Simplex stability

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Abstract

A d-simplex is a collection of $d+1$ sets such that every d of them have nonempty intersection and the intersection of all of them is empty. Fix $k \geq d+2 \geq 3$ and let $\mathcal{G}$ be a family of k-element subsets of an n-element set that contains no d-simplex. We prove that if $|\mathcal{G}| \geq (1-o(1))\binom{n-1}{k-1}$, then there is a vertex x of $\mathcal{G}$ such that the number of sets in $\mathcal{G}$ omitting x is $o(n^{k-1})$ (here $o(1) \to 0$ and $n \to \infty$). A similar result when n/k is bounded from above was recently proved in [11].

Our main result is actually stronger, and implies that if $|\mathcal{G}| > (1+\epsilon)\binom{n-1}{k-1}$ for any $\epsilon > 0$ and n sufficiently large, then $\mathcal{G}$ contains $d+2$ sets $A, A_1, \ldots, A_{d+1}$ such that the A_i's form a d-simplex, and A contains an element of $\cap_{j \neq i} A_j$ for each i. This generalizes, in asymptotic form, a recent result of Vestraëte and the first author [18], who proved it for $d=1, \epsilon=0$ and $n \geq 2k$.

1 Introduction

For any integer $k \geq 2$, we denote the family of all k-element subsets of $[n] := \{1, \ldots, n\}$ by $\binom{[n]}{k}$. A family of sets is a *star* if there is a fixed element that lies in all sets; it is *intersecting* if every two of its sets have nonempty intersection.

Theorem 1 (Erdős-Ko-Rado [1]) *Let $n \geq 2k$ and $\mathcal{G} \subset \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{G}| \leq \binom{n-1}{k-1}$. If $n > 2k$ and equality holds, then $\mathcal{G}$ is a star.*

The forbidden family in Theorem 1 comprises a pair of disjoint sets. A generalization of this structure, with geometric motivation is as follows.

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Definition 2 *Fix $d \geq 1$. A collection of $d+1$ sets $A_1, A_2, ..., A_{d+1}$ is a d-dimensional simplex (or d-simplex) if every d sets have nonempty intersection, and no point lies in all $d+1$ sets.*

Indeed, a 1-simplex is a pair of disjoint edges, and Theorem 1 states that if $\mathcal{G} \subset \binom{[n]}{k}$ with $|\mathcal{G}| > \binom{n-1}{k-1}$, then $\mathcal{G}$ contains a 1-simplex. Perhaps surprisingly, the same threshold for $|\mathcal{G}|$ guarantees a d-simplex, which is a much more complicated configuration.

Conjecture 3 (Chvátal [2]) *Suppose that $k \geq d+1 \geq 2$ and $n > k(d+1)/d$. If $\mathcal{G} \subset \binom{[n]}{k}$ contains no d-simplex, then $|\mathcal{G}| \leq \binom{n-1}{k-1}$. Equality holds if and only if $\mathcal{G}$ is a star.*

The case $d = 2$ of Conjecture 3, which had earlier been asked by Erdős [3], was settled by the first author and Verstraëte in [17]. For large n, Conjecture 3 has been proved by Frankl and Füredi [5]. On the other hand, Keevash and the first author [11] very recently proved Conjecture 3 when k/n and $n/2 - k$ are both bounded away from zero.

There has been a lot of activity recently in extremal combinatorics in proving stability results. Loosely speaking, a stability result for an extremal problem with forbidden configuration F tells us that if our underlying F-free hypergraph has close to maximum size, then its structure is close to that of the example of maximum size. Such results are often independently interesting, although they have predominantly been used to solve classical extremal problems. Indeed, the seminal work of Erdös and Simonovits [20] on graph stability back in the 1960's determined the correct extremal number for graphs without color-critical subgraphs. Moreover, the recent developments for hypergraphs (see, e.g. [7, 8, 9, 12]) seem particularly exciting, since so few exact hypergraph results are known, and it is becoming apparent that no general theory that is strictly analogous to the graph case seems viable.

Within classical intersection type theorems in extremal set theory, there is some evidence that the stability method is trying to compete with the more well established delta system method to prove exact results. For example, a recent conjecture of the author [15] states that if $n > k(d+1)/d$ and $\mathcal{G} \subset \binom{[n]}{k}$ satisfies $|\mathcal{G}| > \binom{n-1}{k-1}$, then $\mathcal{G}$ contains d sets with union of size at most $2k$ and empty intersection. For large n, this was recently proved by Füredi and Ozkahya [6] using delta systems, and simultaneously by the present authors [16] using stability. In other work, Verstraëte and the first author [19] recently proved using the stability approach that if $\mathcal{G} \subset \binom{[n]}{k}$ satisfies $|\mathcal{G}| > \binom{n-1}{k-1}$ and n is sufficiently large, then $\mathcal{G}$ contains a collection of sets such that every point in their union is covered exactly twice. This generalizes the well-known fact that an n-vertex graph with n edges contains a cycle. The delta system method doesn't seem to be well suited for this problem.

The first author proved a stability result for the $d = 2$ case of Conjecture 3, and conjectured that a similar result holds for larger d.

Conjecture 4 (Mubayi [13]) *Fix $k \geq d+1 \geq 3$. For every $\delta > 0$, there exist $\epsilon > 0$ and $n_0 = n_0(\epsilon, k)$ such that the following holds for all $n > n_0$: If $\mathcal{G} \subset \binom{[n]}{k}$ contains no d-simplex and $|\mathcal{G}| > (1-\epsilon)\binom{n-1}{k-1}$, then there exists an $S \subset [n]$ with $|S| = n-1$ such that $|\mathcal{G} \cap \binom{S}{k}| < \delta\binom{n-1}{k-1}$.*

In this paper we settle Conjecture 4 except in the case $k = d+1$. For $k > d+1$, our result is actually stronger, since it guarantees a structure that contains a d-simplex.

Definition 5 *Fix $d \geq 1$. A collection of $d+2$ sets $A, A_1, A_2, ..., A_{d+1}$ is a strong d-simplex if $\{A_1, A_2, ..., A_{d+1}\}$ is a d-simplex, and A contains an element of $\cap_{j \neq i} A_j$ for each $i \in [d+1]$.*

Note that a strong 1-simplex is a collection of three sets A, B, C such that $A \cap B$ and $B \cap C$ are nonempty, and $A \cap C$ is empty. One can also think of this as a path of length three. Our main result below is more conveniently stated and proved using asymptotic notation, where $o(1) \to 0$ as $n \to \infty$.

Theorem 6 (Main Result) *Fix $k \geq d+2 \geq 3$. Let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong d-simplex. If $|\mathcal{G}| \geq (1-o(1))\binom{n-1}{k-1}$, then there is an element $x \in [n]$ such that the number of sets of $\mathcal{G}$ omitting x is $o(n^{k-1})$.*

A similar statement was proved in a recent paper of Keevash and the first author [11] where this problem was considered when $n/2 - k$ and k/n are both bounded away from 0 (thus the sets have size linear in the number of vertices). In [11] the stability result was used to settle Conjecture 3 in this range of n. We were unable to use Theorem 6 to prove the corresponding exact result in the case of n large (this would give a new proof of the result of Frankl and Füredi [5]). Nevertheless, an immediate consequence of the stability result is the following asymptotic result for strong simplices.

Corollary 7 *Fix $k \geq d+2 \geq 3$. Let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong d-simplex. Then $|\mathcal{G}| \leq (1+o(1))\binom{n-1}{k-1}$ as $n \to \infty$.*

It seems that the results of Frankl and Füredi [5] do not transparently imply, even in asymptotic form, the richer configuration guaranteed by Corollary 7 (although the current authors admit that the proof in [5] is very complicated, and it may be possible to modify it to give another proof of Corollary 7). As mentioned above, a strong 1-simplex is just a path of length 3 whose end edges are disjoint. The exact extremal function for this configuration was determined for all $n \geq 2k$ recently by the first author and Verstraëte [18].

The method of our proof requires that $k \geq d+2$. It will be interesting to prove the result for $k = d+1$ (the result is false for $k = d$, since in this range the order of magnitude of $|\mathcal{G}|$ can be as large $\Theta(n^k)$). We further conjecture that an exact result holds for set systems not containing a strong d-simplex. This is a slight strengthening of Chvátal's conjecture.

Conjecture 8 *Let $k \geq d+1 \geq 3$, $n > k(d+1)/d$ and $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong d-simplex. Then $|\mathcal{G}| \leq \binom{n-1}{k-1}$ with equality only for a star.*

The proof of Theorem 6 is by induction on d. The base case $d = 1$ needs a separate argument, and we present this in Section 3. The bulk of the proof is then presented in Section 4. The induction argument has two main steps, which are contained in subsections 4.1 and 4.2.

The basic framework for the proof is the same as that in [13], however, several additional technical steps involving new ideas are needed. The most important of these is the loading of the induction hypothesis, which leads to the definition of strong simplices. Indeed, this is the reason that the base case needs to be proved separately. Furthermore, graphs without strong 1-simplices (i.e., paths of length three) do not have the stability property, so the base case is true only for $k \geq 3$, and this is one reason why our proof works only for $k \geq d+2$.

2 Definitions and Notation

We mostly consider set systems consisting of k-element sets on a ground set $[n] = \{1, 2, ..., n\}$, usually denoted by $\mathcal{G}$. We denote by $V(\mathcal{G})$, the vertex set or the ground set where $V(\mathcal{G}) = \cup_{G \in \mathcal{G}} G$. Subsets are generally denoted by upper case Roman letters, integers by lower case Roman letters and reals by Greek letters.

Suppose $\mathcal{G}$ is a set system consisting of k-element subsets on $[n]$ and $D \subset [n]$ with $|D| \leq k$. The degree $d_{\mathcal{G}}(D)$ is the number of sets $G \in \mathcal{G}$ with $D \subset G$; when $D = \{x\}$, we simply write $d_{\mathcal{G}}(x)$. The trace of a vertex x in $\mathcal{G}$ is defined as $\mathrm{tr}_{\mathcal{G}}(x) = \{S - \{x\} : x \in S \in \mathcal{G}\}$. The sets $A \in \mathrm{tr}_{\mathcal{G}}(x)$ fall into two families : $\mathcal{L}_{\mathcal{G}}(x)$ consists of those A for which there is some $y \neq x$ for which $A \cup \{y\}$ is also in $\mathcal{G}$; $\mathcal{S}_{\mathcal{G}}(x)$ consists of those A for which $A \cup \{y\} \in \mathcal{G}$ implies $y = x$. Note that $d_{\mathcal{G}}(x) = |\mathcal{L}_{\mathcal{G}}(x)| + |\mathcal{S}_{\mathcal{G}}(x)|$.

Say that $x, y \in [n]$ are in the same connected component of $\mathcal{G}$ if there is a sequence $x = x_1, x_2, ..., x_t = y$ for some t such that for every $1 \leq i \leq t-1$, there exists a set $A_i \in \mathcal{G}$ with $\{x_i, x_{i+1}\} \in A_i$.

We say that a function $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. Similarly, $f(n) > (1 - o(1))g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \geq 1$. All asymptotic notation in this paper is taken as $n \to \infty$, where n is the number of vertices.

3 1-simplex stability

In this section, our goal is to prove the $d = 1$ case of Theorem 6. By definition, a strong 1-simplex is a collection of three sets, A, B, C such that $A \cap B = \emptyset$, but $A \cap C \neq \emptyset$ and

$B \cap C \neq \emptyset$. Alternately, a strong 1-simplex is a path of length 3.

Theorem 9 (Stability Result) *Fix $k \geq 3$. Let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong 1-simplex. If $|\mathcal{G}| \geq (1 - o(1))\binom{n-1}{k-1}$, then there exists $x \in [n]$ such that the number of sets omitting x is $o(n^{k-1})$.*

We need the following lemmas in order to prove Theorem 9.

Lemma 10 *Let $n > k \geq 3$ and let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong 1-simplex. Suppose $\mathcal{K}_1, \mathcal{K}_2, \ldots \mathcal{K}_s$ are components of $\mathcal{G}$. Then $\mathcal{K}_i$ is an intersecting family for all $i \in [s]$.*

Proof. Suppose, for contradiction, that $\mathcal{K}_i$ contains disjoint sets A, B. Since $\mathcal{K}_i$ is connected, let $\mathcal{P}$ be the shortest A-B path in $\mathcal{K}_i$. Let C be the set immediately following A and let D (possibly $D = B$) follow C. Since $\mathcal{P}$ is the shortest path, $A \cap D = \emptyset$.
Consequently, the sets A, C, and D form a strong 1-simplex in $\mathcal{G}$. This contradiction implies that $\mathcal{K}_i$ is an intersecting family for all $i \in [s]$. □

Lemma 11 *Let $k \geq 3$. If $n_1, n_2, \ldots, n_s$ are such that $n_1 \geq n_2 \geq \ldots \geq n_s \geq k$ with $\sum_i n_i \leq n$, and*

$$\sum_i \binom{n_i}{k-1} \geq (1 - o(1))\binom{n-1}{k-1},$$

then $n_1 \geq (1 - o(1))n$.

Proof. In order to prove Lemma 11, it suffices to show that for $\epsilon > 0$ and n sufficiently large, if $n_1, n_2, \ldots, n_s$ are such that $n_1 \geq n_2 \geq \ldots \geq n_s \geq k$ with $\sum_i n_i \leq n$ and

$$\sum_i \binom{n_i}{k-1} \geq (1 - \epsilon)\binom{n-1}{k-1},$$

then there exists an $\epsilon' = \epsilon'(\epsilon, k)$ such that $\epsilon' \to 0$ as $\epsilon \to 0$ and $n_1 \geq (1 - \epsilon')n$.
We interpret $\sum_i \binom{n_i}{k-1}$ as the number of $(k-1)$-sets in $\mathcal{H} = \bigcup_{i=1}^{s} \binom{X_i}{k-1}$ where $X_1, X_2, \ldots, X_s$ are disjoint and $|X_i| = n_i$. Then, for every $x \in V(\mathcal{H})$, $d_{\mathcal{H}}(x) \leq \binom{n_1-1}{k-2}$ with equality only when $x \in X_i$ and $|X_i| = n_1$.
Suppose, for contradiction, that $n_1 < (1 - \epsilon')n$. Then, since $n(\mathcal{H}) = \sum_i n_i \leq n$ and $k \geq 3$,

$$\sum_i \binom{n_i}{k-1} = |\mathcal{H}| = \frac{\sum_{x \in V(\mathcal{H})} deg(x)}{k-1} < \frac{n}{k-1}\binom{(1-\epsilon')(n-1)}{k-2} < (1-\epsilon)\binom{n-1}{k-1},$$

where the last inequality follows from an appropriate choice of ϵ' and the fact that n is sufficiently large. This contradiction implies that $n_1 \geq (1 - \epsilon')n$. □

The next result follows immediately from the Hilton-Milner theorem on nontrivial intersecting families. In order to make the proof self contained we give a much simpler argument.

Lemma 12 *Let $n > k \geq 3$ and $\mathcal{G} \subset \binom{X}{k}$ be an intersecting family with $|\mathcal{G}| > 3 + \binom{3k-3}{2}\binom{n-2}{k-2}$. Then $\mathcal{G}$ is a star.*

Proof. First, suppose that there exists $E \in \mathcal{G}$ such that $|E \cap F| \geq 2$ for every $F \in \mathcal{G}$. Then

$$|\mathcal{G}| \leq 1 + \binom{k}{2}\binom{n-2}{k-2} < 3 + \binom{3k-3}{2}\binom{n-2}{k-2},$$

a contradiction. Thus, we may assume that there exists some $x \in V(\mathcal{G})$ and distinct sets $E, E' \in \mathcal{G}$ such that $E \cap E' = \{x\}$. Suppose that $x \notin F$ for some $F \in \mathcal{G}$. Since $\mathcal{G}$ is an intersecting family, E, E', F form a triangle in $\mathcal{G}$ and therefore, every set in $\mathcal{G}$ contains at least 2 elements from $E \cup E' \cup F$. Consequently,

$$|\mathcal{G}| \leq 3 + \binom{|E \cup E' \cup F|}{2}\binom{n-2}{k-2} \leq 3 + \binom{3k-3}{2}\binom{n-2}{k-2},$$

a contradiction. Hence $x \in F$ for every $F \in \mathcal{G}$, and therefore, $\mathcal{G}$ is a star. □

Our final tool is the following exact result for strong 1-simplices.

Theorem 13 (Mubayi-Verstraëte [18]) *Fix $n \geq 2k \geq 6$. Let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong 1-simplex. Then $|\mathcal{G}| \leq \binom{n-1}{k-1}$.*

Proof of Theorem 9. Let $\mathcal{K}_1, \mathcal{K}_2, ...\mathcal{K}_s$ be the components of $\mathcal{G}$. Let $n_i = |V(\mathcal{K}_i)|$. By Lemma 10, $\mathcal{K}_i$ is an intersecting family for each $i \in [s]$. If $n_i \geq 2k$, then by Theorem 1, $|\mathcal{K}_i| \leq \binom{n_i-1}{k-1} \leq \binom{n_i}{k-1}$ and if $n_i \leq 2k-1$, then $|\mathcal{K}_i| \leq \binom{n_i}{k} \leq \binom{n_i}{k-1}$. We may assume that $n_1 \geq n_2 \geq, ... \geq n_s$. Then

$$(1 - o(1))\binom{n-1}{k-1} \leq |\mathcal{G}| = \sum_i |\mathcal{K}_i| \leq \sum_i \binom{n_i}{k-1}.$$

Lemma 11 implies that $n_1 \geq (1 - o(1))n$. Moreover, convexity of binomial coefficients yields

$$|\mathcal{G}| = |\mathcal{K}_1| + \sum_{i=2}^{s} |\mathcal{K}_i| \leq |\mathcal{K}_1| + \sum_{i=2}^{s} \binom{n_i}{k-1} \leq |\mathcal{K}_1| + \binom{n-n_1}{k-1} \leq |\mathcal{K}_1| + o(n^{k-1}). \qquad (1)$$

Since $|\mathcal{G}| \geq (1 - o(1))\binom{n-1}{k-1}$, (1) implies that

$$|\mathcal{K}_1| \geq (1 - o(1))\binom{n-1}{k-1} > 3 + \binom{3k-3}{2}\binom{n-2}{k-2},$$

where the last inequality holds since n is sufficiently large. Since $\mathcal{K}_1$ is an intersecting family of k-element sets, Lemma 12 implies that $\mathcal{K}_1$ is a star. Let x be the center of $\mathcal{K}_1$. Then $d_{\mathcal{G}}(x) \geq (1 - o(1))\binom{n-1}{k-1}$. Finally, Theorem 13 implies that the number of sets in $\mathcal{G}$ omitting x is $o(n^{k-1})$. □

4 d-simplex stability

In this section, we will prove Theorem 6. We need the following crucial lemma to carry out the induction step.

Lemma 14 *If $\mathcal{G} \subset \binom{[n]}{k}$ contains no strong d-simplex and $w \in [n]$, then $\mathcal{L}_{\mathcal{G}}(w)$ contains no strong $(d-1)$-simplex.*

Proof. Suppose, for contradiction, $\mathcal{L}_{\mathcal{G}}(w)$ contains the strong $(d-1)$-simplex $\{A, A_1, ..., A_d\}$, where $\{A_1, ..., A_d\}$ is a $(d-1)$-simplex. Since $A \in \mathcal{L}_{\mathcal{G}}(w)$, there exists $y \in X$ such that $y \neq w$ and $B_{d+1} = A \cup \{y\} \in \mathcal{G}$. Let $B = A \cup \{w\}$ and $B_i = A_i \cup \{w\}$ for all $i \in [d]$. Then, $\{B, B_1, ..., B_{d+1}\}$ is a strong d-simplex in $\mathcal{G}$, a contradiction. □

We now proceed to the proof of Theorem 6 by induction on d. The base case $d = 1$ is Theorem 9, so we let $d \geq 2$. Fix $k \geq d + 2 \geq 4$. Let $\mathcal{G} \subset \binom{[n]}{k}$ contain no strong d-simplex with $|\mathcal{G}| \geq (1 - o(1))\binom{n-1}{k-1}$. We obtain the element $x \in [n]$ in the conclusion of the theorem in two steps:

1) Find a vertex w with $|\mathcal{L}_{\mathcal{G}}(w)|$ large and $|\mathcal{S}_{\mathcal{G}}(w)|$ small, and use induction to conclude that $\mathcal{L}_{\mathcal{G}}(w)$ contains a large star with center x.

2) Show that $|\mathcal{G} - x| = |\{G \in \mathcal{G} : x \notin G\}| = o(n^{k-1})$.

Throughout the next two sections, we will assume that n is sufficiently large wherever required.

4.1 Step 1

Our goal in his subsection is to prove the following claim.

Claim 1 *There exists $w \in [n]$ such that $|\mathcal{L}_{\mathcal{G}}(w)| > (1 - o(1))\binom{n-2}{k-2}$ and $|\mathcal{S}_{\mathcal{G}}(w)| = o(n^{k-1})$.*

Proof. It suffices to show that there exists $w \in [n]$ such that for any $\epsilon > 0$, $|\mathcal{L}_{\mathcal{G}}(w)| > (1 - 3\epsilon)\binom{n-2}{k-2}$ and $|\mathcal{S}_{\mathcal{G}}(w)| < \epsilon\binom{n-1}{k-1} < \epsilon n^{k-1}$. Double counting gives

$$k|\mathcal{G}| = \sum_{x\in[n]} d_{\mathcal{G}}(x) = \sum_{x\in[n]} (|\mathcal{S}_{\mathcal{G}}(x)| + |\mathcal{L}_{\mathcal{G}}(x)|) = \sum_{x\in[n]} |\mathcal{S}_{\mathcal{G}}(x)| + \sum_{x\in[n]} |\mathcal{L}_{\mathcal{G}}(x)|.$$

Since each set in $\mathcal{S}_{\mathcal{G}}(x)$ is counted at most once in the sum, we have $\sum_{x\in[n]} |\mathcal{S}_{\mathcal{G}}(x)| \leq \binom{n}{k-1}$. As we may also assume that $|\mathcal{G}| > (1 - \epsilon)\binom{n-1}{k-1}$, this gives

$$\sum_{x\in[n]} |\mathcal{L}_{\mathcal{G}}(x)| > (1 - \epsilon)k\binom{n-1}{k-1} - \binom{n}{k-1} > (1 - 2\epsilon)n\binom{n-2}{k-2}. \tag{2}$$

Therefore, on average, we have $|\mathcal{L}_\mathcal{G}(x)| > (1-2\epsilon)\binom{n-2}{k-2}$. By Lemma 14, we know that for every $v \in [n]$, $\mathcal{L}_\mathcal{G}(v)$ contains no strong $(d-1)$-simplex, so by induction we conclude that $|\mathcal{L}_\mathcal{G}(v)| < (1+\epsilon)\binom{n-2}{k-2}$. Therefore, most $x \in [n]$ have $|\mathcal{L}_\mathcal{G}(x)|$ close to the average value, and so we can expect to find one which also has $|\mathcal{S}_\mathcal{G}(x)|$ small. We now make this precise.

Let $V = \{x \in [n] : |\mathcal{L}_\mathcal{G}(x)| \le (1-3\epsilon)\binom{n-2}{k-2}\}$. Then it follows that

$$\sum_{x\in[n]} |\mathcal{L}_\mathcal{G}(x)| = \sum_{x\in V} |\mathcal{L}_\mathcal{G}(x)| + \sum_{x\notin V} |\mathcal{L}_\mathcal{G}(x)| \le |V|(1-3\epsilon)\binom{n-2}{k-2} + (n-|V|)(1+\epsilon)\binom{n-2}{k-2}. \quad (3)$$

(2) and (3) imply

$$(1-2\epsilon)n < |V|(1-3\epsilon) + (1+\epsilon)(n-|V|).$$

A simple calculation yields $|V| < 3n/4$. This means that at least $n/4$ elements $x \notin V$ satisfy $|\mathcal{L}_x(\mathcal{G})| > (1-3\epsilon)\binom{n-2}{k-2}$. Suppose that $|\mathcal{S}_\mathcal{G}(x)| > \epsilon\binom{n-1}{k-1}$ for all $x \notin V$. Then, we have

$$\sum_{x\in[n]} |\mathcal{S}_\mathcal{G}(x)| \ge \sum_{x\notin V} |\mathcal{S}_\mathcal{G}(x)| > \epsilon\frac{n}{4}\binom{n-1}{k-1} > \binom{n}{k-1},$$

which is a contradiction. Therefore, there exists $w \notin V$ such that $|\mathcal{S}_\mathcal{G}(w)| < \epsilon\binom{n-1}{k-1} < \epsilon n^{k-1}$. Since $|\mathcal{L}_\mathcal{G}(w)| > (1-3\epsilon)\binom{n-2}{k-2}$ as well, this completes the proof. □

Let w be as in Claim 1. By Lemma 14, $\mathcal{L}_\mathcal{G}(w)$ is a family of $(k-1)$-element sets that contains no strong $(d-1)$-simplex. Also, $|\mathcal{L}_\mathcal{G}(w)| > (1-o(1))\binom{n-2}{k-2} = (1-o(1))\binom{(n-1)-1}{(k-1)-1}$. Hence, induction applies and we conclude that there exists $x \in [n] - \{w\}$ such that

$$d_{\mathcal{L}_\mathcal{G}(w)}(x) > (1-o(1))\binom{n-2}{k-2}. \quad (4)$$

This implies that

$$|\mathcal{L}_\mathcal{G}(w) - x| = |\{L \in \mathcal{L}_\mathcal{G}(w) : x \notin L\}| = o(n^{k-2}). \quad (5)$$

4.2 Step 2

Our goal in this subsection is to complete the proof by showing that

$$|\mathcal{G} - x| = |\{G \in \mathcal{G} : x \notin G\}| = o(n^{k-1}).$$

Let

$$\mathcal{G}_{w,x} = \left\{E \in \binom{[n]-\{w,x\}}{k-2} : E \cup \{w,x\} \in \mathcal{G}\right\}.$$

The following result about matchings due to Frankl is a useful tool in proving Claim 2.

Theorem 15 (Frankl [4]) *Let $\mathcal{F} \subset \binom{n}{k}$ contain no s pairwise disjoint sets. Then $|\mathcal{F}| \leq (s-1)\binom{n-1}{k-1}$ for all $n \geq sk$.*

Claim 2 *There exist d pairwise disjoint $(k-d-1)$-element sets $A_1, A_2, \dots, A_d$, such that for each $i \in [d]$, $A_i \subset V(\mathcal{L}_\mathcal{G}(w)) - \{x\}$ and*

$$\left|\left\{E \in \binom{V(\mathcal{L}_\mathcal{G}(w)) - \{x\}}{d-1} : A_i \cup E \in \mathcal{G}_{w,x}\right\}\right| > (1-o(1))\binom{n-2}{d-1}.$$

Proof. Let $\epsilon > 0$. Since k is fixed and ϵ is arbitrary, it suffices to show that there exist $(k-d-1)$-element sets $A_1, A_2, \dots, A_d$ that are pairwise disjoint and satisfy the following inequality for each $i \in [d]$:

$$\left|\left\{E \in \binom{V(\mathcal{L}_\mathcal{G}(w)) - \{x\}}{d-1} : A_i \cup E \in \mathcal{G}_{w,x}\right\}\right| > (1-k\epsilon)\binom{n-2}{d-1}. \tag{6}$$

Let t be the number of $(k-d-1)$-element sets $T \subset V(\mathcal{L}_\mathcal{G}(w)) - \{x\}$ such that

$$\left|\left\{E \in \binom{V(\mathcal{L}_\mathcal{G}(w)) - \{x\}}{d-1} : T \cup E \in \mathcal{G}_{w,x}\right\}\right| > (1-k\epsilon)\binom{n-2}{d-1}.$$

Let

$$P = \{(T,E) : |T| = k-d-1, |E| = d-1, T \cup E \in \mathcal{G}_{w,x}\}.$$

For each S satisfying $x \in S \in \mathcal{L}_\mathcal{G}(w)$ we obtain $\binom{k-2}{d-1}$ pairs $(T,E) \in P$ by choosing any $E \in \binom{S-\{x\}}{d-1}$ and $T = S - \{x\} - E$. Hence

$$|P| \geq \binom{k-2}{d-1} d_{\mathcal{L}_\mathcal{G}(w)}(x) > \binom{k-2}{d-1}(1-\epsilon)\binom{n-2}{k-2},$$

where the last inequality holds because of (4). On the other hand, the definition of t yields

$$|P| \leq t\binom{n-2}{d-1} + \left[\binom{n-2}{k-d-1} - t\right](1-k\epsilon)\binom{n-2}{d-1}.$$

Putting these two bounds together gives

$$\binom{k-2}{d-1}(1-\epsilon)\binom{n-2}{k-2} < t\binom{n-2}{d-1} + \left[\binom{n-2}{k-d-1} - t\right](1-k\epsilon)\binom{n-2}{d-1}.$$

Rearranging and solving for t, we obtain

$$t > \frac{\binom{k-2}{d-1}(1-\epsilon)\binom{n-2}{k-2} - (1-k\epsilon)\binom{n-2}{k-d-1}\binom{n-2}{d-1}}{\epsilon k\binom{n-2}{d-1}}.$$

On further simplification, the above expression yields

$$t > k^2\binom{n-3}{k-d-2} > (d-1)\binom{n-3}{k-d-2}.$$

Using Theorem 15, we obtain a collection of d disjoint sets $A_1, A_2, \ldots, A_d$, as required by the claim. $\square$

Claim 3 *There exists a $(k-d-2)$-element set A_{d+1} disjoint from $\cup_{i=1}^{d} A_i$ such that*

$$\left|\left\{E \subset \binom{V(\mathcal{L}_{\mathcal{G}}(w)) - \{x\}}{d} : A_{d+1} \cup E \in \mathcal{G}_{w,x}\right\}\right| > (1-o(1))\binom{n-2}{d}$$

Proof. Let $\epsilon > 0$. As is the previous claim, it suffices to show that

$$\left|\left\{E \subset \binom{V(\mathcal{L}_{\mathcal{G}}(w)) - \{x\}}{d} : A_{d+1} \cup E \in \mathcal{G}_{w,x}\right\}\right| > (1-k\epsilon)\binom{n-2}{d} \tag{7}$$

We let t be the number of $(k-d-2)$-element sets T satisfying (7). Then, by a similar argument as in the proof of Claim 2, we obtain

$$\binom{k-2}{d}(1-\epsilon)\binom{n-2}{k-2} < t\binom{n-2}{d} + \left[\binom{n-2}{k-d-2} - t\right](1-k\epsilon)\binom{n-2}{d}.$$

Solving for t gives

$$t > \frac{\binom{k-2}{d}(1-\epsilon)\binom{n-2}{k-2} - (1-k\epsilon)\binom{n-2}{k-d-2}\binom{n-2}{d}}{\epsilon k\binom{n-2}{d}}.$$

Since k and d are fixed and $\epsilon > 0$ is arbitrary, the above expression yields

$$t > k^2\binom{n-3}{k-d-3} > dk\binom{n-3}{k-d-3}.$$

The number of $(k-d-2)$-element sets T having at least one point in $\cup_{i=1}^{d} A_i$ is at most $dk\binom{n-3}{k-d-3}$. Since $t > dk\binom{n-3}{k-d-3}$, we conclude that there exists at least one set A_{d+1} satisfying (7) that is disjoint from $A_1, A_2, \ldots, A_d$. $\square$

For $i = 1, 2, \ldots, d$, define

$$\mathcal{H}_i = \left\{E \in \binom{V(\mathcal{L}_{\mathcal{G}}(w))}{d-1} : A_i \cup E \in \mathcal{G}_{w,x}\right\}$$

and

$$\mathcal{H}_{d+1} = \left\{E \in \binom{V(\mathcal{L}_{\mathcal{G}}(w))}{d} : A_{d+1} \cup E \in \mathcal{G}_{w,x}\right\}.$$

By Claim 2, for each $i = 1, 2, \ldots, d$,

$$|\mathcal{H}_i| > (1-o(1))\binom{n-2}{d-1}.$$

Let

$$\mathcal{H}_0 = \bigcap_{i=1}^{d} \mathcal{H}_i.$$

Since d is fixed, it follows that

$$|\mathcal{H}_0| > (1 - o(1))\binom{n-2}{d-1}. \tag{8}$$

Let $V = V(\mathcal{H}_0)$.

Claim 4 *$|V| > (1 - o(1))n$.*

Proof. Suppose for contradiction, that there exists $\epsilon \in (0, \frac{1}{d^2})$ such that $|V| < (1-\epsilon)(n-2)$. Then,

$$\begin{aligned} |\mathcal{H}_0| &< \binom{(1-\epsilon)(n-2)}{d-1} \\ &< (1-\epsilon)^{d-1}\binom{n-2}{d-1} \\ &< \left(1 - (d-1)\epsilon + \binom{d-1}{2}\epsilon^2\right)\binom{n-2}{d-1} \\ &< (1 - (d-2)\epsilon)\binom{n-2}{d-1}. \end{aligned}$$

This contradicts (8), hence proving the claim. □

Let $V' = V \cup V(\mathcal{H}_{d+1}) \cup \{w, x\}$. Then, $B = [n] - V'$ satisfies $|B| = o(n)$. We partition $\mathcal{G} - x$ into $\mathcal{G}_1$ and $\mathcal{G}_2$ defined as follows:

$$\mathcal{G}_1 = \{E \in \mathcal{G} - x : |E \cap B| \le 1\}$$

and

$$\mathcal{G}_2 = \{E \in \mathcal{G} - x : |E \cap B| > 1\}.$$

We will show that

$$|\mathcal{G} - x| = |\mathcal{G}_1 \cup \mathcal{G}_2| = |\mathcal{G}_1| + |\mathcal{G}_2| = o(n^{k-1}).$$

We first focus on $\mathcal{G}_1$. Let us consider $\mathrm{tr}_{\mathcal{G}_1}(w)$, i.e., the collection of $(k-1)$-element sets E such that $E \cup \{w\} \in \mathcal{G}_1$. By definition, $\mathrm{tr}_{\mathcal{G}_1}(w) \subset \mathcal{L}_{\mathcal{G}}(w) \cup \mathcal{S}_{\mathcal{G}}(w)$ and therefore

$$|\mathrm{tr}_{\mathcal{G}_1}(w)| \le |\mathrm{tr}_{\mathcal{G}_1}(w) \cap \mathcal{L}_{\mathcal{G}}(w)| + |\mathrm{tr}_{\mathcal{G}_1}(w) \cap \mathcal{S}_{\mathcal{G}}(w)|.$$

From (5), we have $|\mathcal{L}_{\mathcal{G}}(w) - x| = o(n^{k-2})$. Therefore, it follows that

$$|\mathrm{tr}_{\mathcal{G}_1}(w) \cap \mathcal{L}_{\mathcal{G}}(w)| \le |\mathcal{L}_{\mathcal{G}}(w) - x| = o(n^{k-2}).$$

By Claim 1, it follows that

$$|\mathrm{tr}_{\mathcal{G}_1}(w) \cap \mathcal{S}_{\mathcal{G}}(w)| \leq |\mathcal{S}_{\mathcal{G}}(w)| = o(n^{k-1}).$$

Thus,

$$d_{\mathcal{G}_1}(w) = |\mathrm{tr}_{\mathcal{G}_1}(w)| = o(n^{k-2}) + o(n^{k-1}) = o(n^{k-1}). \tag{9}$$

We will now bound the size of $\mathcal{G}_1 - w$, i.e, the sets in $\mathcal{G}_1$ that do not contain w.

Claim 5 *For each $E \in \mathcal{G}_1 - w$, there exists a d-element set $D \subset E - B$ such that $d_{\mathcal{G}_1 - w}(D) < kd\binom{n-2}{k-d-1}$.*

Proof. Let $E \in \mathcal{G}_1 - w$. Suppose, for contradiction, that every d-element subset of $E - B$ is contained in more than $kd\binom{n-2}{k-d-1}$ sets of $\mathcal{G}_1 - w$. Let $D' = \{a_1, a_2, \ldots, a_{d+1}\} \subset E - B$. Such a choice of D' is possible since $k \geq d+2$. For $i = 1, 2, \ldots, d+1$, define

$$D_i = D' - \{a_i\}.$$

Choose $E_1 \neq E$ such that $D_1 \subset E_1 \in \mathcal{G}_1 - w$. Now consider the sets $E' \in \mathcal{G}_1 - w$ such that $D_2 \subset E' \neq E$ and $E' \cap (E_1 - D_1) \neq \emptyset$. The number of these is at most $(k-d)\binom{n-2-(d-1)}{k-(d+1)} < (k-d)\binom{n-2}{k-d-1}$. As D_2 is contained in more than $kd\binom{n-2}{k-d-1}$ sets of $\mathcal{G}_1 - w$, there exists a set $E_2 \in \mathcal{G}_1 - w$ such that $D_2 \subset E_2 \neq E$ and $(E_2 - D_2) \cap (E_1 - D_1) = \emptyset$. Repeating this argument, we can find sets $E_3, E_4, \ldots, E_{d+1}$ so that for each $i = 1, 2, \ldots, d+1$, $D_i \subset E_i \neq E$ and the sets $E_i - D_i$ are pairwise disjoint. Now consider the sets $E_1, E_2, \ldots, E_{d+1} \in \mathcal{G}_1 - w$. Clearly, this is a collection of $d+1$ sets where every d sets have a non-empty intersection, but no point lies in the intersection of all $d+1$ of them. Alternatively, $\{E_1, E_2, \ldots, E_{d+1}\}$ is a d-simplex. Together with E, this collection forms a strong d-simplex in $\mathcal{G}$, which is a contradiction. □

Claim 6 $|\mathcal{G}_1 - w| = o(n^{k-1})$.

Proof. Suppose, for contradiction, that there exists $\epsilon > 0$ such that $|\mathcal{G}_1 - w| > \epsilon\binom{n-2}{k-1}$. From Claim 5, we know that for each $E \in \mathcal{G}_1 - w$, there exists a d-element subset $D \subset E - B$ such that $\mathrm{d}_{\mathcal{G}_1 - w}(D) < kd\binom{n-2}{k-d-1}$. If t is the number of such d-element subsets, then

$$t \geq \frac{\epsilon\binom{n-2}{k-1}}{kd\binom{n-2}{k-d-1}} > \epsilon'\binom{n-2}{d}, \tag{10}$$

where $\epsilon' = \frac{\epsilon}{k^d}$. Recall that

$$\mathcal{H}_{d+1} = \left\{ E \in \binom{V(\mathcal{L}_{\mathcal{G}}(w))}{d} : A_{d+1} \cup E \in \mathcal{G}_{w,x} \right\}.$$

Claim 3 implies that $|\mathcal{H}_{d+1}| > (1-o(1))\binom{n-2}{d}$. Choose $\delta \in (0, \frac{\epsilon'}{1+d^2})$. Then, we have

$$|\mathcal{H}_{d+1}| > (1-\delta)\binom{n-2}{d}.$$

Therefore, we conclude that at least $(\epsilon' - \delta)\binom{n-2}{d}$ of the d-element subsets counted in (10) are members of $\mathcal{H}_{d+1}$. Let us denote this family by $\mathcal{H}'$. We now argue that there exists a set $D = \{a_1, a_2, \ldots, a_d\} \in \mathcal{H}'$ such that $D_i = D - \{a_i\} \in \mathcal{H}_0$ for each $i = 1, 2, \ldots, d$. Otherwise, for every $D \in \mathcal{H}'$, there exists at least one $(d-1)$-element subset $D' \subset D$ such that $D' \notin \mathcal{H}_0$. Let s be the number of these sets D'. Since d is fixed and δ is arbitrary, (8) implies that $|\mathcal{H}_0| > (1 - d\delta)\binom{n-2}{d-1}$. In other words, the number of $(d-1)$-element sets that are not contained in $\mathcal{H}_0$ is less than $d\delta\binom{n-2}{d-1}$. Then certainly,

$$d\delta\binom{n-2}{d-1} > s > \frac{(\epsilon'-\delta)\binom{n-2}{d}}{n} = \frac{(\epsilon'-\delta)(n-d-1)}{nd}\binom{n-2}{d-1},$$

which is a contradiction since $\delta < \frac{\epsilon'}{1+d^2}$. Choose $E \in \mathcal{G}_1 - w$ such that $D \subset E - B$. For $i = 1, 2, \ldots, d$, let

$$E_i = A_i \cup D_i \cup \{w, x\},$$

where $A_1, A_2, \ldots, A_d$ are obtained from Claim 2 and

$$E_{d+1} = A_{d+1} \cup D \cup \{w, x\},$$

where A_{d+1} is obtained from Claim 3. By definition, $E_i \in \mathcal{G}$ for each $i = 1, \ldots, d+1$. Consider the collection $\{E, E_1, E_2, \ldots, E_d\}$. The intersection of all of these sets is empty, since $\{w, x\} \cap E = \emptyset$, the A_i's are pairwise disjoint, and $a_i \notin D_i$. On the other hand, all the E_i's contains both w and x, and every $d-1$ of the E_i's together with E have nonempty intersection as well. Thus every d of these sets have nonempty intersection. Therefore $\{E, E_1, E_2, \ldots, E_d\}$ is a d-simplex. Together with E_{d+1}, this collection forms a strong d-simplex in $\mathcal{G}$, which is a contradiction. Hence, we have proved that there exists no $\epsilon > 0$ for which $|\mathcal{G}_1 - w| > \epsilon\binom{n-2}{k-1}$. Alternately, $|\mathcal{G}_1 - w| = o(\binom{n-2}{k-1}) = o(n^{k-1})$. □

Our final task is to bound $|\mathcal{G}_2|$. Since every $E \in \mathcal{G}_2$ contains at least two points in B, we have

$$2|\mathcal{G}_2| \le \sum_{x \in B} d_{\mathcal{G}_2}(x) = \sum_{x \in B}(|\mathcal{S}_{\mathcal{G}_2}(x)| + |\mathcal{L}_{\mathcal{G}_2}(x)|) = \sum_{x \in B}|\mathcal{S}_{\mathcal{G}_2}(x)| + \sum_{x \in B}|\mathcal{L}_{\mathcal{G}_2}(x)|. \qquad (11)$$

Define $\partial\mathcal{G}_2 = \{S \in \binom{[n]}{k-1} : \text{there exists some } T \in \mathcal{G}_2 \text{ with } S \subset T\}$. Then, for any $E' \in \partial\mathcal{G}_2$, $|E' \cap B| \ge 1$. Since $\cup_{x \in B}\mathcal{S}_{\mathcal{G}_2}(x) \subset \partial\mathcal{G}$, the same conclusion holds for sets in $\cup_{x \in B}\mathcal{S}_{\mathcal{G}_2}(x)$.

Also, by definition, for every $E \in \cup_{x\in B}\mathcal{S}_{\mathcal{G}_2}(x)$, there is exactly one x for which $E \in \mathcal{S}_{\mathcal{G}_2}(x)$. Therefore,

$$\sum_{x\in B} |\mathcal{S}_{\mathcal{G}_2}(x)| \leq |B|\binom{n-1}{k-2} = o(n)\binom{n-1}{k-2} = o(n^{k-1}). \tag{12}$$

Lemma 14 and induction imply that $|\mathcal{L}_{\mathcal{G}_2}(x)| < (1+\epsilon)\binom{n-2}{k-2}$ for every $x \in B$ and some $\epsilon > 0$. This yields

$$\sum_{x\in B} |\mathcal{L}_{\mathcal{G}_2}(x)| < |B|(1+\epsilon)\binom{n-2}{k-2} = o(n)(1+\epsilon)\binom{n-2}{k-2} = o(n^{k-1}). \tag{13}$$

From (12) and (13), we have

$$2|\mathcal{G}_2| = o(n^{k-1}) \Rightarrow |\mathcal{G}_2| = o(n^{k-1}). \tag{14}$$

Finally, we conclude that

$$\begin{aligned} |\mathcal{G} - x| &= |\mathcal{G}_1| + |\mathcal{G}_2| \\ &= d_{\mathcal{G}_1}(w) + |\mathcal{G}_1 - w| + |\mathcal{G}_2| \\ &= o(n^{k-1}) + o(n^{k-1}) + o(n^{k-1}) \\ &= o(n^{k-1}), \end{aligned}$$

where the respective bounds follow from (9), Claim 6, and (14). This completes the proof. □

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