_1, \ldots, x_{r-2}, x\}$. Since $E \in \mathcal{L}_w$, there exists $y \notin \{x, w\}$ such that $E \cup \{y\} \in \mathcal{G}$. Because $\delta_r(n-2) > 2r$, there is an element $z \in \mathrm{tr}_{\mathcal{G}_w}(T_0) - E$. In particular, $T_0 \cup \{w, z\} \in \mathcal{G}$. Consider the d sets

$$E \cup \{w\}, E \cup \{y\}, T_0 \cup \{w, z\}, T_0 \cup \{w, x_1\}, \ldots, T_0 \cup \{w, x_{d-3}\}.$$

All these sets are in $\mathcal{G}$, and if $d = 3$, we consider only the first three. Because $T_0 \cap E = \emptyset$, these three sets have empty intersection. On the other hand, the union of these d sets is at most

$$|E| + |T_0| + 3 = (r-1) + (r-2) + 3 = 2r.$$

This contradicts the fact that $\mathcal{G}$ is a $K(d)$-family.

From the above argument, we conclude that no $E \in \mathcal{L}_w$ with $x \in E$ satisfies $E \subset \mathrm{tr}_{\mathcal{G}_w}(T_0)$. Consequently,

$$\deg_{\mathcal{L}_w}(x) \leq \binom{n-2}{r-2} - \binom{\deg_{\mathcal{G}_w}(T_0)}{r-2} < \binom{n-2}{r-2} - \binom{\frac{1}{2}\delta_r(n-2)}{r-2} < \left(1 - \left(\frac{2}{5}\delta_r\right)^{r-2}\right)\binom{n-2}{r-2},$$

where the last inequality follows from (4). By the choice of δ_{r-1} from (1), this is upper bounded by $(1 - 2\delta_{r-1})\binom{n-2}{r-2}$, a contradiction to (6). □

Before turning our attention to $\mathcal{G}_2$, we need the following definition and result. Let

$$\mathcal{G}_{w,x} = \mathrm{tr}_{\mathcal{G}}(\{w,x\}) = \left\{ E \in \binom{X - \{w,x\}}{r-2} : E \cup \{w,x\} \in \mathcal{G} \right\}.$$

Claim 2. There are disjoint $(r-3)$-sets $S_1, S_2 \subset V(\mathcal{L}_w)$ such that for each $i \in \{1,2\}$,

$$|\{y \in V(\mathcal{L}_w) - \{x\} : S_i \cup \{w,x,y\} \in \mathcal{G}\}| \geq (1 - 4\delta_{r-1})(n - r + 1).$$

Proof. Let t be the number of $(r-3)$-sets $T \subset V(\mathcal{L}_w) - \{x\}$ satisfying

$$\mathrm{tr}_{\mathcal{G}_{w,x}}(T) = |\{y \in V(\mathcal{L}_w) - \{x\} : T \cup \{w,x,y\} \in \mathcal{G}\}| \geq (1 - 4\delta_{r-1})(n - r + 1).$$

Since each set of $\mathrm{tr}_{\mathcal{L}_w}(x)$ contributes to $\mathrm{tr}_{\mathcal{G}_{w,x}}(T)$, when we sum we obtain

$$\begin{aligned}
(r-2)\mathrm{deg}_{\mathcal{L}_w}(x) = \binom{r-2}{r-3}\mathrm{deg}_{\mathcal{L}_w}(x) \quad &\leq \sum_{T' \in \binom{V(\mathcal{L}_w) - \{x\}}{r-3}} \mathrm{deg}_{\mathcal{G}_{w,x}}(T') \\
&\leq t(n - r + 1) + \left[\binom{n-2}{r-3} - t \right] (1 - 4\delta_{r-1})(n - r + 1).
\end{aligned}$$

This implies that

$$t \geq \frac{(r-2)\mathrm{deg}_{\mathcal{L}_w}(x) - \binom{n-2}{r-3}(1 - 4\delta_{r-1})(n - r + 1)}{4\delta_{r-1}(n - r + 1)}.$$

By (6), this is at least

$$\begin{aligned}
&\frac{(r-2)(1 - 2\delta_{r-1})\binom{n-2}{r-2} - \binom{n-2}{r-3}(1 - 4\delta_{r-1})(n - r + 1)}{4\delta_{r-1}(n - r + 1)} \\
= \; &\frac{1 - 2\delta_{r-1}}{4\delta_{r-1}}\binom{n-2}{r-3} - \frac{1 - 4\delta_{r-1}}{4\delta_{r-1}}\binom{n-2}{r-3} \\
= \; &\frac{1}{2}\binom{n-2}{r-3} \\
> \; &\binom{n-3}{r-4} = \binom{(n-2)-1}{(r-3)-1},
\end{aligned}$$

where the last inequality holds since $n > n_0 > 2r$. Thus the Erdős-Ko-Rado theorem applies to give the sets S_1 and S_2. □

Define, for $i \in \{1,2\}$,

$$A_i = \{y \in V(\mathcal{L}_w) - \{x\} : S_i \cup \{w,x,y\} \in \mathcal{G}\},$$

and let

$$A = A_1 \cap A_2 = \{y \in V(\mathcal{L}_w) : S_i \cup \{w,x,y\} \in \mathcal{G} \text{ for } i = 1,2\}.$$

Set

$$B = X - A - \{w, x\}.$$

Note that $S_1 \cup S_2 \subset B$. By Claim 2, $|A_i| \geq (1 - 4\delta_{r-1})(n - r + 1)$ for $i = 1, 2$. Therefore

$$|A| = |A_1 \cap A_2| \geq 2(1 - 4\delta_{r-1})(n - r + 1) - n.$$

It now follows that $|B| \leq n - |A| \leq 2n - 2(1 - 4\delta_{r-1})(n - r + 1) \leq 9\delta_{r-1}n$. By adding points from A arbitrarily to B, we may assume that

$$|B| = \lfloor 9\delta_{r-1}n \rfloor.$$

Claim 3. $|\mathcal{G}_2| \leq 18\delta_{r-1}n\binom{n-1}{r-2}$.

Proof. Partition $\mathcal{G}_2$ into $\mathcal{G}_{21} \cup \mathcal{G}_{22}$, where

$$\mathcal{G}_{21} = \{E \in \mathcal{G}_2 : |E \cap B| \leq 1\} \quad \text{and} \quad \mathcal{G}_{22} = \{E \in \mathcal{G}_2 : |E \cap B| \geq 2\}.$$

Subclaim 3.1. $|\mathcal{G}_{22}| < 2|B|\binom{n-1}{r-2}$.

Proof. In what follows, the families $\mathcal{S}_y$ and $\mathcal{L}_y$ are taken with respect to $\mathcal{G}_{22}$. Since each set in $\mathcal{G}_{22}$ contains at least two elements in B,

$$2|\mathcal{G}_{22}| \leq \sum_{y \in B} \deg_{\mathcal{G}_{22}}(y) = \sum_{y \in B}(|\mathcal{S}_y| + |\mathcal{L}_y|) = \sum_{y \in B} |\mathcal{S}_y| + \sum_{y \in B} |\mathcal{L}_y|. \tag{8}$$

Recall that $\mathcal{S}_B = \sum_{y \in B} \mathcal{S}_y \subset \partial\mathcal{G}_{22}$. The definition of $\mathcal{G}_{22}$ implies that every $E \in \mathcal{S}_B$ satisfies $|E \cap B| \geq 1$. Moreover, for every $E \in \mathcal{S}_B$, there is exactly one y for which $E \in \mathcal{S}_y$. Therefore

$$\mathcal{S}_B \subset \left\{E \in \binom{X}{r-1} : |E \cap B| \geq 1\right\},$$

and consequently

$$\sum_{y \in B} |\mathcal{S}_y| = |\mathcal{S}_B| \leq |B|\binom{n-1}{r-2}. \tag{9}$$

Since $\mathcal{G}_{22}$ is a $K(d)$-family, the argument in Step 2 implies that $\mathcal{L}_y$ is a $K(d-1)$-family for every $y \in B$. Since $n_0 - 1 > n_0(\epsilon_{r-1}, r-1)$, and $2 \leq d - 1 \leq r - 1$, the induction hypothesis applies to $\mathcal{L}_y$ and $|\mathcal{L}_y| \leq (1 + \delta_{r-1})\binom{n-2}{r-2} < 2\binom{n-2}{r-2}$ for every $y \in B$. Therefore

$$\sum_{y \in B} |\mathcal{L}_y| < 2|B|\binom{n-2}{r-2}. \tag{10}$$

Now (8), (9) and (10) imply

$$2|\mathcal{G}_{22}| < 3|B|\binom{n-1}{r-2}.$$

This gives the required bound (with room to spare) on $|\mathcal{G}_{22}|$. $\square$

Together with the fact that $|B| = \lfloor 9\delta_{r-1}n \rfloor$, Subclaim 3.1 implies that

$$|\mathcal{G}_{22}| < 18\delta_{r-1}n\binom{n-1}{r-2}. \tag{11}$$

Subclaim 3.2. $\mathcal{G}_{21} = \emptyset$.

Proof. Suppose on the contrary that $E \in \mathcal{G}_{21}$. Since $|E \cap B| \leq 1$ and $S_1 \cup S_2 \subset B$, we have $|E \cap (S_1 \cup S_2)| \leq 1$. Furthermore, by Claim 2, we know that S_1 and S_2 are disjoint. Therefore E has empty intersection with at least one S_i, say S_1. Since $d \leq r$, we may choose $d-1$ distinct elements $y_1, \ldots, y_{d-1} \in A \cap E$. By the definition of A, we know that

$$A_i = S_1 \cup \{w, x, y_i\} \in \mathcal{G} \quad \text{for all} \qquad i = 1, \ldots, d-1.$$

The $d-1$ sets above together with $A_d = E$ yield d sets $A_1, \ldots, A_d \in \mathcal{G}$. Because $E \cap (S_1 \cup \{w, x\}) = \emptyset$, it is easy to see that $\bigcap_{i=1}^{d} A_i = \emptyset$. Also, $|\bigcup_{i=1}^{d} A_i| = |E| + |S_1| + \{w, x\} = 2r-1$. This contradicts the fact that $\mathcal{G}$ is a $K(d)$-family. $\square$

Subclaim 3.2 and (11) yield $|\mathcal{G}_2| < 18\delta_{r-1}n\binom{n-1}{r-2}$, which finishes the proof of Claim 3. $\square$

Since $n > 2r$ and $\delta_{r-1} \leq \delta_r/[72(r-1)]$ by (1),

$$18\delta_{r-1}n\binom{n-1}{r-2} = 18\delta_{r-1}n\frac{r-1}{n-r+1}\binom{n-1}{r-1} < 36\delta_{r-1}(r-1)\binom{n-1}{r-1} < \frac{\delta_r}{2}\binom{n-1}{r-1}.$$

Consequently, Claims 1 and 3 give

$$|\mathcal{G} - x| = |\mathcal{G}_1| + |\mathcal{G}_2| \leq \frac{\delta_r}{2}\binom{n-1}{r-1} + \frac{\delta_r}{2}\binom{n-1}{r-1} = \delta_r\binom{n-1}{r-1},$$

and the proof of the theorem is complete. $\square$

4 From stability to an exact result

In this section we use the stability result proved in the last section to give the exact result in Theorem 3 for large n.

Proof of Theorem 3. Let $b_0 = b_0(r)$ be the threshold from Theorem 2 with $d = 4$ and $\delta = 1$. In other words, every $K(4)$-family on $b \geq b_0$ vertices has size at most $2\binom{b-1}{r-1}$. We may also assume that

$$b_0 > 10r^2. \tag{12}$$

Now select

$$\delta = \frac{(r-1)!}{r^{4r}}. \tag{13}$$

Let n_0 be the output from Theorem 2 with $d = 4$ for this δ (Theorem 2 also outputs ϵ but this is not relevant for us). Finally, choose N so that

$$N > \max\{2r^3, n_0\} \qquad \text{and} \qquad \binom{N-r-1}{r-2} > 2b_0^r. \tag{14}$$

Suppose that $n > N$ and $\mathcal{G} \subset \binom{X}{r}$ is a $K(4)$-family ($|X| = n$) with $|\mathcal{G}| = \binom{n-1}{r-1}$. We will show that $\mathcal{G}$ is a star. Since a star is a maximal $K(4)$-family, this proves the required bound on $\mathcal{G}$, with the characterization of equality as well. As $n > N > n_0$, there exists $x \in X$ such that

$$m := |\mathcal{G} - x| < \delta \binom{n-1}{r-1}. \tag{15}$$

If $m = 0$, then $\mathcal{G}$ is a star and we are done, hence we may assume that $m > 0$.

Let

$$\mathcal{G}_x = \mathrm{tr}_{\mathcal{G}}(x) = \left\{ E \in \binom{X}{r-1} : E \cup \{x\} \in \mathcal{G} \right\}.$$

Since each set in $\mathcal{G}_x$ corresponds bijectively to an edge containing x,

$$|\mathcal{G}_x| = \binom{n-1}{r-1} - m. \tag{16}$$

Claim 1. There are pairwise disjoint $(r-2)$-sets $S_1, S_2, \ldots, S_r \in \binom{X-\{x\}}{r-2}$ such that

$$\deg_{\mathcal{G}_x}(S_i) = |\{y \in X : S_i \cup \{x, y\} \in \mathcal{G}\}| \geq n - r + 1 - \frac{2rm}{\binom{n-1}{r-2}}.$$

Proof. Let t be the number of $(r-2)$-sets $T \subset X - \{x\}$ satisfying

$$\deg_{\mathcal{G}_x}(T) \geq n - r + 1 - \frac{2rm}{\binom{n-1}{r-2}}.$$

Then

$$\begin{aligned}
(r-1)|\mathcal{G}_x| = \binom{r-1}{r-2}|\mathcal{G}_x| \quad &= \sum_{T' \in \binom{X-\{x\}}{r-2}} \deg_{\mathcal{G}_x}(T') \\
&\leq \quad t(n-r+1) + \left(\binom{n-1}{r-2} - t\right)\left(n - r + 1 - \frac{2rm}{\binom{n-1}{r-2}}\right).
\end{aligned}$$

This implies that

$$t\frac{2rm}{\binom{n-1}{r-2}} \geq (r-1)|\mathcal{G}_x| - \binom{n-1}{r-2}\left(n-r+1-\frac{2rm}{\binom{n-1}{r-2}}\right).$$

By (16), the RHS is

$$(r-1)\left(\binom{n-1}{r-1}-m\right) - \binom{n-1}{r-2}\left(n-r+1-\frac{2rm}{\binom{n-1}{r-2}}\right) = -(r-1)m + 2rm.$$

Hence

$$t \geq \left(1-\frac{r-1}{2r}\right)\binom{n-1}{r-2} > \frac{1}{2}\binom{n-1}{r-2}.$$

Now consider the family of all $(r-2)$-sets described above, and let $T_1, \ldots, T_l$ be a maximum matching in this family. If $l < r$, then all other sets of this family have an element within $\cup_i T_i$, which implies that $t \leq (r-1)(r-2)\binom{n-1}{r-3} < r^2\binom{n-1}{r-3} < \binom{n-1}{r-2}/2$, because $n > 2r^3$ from (14). This contradiction shows that $l \geq r$ and the claim is proved. □

By Claim 1, for every $1 \leq i \leq r$

$$|\{y \in X : S_i \cup \{x, y\} \notin \mathcal{G}\}| < r + \frac{2rm}{\binom{n-1}{r-2}}.$$

Let

$$B = \{y \in X : S_i \cup \{x, y\} \notin \mathcal{G} \text{ for some } i \in [r]\}.$$

Then $|B| < r^2 + 2r^2m/\binom{n-1}{r-2}$. By adding points arbitrarily to B, we may assume that

$$|B| = r^2 + \left\lfloor \frac{2r^2m}{\binom{n-1}{r-2}} \right\rfloor. \tag{17}$$

Now define, for each $i \in \{0, \ldots, r\}$,

$$\mathcal{T}_i = \{T \in \mathcal{G} - x : |T \cap B| = i\}.$$

Note that $\mathcal{T}_0 \cup \cdots \cup \mathcal{T}_r$ is a partition of $\mathcal{G} - x$. In the remainder of the proof, we will show that $|\mathcal{G} - x| = |\bigcup_{i=0}^r \mathcal{T}_i| < m$, thereby contradicting (15). We first need two more Claims.

Claim 2. $\mathcal{T}_p = \emptyset$ for $0 \leq p < r-1$.

Proof. If $T \in \mathcal{T}_p$, then write $T = T_1 \cup T_2$, where $T_1 = T \cap B$ (so $|T_1| = p$) and $T_2 = T - T_1$. Since $|T_1| < r-1$, and the sets $S_1, \ldots, S_r$ are pairwise disjoint, we may assume that $S_i \cap T_1 = \emptyset$ for some i (note also that each $S_j \subset B$). Since $p \leq r-2$, we have $|T_2| \geq 2$. Let y, z, w be three elements outside B, at least two of which are in T_2. Then $S_i \cup \{x, y\}, S_i \cup \{x, z\}, S_i \cup \{x, w\}$ are all sets in $\mathcal{G}$. Together with T this yields four sets whose union is at most $2r$ and intersection is empty. This contradicts the hypothesis that $\mathcal{G}$ is a $\mathcal{K}(4)$-family. $\square$

Claim 2 implies that $\mathcal{G} - x = \mathcal{T}_r \cup \mathcal{T}_{r-1}$. We next estimate the size of $\mathcal{T}_{r-1}$.

Claim 3. $|\mathcal{T}_{r-1}| \leq \binom{|B|}{r-1}$.

Proof. Suppose there exists an $(r-1)$-set $E \subset B$ and elements $y, z \notin B$ such that $E \cup \{y\}, E \cup \{z\} \in \mathcal{T}_{r-1}$. Since $|E| = r-1$, as before we may assume that $S_i \cap E = \emptyset$ for some i. By Claim 1, we have $S_i \cup \{x, y\}, S_i \cup \{x, z\} \in \mathcal{G}$. Together with $E \cup \{y\}$ and $E \cup \{z\}$, this yields four sets in $\mathcal{G}$ whose union is $2r$ and intersection is empty. This contradicts the fact that $\mathcal{G}$ is a $K(4)$-family. Consequently, we may count sets in $\mathcal{T}_{r-1}$ by their intersection with B. This yields $|\mathcal{T}_{r-1}| \leq \binom{|B|}{r-1}$. $\square$

We now consider two cases, depending on the size of B.

Case 1: $|B| < b_0$**.** Clearly $|\mathcal{T}_r|$ and can be bounded by $\binom{|B|}{r}$. Together with Claim 3, this gives

$$m = |\mathcal{G} - x| = |\mathcal{T}_r| + |\mathcal{T}_{r-1}| \leq \binom{|B|}{r} + \binom{|B|}{r-1} < 2|B|^r < 2b_0^r.$$

Call each r-set in $\mathcal{G} - x$ *bad*, and each r-set containing x that is absent from $\mathcal{G}$ *missing*. Thus both the number of bad edges and missing edges is m. Now pick a bad edge S. Then for each $(r-2)$-set $E \subset X - \{S \cup \{x\}\}$, at least one of the r-sets $E \cup \{x, v\}$ for $v \in S$ must be missing, since otherwise three of these sets together with S would imply that $\mathcal{G}$ is not a $K(4)$-family. Consequently, to each bad set we may associate $\binom{n-r-1}{r-2}$ missing sets. By (14), this is already greater than $2b_0^r > m$, and so we can have no bad sets at all. Thus in this case $m = 0$ which is a contradiction.

Case 2: $|B| \geq b_0$**.** In this case, the argument works because we can bound the size of $\mathcal{T}_r$. Since $\mathcal{T}_r$ is itself a $K(4)$-family, the choice of b_0 implies that $|\mathcal{T}_r| \leq 2\binom{|B|-1}{r-1}$. Recalling from (17) that $|B| = \lfloor 2r^2 m / \binom{n-1}{r-2} \rfloor + r^2$, and using $|B| \geq b_0 > 10r^2$ from (12), we obtain $|B| < 3r^2 m / \binom{n-1}{r-2}$. Consequently,

$$m = |\mathcal{G} - x| = |\mathcal{T}_r| + |\mathcal{T}_{r-1}| < 2\binom{|B|-1}{r-1} + \binom{|B|}{r-1} < \frac{3|B|^{r-1}}{(r-1)!} < \frac{r^{2r} m^{r-1}}{\binom{n-1}{r-2}^{r-1}}.$$

Simplifying,

$$m^{r-2} > \binom{n-1}{r-2}^{r-1} \frac{1}{r^{2r}} > \left(\frac{n-1}{r-2}\right)^{(r-2)(r-1)} \frac{1}{r^{2r}} > \frac{(n-1)^{(r-2)(r-1)}}{r^{(r-2)(r-1)+2r}}.$$

This implies that

$$m > \frac{(n-1)^{r-1}}{r^{r-1+2r/(r-2)}} > \frac{(n-1)^{r-1}}{r^{4r}}.$$

On the other hand, by (15) we know that $m < \delta\binom{n-1}{r-1} < \delta(n-1)^{r-1}/(r-1)!$. Putting these together yields $\delta > (r-1)!/r^{4r}$ which contradicts (13) and completes the proof. $\square$

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