Partitioning ordered hypergraphs

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Abstract

An ordered r-graph is an r-uniform hypergraph whose vertex set is linearly ordered. Given 2 ≤ k ≤ r, an ordered r-graph H is interval k-partite if there exist at least k disjoint intervals in the ordering such that every edge of H has nonempty intersection with each of the intervals and is contained in their union.

Our main result implies that for each α > k − 1 and d > 0, every n-vertex ordered r-graph with d n^α edges has for some m ≤ n an m-vertex interval k-partite subgraph with Ω(dm^α) edges. This is an extension to ordered r-graphs of the observation by Erdős and Kleitman that every r-graph contains an r-partite subgraph with a constant proportion of the edges. The restriction α > k − 1 is sharp. We also present applications of the main result to several extremal problems for ordered hypergraphs.

1 Introduction

We let [n] = {1, . . . , n} and use standard asymptotic notation; in particular, given functions f, g : \mathbb{Z}^+ → \mathbb{R}^+, we write f(n) = Ω(g(n)) if there exists c > 0 such that f(n) ≥ cg(n) for all n ≥ 1. We associate a hypergraph H with its edge set and write e(H) for the number of the edges and v(H) for the number of the vertices in H.

An r-graph is a hypergraph with all edges of size r; it is r-partite if there is a partition of the vertex set into r parts such that every edge has exactly one vertex in each part. The following observation is due to Erdős and Kleitman:

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**Proposition A.** (Erdős-Kleitman [6]) Every $r$-graph contains an $r$-partite subgraph with at least $\frac{r!}{r^r}$ proportion of its edges.

In particular, any extremal problem for $r$-graphs can be reduced to the corresponding extremal problem where the underlying $r$-graph is $r$-partite with the loss of only a constant multiplicative factor. In this paper, we consider analogs of this result in the ordered hypergraph setting and illustrate their use on some ordered extremal hypergraph problems.

An ordered hypergraph is a hypergraph together with a linear ordering of its vertex set. Extremal problems on ordered hypergraphs arose from several sources, in particular, from combinatorial geometry, enumeration of permutations with forbidden subpermutations, and the study of matrices with forbidden submatrices – see for instance Anstee [1, 2], Füredi and Hajnal [11], Pach and Tardos [19], Marcus and Tardos [16], Tardos [21], Fox [8].

Let $V$ be a linearly ordered set. An interval in $V$ is a set of consecutive elements in the ordering. For $A, B \subset V$, we write $A < B$ to mean that $a < b$ for every $a \in A, b \in B$. A key definition in our work is the following:

**Definition 1.** Let $k$ be a positive integer. An ordered $r$-graph $H$ is interval $k$-partite if for some $\ell \geq k$ there are intervals $I_1 < I_2 < \cdots < I_\ell$ such that every edge of $H$ is contained in $I_1 \cup \ldots \cup I_\ell$ and has nonempty intersection with $I_j$ for each $1 \leq j \leq \ell$.

In particular, an ordered $r$-graph $H$ is interval $r$-partite if there exist intervals $I_1 < I_2 < \cdots < I_r$ in $V(G)$ such that every edge of $H$ contains exactly one vertex from each $I_i$. In these terms, the Erdős–Kleitman observation, Proposition A, does not hold for ordered graphs as witnessed by the following simple example: every interval bipartite subgraph of the ordered graph with vertex set $[2n]$ and edge set $\{\{2i − 1, 2i\}: 1 \leq i \leq n\}$ has at most one edge. However, Pach and Tardos [19] showed that dense ordered graphs contain relatively dense interval bipartite graphs using the following result:

**Theorem B.** (Pach and Tardos [19]). Each ordered $n$-vertex graph $G$ is the union of edge-disjoint subgraphs $G_i$ for $0 \leq i \leq \lfloor \log_2 n \rfloor$ such that each $G_i$ is a union of at most $2^i$ interval bipartite graphs with parts of size at most $\lceil n/2^i \rceil$.

Our first main result is the following ordered hypergraph analog of Theorem B:

**Theorem 1.1.** Let $2 \leq k \leq r \leq n$ be integers. Then every ordered $n$-vertex $r$-graph $H$ is the union of edge-disjoint ordered $r$-graphs $H_i$ for $0 \leq i \leq \lfloor \log_2 n \rfloor$ such that each $H_i$ is a union of at most $\frac{1}{(k-1)!} \sum_{j=k}^{r} \binom{2k-2}{j} \cdot 2^{i(k-1)}$ interval $k$-partite $r$-graphs with parts of size at most $\lceil n/2^i \rceil$.

For $k = r = 2$, Theorem 1.1 corresponds to Theorem B. Note that Theorem B easily implies the following, which appears implicitly in Pach and Tardos [19]:

**Theorem C.** For each real $\alpha \geq 1$, $d > 0$ and $n > 1$, if $G$ is an ordered $n$-vertex graph with $e(G) = dn^\alpha$, ...
then for some \( m \in [n] \), \( G \) contains an interval bipartite subgraph \( G' \) with parts of size at most \( m \) and

\[
e(G') = \begin{cases} 
\Omega\left( \frac{dm^\alpha}{\log_2 n} \right) & \text{if } \alpha = 1 \\
\Omega(dm^\alpha) & \text{if } \alpha > 1 
\end{cases}
\]

(1)

As observed by Pach and Tardos [19], the logarithmic factor in (1) for \( \alpha = 1 \) is necessary: for the ordered path \( P \) with edges \( \{v_i, v_{i+1}\} : 1 \leq i \leq 4 \) such that \( v_2 < v_4 < v_3 < v_1 < v_5 \), extremal \( n \)-vertex ordered \( P \)-free graphs have \( n \log n + O(n) \) edges, whereas an extremal \( n \)-vertex interval bipartite \( P \)-free graph has \( \Theta(n) \) edges (see Füredi [10], Bienstock and Győri [3], and Tardos [22]).

Our second main result is the following generalization of Theorem C to ordered \( r \)-graphs:

**Theorem 1.2.** Let \( 2 \leq k \leq r \leq n \) be integers and let \( \alpha \) be a real number with \( k - 1 \leq \alpha \leq r \). Then every ordered \( r \)-graph \( H \) with \( n \) vertices and \( dn^\alpha \) edges has an interval \( k \)-partite subgraph \( H' \) with parts of size at most \( m \) for some \( m \in [n] \) and

\[
e(H') = \begin{cases} 
\Omega\left( \frac{dm^\alpha}{\log_2 n} \right) & \text{if } \alpha = k - 1 \\
\Omega(dm^\alpha) & \text{if } \alpha > k - 1 
\end{cases}
\]

(2)

The case \( k = r = 2 \) is Theorem C.

**Remarks.**

- Theorem 1.2 is sharp in that for \( 2 \leq k < r \) and \( \alpha = k - 1 \), there exist \( n \)-vertex \( r \)-graphs \( H \) with \( e(H) = dn^\alpha \) where every interval \( k \)-partite subgraph \( H' \) with parts of size \( m \) has \( e(H') = O(dm^\alpha / \log n) \), and for \( \alpha < k - 1 \), there exist \( n \)-vertex \( r \)-graphs \( H \) with \( e(H) = dn^\alpha \) where every interval \( k \)-partite subgraph \( H' \) has \( e(H') = O(dn^{\alpha-a}) \) where \( a = \min\{1, k - 1 - \alpha\} > 0 \). We will prove this in Section 3 (see Constructions 1 and 2).

- For \( \alpha > k - 1 \), Theorem 1.2 guarantees that an \( n \)-vertex ordered \( r \)-graph with \( \Theta(n^\alpha) \) edges has an interval \( k \)-partite subgraph parts of size \( m \) and \( \Theta(m^\alpha) \) edges for some \( m \in [n] \). In sharp contrast with the Erdős–Kleitman Lemma, Proposition A, the value of \( m \) may be necessarily be small relative to the number of vertices in the host \( r \)-graph: we give a construction in Section 2 (see Construction 3) where we need \( m = O(n^{1-1/\alpha}) \) for \( \alpha > k - 1 \).

- We do not optimize the constant \( c = c(\alpha, k, r) \) in the bound \( e(H') \geq cdm^\alpha \) for \( \alpha > k - 1 \) in Theorem 1.2. The proof of Theorem 1.2 gives

\[
c(\alpha, k, r) \geq \frac{(k-1)!(1-2^{k-1-\alpha})}{\sum_{j=k}^{r} (\binom{2k-2}{j})}.
\]

(3)
In particular, \( c(r, r, r) \geq (r - 1)!4^{-r} \), whereas for every \( r \)-partite subgraph \( H' \) of \( K_n^r \) with parts of size \( m \), \( e(H') \leq m^r \), and so \( c(r, r, r) \leq r! \).

- For each partition \( \pi \) of \( r \), one can extend Theorem 1.2 to the setting of interval \( \pi \)-partite subgraphs – here \( \pi \) specifies the number of vertices of an edge in each part – by replacing the range of \( \alpha \) to \( \alpha \geq f(\pi) \) where \( f(\pi) \) is the maximum length of a partition that is not a refinement of \( \pi \). For example, if \( \pi = 1 + 1 + \cdots + 1 \), then \( f(\pi) = r - 1 \), if \( \pi = 2 + 1 + \cdots + 1 \), then \( f(\pi) = r - 2 \), and if \( \pi = 1 + (r - 1) \), then \( f(\pi) = \lfloor r/2 \rfloor \). This has other interesting consequences which we will explore in forthcoming work.

### 1.1 Applications of Theorem 1.2

We next describe how to apply Theorem 1.2 to a variety of ordered extremal problems and convex geometric extremal problems for families of \( r \)-graphs. This enables us to transfer classical extremal problems to the ordered setting via Theorem 1.2. The following definition is needed:

**Definition 2.** For an \( r \)-partite \( r \)-graph \( F \), \( \text{ord}(F) \) denotes the family of interval \( r \)-partite \( r \)-graphs isomorphic to \( F \). For a family \( \mathcal{F} \) of \( r \)-partite \( r \)-graphs, \( \text{ord}(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \text{ord}(F) \).

Note that \( \text{ord}(\mathcal{F}) \) may be empty but not in the cases we investigate. A first and natural example is the case that \( \mathcal{F} \) consists of the \( r \)-graph of two disjoint edges. The Erdős-Ko-Rado Theorem [7] states that for \( n \geq 2r + 1 \), the unique extremal \( n \)-vertex \( r \)-graph without two disjoint edges consists of all \( r \)-sets containing one vertex, with \( \binom{n-1}{r-1} \) edges. In [13], the following ordered version of the Erdős-Ko-Rado Theorem is proved:

**Theorem 1.3.** ([13]) Let \( r \geq 3 \) and \( n \geq 2r + 1 \). Then the maximum number of edges in an ordered \( n \)-vertex \( r \)-graph that does not contain two edges of the form \( \{v_1, v_2, \ldots, v_r\} \) and \( \{w_1, w_2, \ldots, w_r\} \) such that \( v_1 < w_1 < v_2 < w_2 < \cdots < v_r < w_r \) is exactly \( \binom{n}{r} - \binom{n-r}{r} \).

For an ordered \( r \)-graph \( F \), let \( \text{ex}_{\rightarrow}(n, F) \) denote the maximum number of edges in an \( n \)-vertex ordered \( r \)-graph that does not contain \( F \). For a family \( \mathcal{F} \) of ordered \( r \)-graphs, let \( \text{ex}_{\rightarrow}(n, \mathcal{F}) = \min_{F \in \mathcal{F}} \text{ex}_{\rightarrow}(n, F) \). In this language Theorem 1.3 implies that for \( n \geq 2r + 1 \),

\[
\text{ex}_{\rightarrow}(n, \text{ord}(\mathcal{F})) \leq \binom{n}{r} - \binom{n-r}{r},
\]

where \( F \) is the \( r \)-graph comprising two disjoint edges (in fact, it applies to a particular member of \( \text{ord}(\mathcal{F}) \)). Results for hypergraph matchings (i.e., for sets of disjoint edges) by Klazar and Marcus [15] show that for each interval \( r \)-partite matching \( M \), \( \text{ex}_{\rightarrow}(n, M) = O(n^{r-1}) \), thereby extending the celebrated Marcus-Tardos [16] theorem for matchings in ordered graphs to ordered \( r \)-graphs. We now give some further examples where classical extremal problems are transferred to the ordered setting via Theorem 1.2.
1.1.1 Simplices

A $d$-dimensional $r$-simplex is an $r$-graph of $d + 1$ edges such that any $d$ of the edges have non-empty intersection, but all $d + 1$ edges have empty intersection. Denote by $S_d^r$ the family of $d$-dimensional $r$-simplices. The set $S_d^r$ is non-empty if $r \geq d$. The study of these abstract simplices in the context of extremal hypergraph theory was first initiated by Chvátal who posed the following conjecture.

Conjecture 1. (Chvátal [4]) Let $r \geq d + 1 \geq 3$ and $n \geq r(d + 1)/d$. Then $\text{ex}(n, S_d^r) = \binom{n-1}{r-1}$.

Frankl and Füredi [9] proved Conjecture 1 for large $n$ (Keller and Lifschitz [14] improved the bounds on $n$) and Mubayi and Verstraëte [17] proved it for $d = 2$, which was a problem of Erdős. Very recently, Currier [5] proved the conjecture for $n \geq 2r$. We prove the following theorem.

Theorem 1.4. For all fixed $r \geq d + 1 \geq 3$,

$$\text{ex}(n, \text{ord}(S_d^r)) = \Theta(n^{r-1}).$$

1.1.2 Expansions

Our next example is more general. If $\mathcal{F}$ is a family of $(r - 1)$-graphs, let $\mathcal{F}^+$ denote the family of $r$-graphs $F^+$ obtained from each $F \in \mathcal{F}$ by adding a vertex $v_e$ to edge $e \in F$ such that all the vertices $v_e : e \in F$ are distinct from each other and from the vertices of $F$. A study of extremal problems for families $\mathcal{F}^+$ is given in [18], where $\mathcal{F}^+$ is referred to as an expansion of $\mathcal{F}$. Such families lend themselves naturally to an application of Theorem 1.2.

Theorem 1.5. Let $r \geq 3$ and $\mathcal{F}$ be a family of $(r - 1)$-graphs with $\text{ex}(n, \text{ord}(\mathcal{F})) = O(n^{r-2})$. Then

$$\text{ex}(n, \text{ord}(\mathcal{F}^+)) = O(n^{r-1}).$$

Actually, our proof yields a stronger fact. Recall that for a vertex $v$ in an $r$-graph $G$, the link of $v$ in $G$ is the $(r - 1)$-graph $G(v)$ whose edge set is $\{A - v; v \in A \in E(G)\}$. Our proof shows that for some $C$ each ordered $n$-vertex $r$-graph with at least $Cn^{r-1}$ edges contains an $r$-graph obtained from an $F_0 \in \text{ord}(\mathcal{F}^+)$ by choosing some vertex $v_0$ of degree 1 in $F_0$ and adding all the edges of the kind $L \cup v_0$ where $L$ is an $(r - 1)$-subset of an edge in $F_0$. This stronger fact implies that

$$\text{ex}(n, \text{ord}(T_r)) = O(n^{r-1}),$$

where $T_r = \{e, f, g\}$ is the loose $r$-uniform triangle i.e. $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. 

1.1.3 Hypergraph forests

Our next application concerns hypergraph forests. The shadow $\partial H$ of an $r$-graph $H$ is the collection of $(r-1)$-sets contained in some edge of $H$. We follow Frankl and Füredi [9] for an inductive definition of trees in hypergraphs: a single edge is a tree, and given any tree $T$ with edges $e_1, e_2, \ldots, e_h$, a tree with $h+1$ edges is obtained by selecting $f \in \partial T$ and a vertex $x$ not in $T$, and adding the edge $f \cup \{x\}$. A forest is a subgraph of a tree. By definition, each 2-uniform tree (respectively, 2-uniform forest) is a tree (respectively, forest) in the usual sense. Using Theorem 1.2, we prove the following:

**Theorem 1.6.** Fix $r \geq 2$ and let $F$ be an $r$-uniform forest. Then $\lim_{n \to \infty} \frac{\text{ex}(n, F)}{\text{ord}(F)} = 0$.

**Remarks.**
- A conjecture of Pach and Tardos [19] would imply $\lim_{n \to \infty} \frac{\text{ex}(n, T)}{n^{1+o(1)}}$ for every 2-interval-partite tree $T$ with at least two edges. Theorems 1.5 and 1.6 suggest that perhaps for every interval $r$-partite $r$-uniform tree $T$, $\lim_{n \to \infty} \frac{\text{ex}(n, T)}{n^{r-1+o(1)}}$.
- It remains an intriguing open problem to determine for which $r$-graph families $\mathcal{F}$

$$\lim_{n \to \infty} \frac{\text{ex}(n, \mathcal{F})}{n^{r-1}} = 0 \implies \lim_{n \to \infty} \frac{\text{ex}(n, \text{ord}(\mathcal{F}))}{n^{r-1}} = 0.$$  \hspace{1cm} (4)

According to Theorem 1.6, this is true for $r = 2$. Since for every $r$-uniform forest $F$, $\lim_{n \to \infty} \frac{\text{ex}(n, F)}{n^{r-1}} = 0$, Theorem 1.6 yields that the above implication is also true if $\mathcal{F}$ contains an $r$-uniform forest. We do not know any explicit example for $r \geq 3$ for which (4) fails, although we believe that many such examples exist.

- In [12], we heavily used the $k = r - 1$ case of Theorem 1.2 to prove that the extremal function of so called crossing paths in convex geometric hypergraphs has order $n^{r-1}$ or $n^{r-1} \log n$.

1.1.4 Ordered Ruzsa-Szemerédi Theorem

We consider the ordered version of the famous Ruzsa-Szemerédi (6,3)-Theorem [20] which states that the maximum number of edges in an $n$-vertex 3-graph with no 6 vertices spanning 3 edges is $o(n^2)$. This is equivalent to the statement $\lim_{n \to \infty} \frac{\text{ex}(n, \mathcal{F}_{RS})}{n^{2}} = o(1)$ where $\mathcal{F}_{RS} = \{I_2, T_3\}$ and $I_2$ is the 3-graph comprising two edges sharing exactly two points.

**Theorem 1.7.** Let $\mathcal{F}_{RS} = \{I_2, T_3\}$. Then $\lim_{n \to \infty} \frac{\text{ex}(n, \text{ord}(\mathcal{F}_{RS}))}{n^2} = o(1)$.

1.1.5 Forbidden ordered intersections

Our final example addresses an $r$-graph problem whose answer has order of magnitude $n^\alpha$ where $\alpha \neq r - 1$. Let $I'(\ell)$ denote the $r$-graph consisting of two edges sharing exactly $\ell$ vertices. The study
Lemma 2.1. Let

\[ \text{ex}(n, I^r(\ell)) = \Theta(n^{\max\{\ell, r-\ell-1\}}) \quad \text{for } 0 \leq \ell \leq r - 1. \]

We are able to extend this result to the ordered setting using Theorem 1.2.

**Theorem 1.8.** For \( r \geq 2 \) and \( 1 \leq \ell \leq r - 1 \), and \( \alpha = \max\{\ell, r - \lceil (\ell + 1)/2 \rceil\} \)

\[ \Omega(n^\alpha) = \text{ex}_{\rightarrow}(n, \text{ord}(I^r(\ell))) = \begin{cases} O(n^\alpha) & \text{if } \ell \text{ is odd} \\ O(n^\alpha \log n) & \text{if } \ell \text{ is even.} \end{cases} \]

Note that the \( \ell = 0 \) case is covered by Theorem 1.3 which gives \( \text{ex}_{\rightarrow}(n, \text{ord}(I^r(0))) = \Theta(n^{r-1}) \). A construction of a dense \( \text{ord}(I^r(\ell)) \)-free ordered \( r \)-graph is given in Construction 4. We believe the \( \log n \) factor when \( \ell \) is even can be removed, so that \( \text{ex}_{\rightarrow}(n, \text{ord}(I^r(\ell))) = \Theta(n^\alpha) \).

We present four constructions in Section 2 and prove Theorem 1.2 in Section 3. Theorems 1.4-1.8 are proved in Section 4.

## 2 Constructions

Our first construction requires the following lemma.

**Lemma 2.1.** Let \( H_{n_1,n_2} \) be the ordered bipartite graph with vertex set \([n_1 + n_2]\) and parts \( A = [n_1] \) and \( B = \{n_1 + 1, \ldots, n_1 + n_2\} \) such that for \( i < j \), the pair \( ij \) is an edge in \( H_{n_1,n_2} \) iff \( 1 \leq i \leq n_1 < j \leq n_1 + n_2 \) and \( j - i \) is a power of 2. Then for each \( A' \subseteq A \) and \( B' \subseteq B \), the number of edges in \( H_{n_1,n_2}[A' \cup B'] \) is at most \( 2|A' \cup B'| \).

**Proof.** Suppose for some \( A' \subseteq A \) and \( B' \subseteq B \), graph \( H := H_{n_1,n_2}[A' \cup B'] \) has more than \( 2(|A'| + |B'|) \) edges. Let us assume \( A' = \{a_1, a_2, \ldots, a_l\} \) and \( B' = \{b_1, b_2, \ldots, b_m\} \) where

\[ a_1 < a_2 < \cdots < a_l < b_1 < b_2 < \cdots < b_m. \]

For each vertex \( a \in A' \), remove from \( H \) the edges \( \{a, b_i\} \) and \( \{a, b_j\} \) where \( i \) is minimum index for which such an edge exists, and \( j \) is maximum index for which such an edge exists. Repeat this procedure for \( b \in B' \) with respect to vertices in \( A' \). Since \( H \) has more than \( 2(|A'| + |B'|) \) edges, and we removed at most \( 2(|A'| + |B'|) \) edges, the remaining graph \( H' \) has an edge \( \{a, b\} \) with \( a \in A', b \in B' \). Now there exist vertices \( a' \) and \( b' \) such that \( \{a', b\} \) and \( \{a, b'\} \) are edges and \( a < a' < b' < b \). However, it is not possible for \( b - a, b' - a \), and \( b - a' \) all to be powers of 2.

Our first construction shows the logarithmic factor in the first bound in Theorem 1.2 is necessary.
Construction 1: An $n$-vertex $r$-graph $H_r(n)$ with $dn^{k-1}$ edges such that for every $m \leq n$

$$e(H') = O\left(\frac{dm^{k-1}}{\log n}\right)$$

for every interval $k$-partite $H' \subset H_r(n)$ with parts of size $m$.

The vertex set of $H_r(n)$ is $[n]$ ordered as $1 < 2 < \cdots < n$. The edges of $H_r(n)$ are the sets \{v_1, v_2, \ldots, v_r\} with $v_1 < v_2 < \cdots < v_r$ such that the difference $v_{i+1} - v_i$ is a power of 2 for $i = 1, \ldots, r - k + 1$. Then $e(H_r(n)) = \Theta(n^{k-1}(\log n)^{r-k+1}) = \Theta(dn^{k-1})$ where $d = (\log n)^{r-k+1}$.

Let $H'$ be any interval $k$-partite subgraph of $H_r(n)$, with ordered parts $I_1 < I_2 < \cdots < I_k$ each of size $m$, and $G'$ be the bipartite graph whose edges are pairs \{v, w\} $\in e = \{v_1, v_2, \ldots, v_r\} \in H'$ where $v$ is the largest vertex in $I_1$ and $w$ is the smallest vertex in $I_2$. Note, crucially, that $w - v$ must be a power of 2, since otherwise the $r - k + 2$ smallest vertices of $e$ lie in $I_1$, which means that some $I_j$ is empty. Lemma 2.1 now yields that $e(G') \leq 2v(G') \leq 4m$. But then $e(H') = O(m^{k-1}(\log n)^{r-k})$, so $e(H') = O(dm^{k-1}/\log n)$.

Our next construction shows that the bound $\alpha \geq k - 1$ in Theorem 1.2 cannot be improved.

Construction 2: An $rn$-vertex $r$-graph $H'_n(r)$ with $dn^\alpha$ edges such that for $1 \leq \alpha < k - 1 \leq r - 1$

$$e(H') = O(dm^\alpha/n^\alpha)$$

for every interval $k$-partite $H' \subset H'_n(r)$ with parts of size $m$.

For $k \leq r$ and $1 \leq j \leq n$, let $I_j = \{(r-k+2)j - (r-k+1), (r-k+2)j - (r-k+1) + 1, \ldots, (r-k+2)j\}$, so that $|I_j| = r - k + 2$, and let $H'_n(r)$ be the ordered $r$-graph with vertex set $[rn]$ and edge set

$$\{I_j \cup \{ar_{r-k+3}, ar_{r-k+4}, \ldots, ar\} : 1 \leq j \leq n, \text{ and } (\ell - 1)n < a\ell \leq \ell n \text{ for } r - k + 3 \leq \ell \leq r\}.$$

By definition, $e(H'_n(r)) = n^{k-1} = dn^\alpha$ where $d = n^{k-1-\alpha}$. On the other hand, let $H'$ be an interval $k$-partite subgraph $H'$ of $H'_n(r)$ with parts of size $m$. Then there exists $j$ such that every edge of $H'$ contains $I_j$. In particular,

$$e(H') = O(m^{k-2}) = O\left(dm^\alpha \cdot \frac{m^{k-2-\alpha}}{n^{k-1-\alpha}}\right).$$

If $\alpha \leq k - 2$, then $m^{k-2-\alpha}/n^{k-1-\alpha} = O(1/n)$. If $\alpha > k - 2$, then $m^{k-2-\alpha}/n^{k-1-\alpha} = O(n^{k-1-\alpha})$, as claimed in (7).

The next construction shows that the interval $k$-partite subgraph guaranteed by Theorem 1.2 may have few vertices.

Construction 3: Let $k - 1 < \alpha \leq r$. We give an ordered $n$-vertex $r$-graph $H(n,r)$ with $dn^\alpha$ edges
such that for every interval $k$-partite subgraph $H'$ of $H(n, r)$ with parts of size $m$ and $e(H') = \Omega(dm^\alpha)$,

$$m = O(n^{1-1/\alpha}).$$

Consider the ordered $r$-graph $H = H(n, r)$ with vertex set $[n]$ and edge set $\{[i, i + r - 1] : 1 \leq i \leq n - r + 1\}$. Then

$$e(H) = n - r + 1 = dn^\alpha$$

where $d = \Theta(n^{1-\alpha})$. On the other hand, if $H'$ is an interval $k$-partite subgraph with parts of size $m$, then $H'$ cannot contain two disjoint edges, so $e(H') \leq r$. So if $e(H') = \Omega(dm^\alpha)$, then $dm^\alpha = O(r)$ so $m^\alpha = O(1/d) = O(n^{\alpha-1})$.

Our final construction provides a lower bound on $\text{ex}_*(n, \text{ord}(I^r(\ell)))$ for Theorem 1.8.

**Construction 4:** For $0 \leq \ell \leq r - 1$ and $\alpha = \max\{\ell, r - \lceil(\ell + 1)/2\rceil\}$, we give an ordered $6n$-vertex $r$-graph $H(n, r, \ell)$ with $\Omega(n^\alpha)$ edges not containing any member of $\text{ord}(I^r(\ell))$.

For $\alpha = \ell$, let $H(n, r, \ell)$ be a Steiner $(n, r, \ell)$-system with any ordering of the vertices. Since $H(n, r, \ell)$ is $I^r(\ell)$-free, it is also $\text{ord}(I^r(\ell))$-free. If $\alpha = r - \lceil(\ell + 1)/2\rceil$, let $H(n, r, \ell)$ be defined as follows. The vertex set of $H(n, r, \ell)$ is $[6n]$. Let $M_2$ be the set of pairs $\{2i - 1, 2i\}$ for $1 \leq i \leq n$, and let $M_3$ be the set of triples $\{2n + 3i - 2, 2n + 3i - 1, 2n + 3i\}$ for $1 \leq i \leq n$. If $\ell$ is odd, then the edges of $H(n, r, \ell)$ consist of $r - \ell - 1$ vertices from $[5n, 6n]$, and $(\ell + 1)/2$ pairs from $M_2$. If $\ell$ is even, each edge of $H(n, r, \ell)$ consists of one triple from $M_3$, $\ell/2 - 1$ pairs from $M_2$, and $r - \ell - 1$ vertices from $[5n, 6n]$. If $\ell$ is odd, then

$$e(H(n, r, \ell)) = \binom{n}{(\ell + 1)/2} \cdot \binom{n}{r - \ell - 1} = \Theta(n^\alpha).$$

If $\ell$ is even, then

$$e(H(n, r, \ell)) = n \cdot \binom{n}{\ell/2 - 1} \cdot \binom{n}{r - \ell - 1} = \Theta(n^\alpha).$$

Furthermore, $H(n, r, \ell)$ contains no member of $\text{ord}(I^r(\ell))$: if $\{e, f\} \in \text{ord}(I^r(\ell))$ and $e, f \in H(n, r, \ell)$, then $e \cap [5n] = f \cap [5n]$ and so $|e \cap f| = \ell + 1$, a contradiction.

### 3 Proof of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Let $g = [\log_2 n]$, so that $2^g \leq n < 2^{g+1}$. For $0 \leq i \leq g$, let $I_i$ be the partition of $V(H)$ into intervals of length $2^{g-i}$ plus one interval of length at most $2^{g-i}$ containing the vertex $n$. Note $I_g$ is the partition into singletons, so for each $e \in H$, there exists a minimum $i(e)$ such that $e$ intersects at least $k$ intervals in $I_{i(e)}$. For $0 \leq i \leq g$, let

$$H_i = \{e \in H : i(e) = i\}$$
so that $H = \bigsqcup_{i=0}^{g} H_i$ – the $H_i$ are edge-disjoint. Theorem 1.4 follows from the following claim: for $0 \leq i \leq g$, $H_i$ is a union of $t_i \leq \sum_{j=k}^{r} \binom{2k-2}{j} \cdot 2^{i(k-1)}$ interval $k$-partite hypergraphs $H_{ij}: 1 \leq j \leq t_i$ with parts from $\mathcal{I}_i$ – note that the parts in $\mathcal{I}_i$ each have size at most $2^{\alpha - i} \leq \lceil n/2^i \rceil$. The claim is trivial for $i = 0$, since $H_0$ is empty unless $k = 2$ and $n > 2^g$, in which case $H_0$ is $k$-partite and $t_0 = 1$. To see the claim for $i \geq 1$, note that for $e \in H_i$, there are $s \leq k - 1$ intervals $I_1, I_2, \ldots, I_s \in \mathcal{I}_{i-1}$ such that $e = \bigcup_{j=1}^{s} (e \cap I_j) \subset I = \bigcup_{j=1}^{s} I_j$, by the definition of $i(e) = i$. If $|\mathcal{I}_{i-1}| \geq k - 1$, then by adding intervals if needed we can assume $s = k - 1$. In this case, there are at most $\binom{|\mathcal{I}_{i-1}|}{k-1} \leq \frac{|\mathcal{I}_{i-1}|^{k-1}}{(k-1)!} \leq 2^{(k-1)-1}$ choices for these $k - 1$ intervals, and then at most $\sum_{j=k}^{r} \binom{2k-2}{j}$ choices for the intervals from $\mathcal{I}_i$ contained in $I$ and intersecting $e$.

If $|\mathcal{I}_{i-1}| \leq k - 2$, then again by adding intervals if needed we can assume $s = |\mathcal{I}_{i-1}|$, $I = [n]$, and have at most $\sum_{j=k}^{r} \binom{2s}{j} \leq \sum_{j=k}^{r} \binom{2k-4}{j}$ choices for the intervals from $\mathcal{I}_i$ contained in $I$ and intersecting $e$. Since $e$ intersects at least $k$ intervals in $\mathcal{I}_{i(e)}$, $k \leq 2s \leq 2i+1$, and $2^{i(k-1)} \geq \binom{k}{2}^{k-1} \geq (k - 1)!$; so the bound of the theorem holds again. □

**Proof of Theorem 1.2** We derive Theorem 1.2 from Theorem 1.1 using the notation of its proof. Let $H$ be an $n$-vertex ordered $r$-graph with $dn^\alpha$ edges. Let $C = \sum_{j=k}^{r} \binom{2k-2}{j}$ and $m_i = 2^{\alpha - i}$. We prove that some $H_{ij}$ has

$$e(H_{ij}) \geq \begin{cases} 
\frac{1}{C} \cdot \frac{dm_i^\alpha}{1 + \log_2 n} & \text{if } \alpha = k - 1 \\
\frac{1 - 2^{k-\alpha}}{C} \cdot dm_i^\alpha & \text{if } \alpha > k - 1
\end{cases}$$

Note that $H_{ij}$ is $k$-partite and the parts of $H_{ij}$ have size at most $m_i$, so the above statements imply Theorem 1.2. Suppose, for a contradiction, that no $H_{ij}$ satisfies the above bounds.

**Case 1.** $\alpha = k - 1$. Then recalling $2^g \leq n$ and $t_i \leq C \cdot 2^{i(k-1)}$,

$$e(H) = \sum_{i=0}^{g} e(H_i) \leq \sum_{i=0}^{g} \sum_{j=1}^{t_i} e(H_{ij}) < \sum_{i=0}^{g} \sum_{j=1}^{t_i} \frac{1}{C} \cdot \frac{dm_i^\alpha}{1 + \log_2 n} = \sum_{i=0}^{g} \frac{d \cdot 2^{i(k-1)(g-i)}}{C(1 + \log_2 n)} \leq \frac{d n^{k-1}}{1 + \log_2 n} \sum_{i=0}^{g} \frac{1}{1 + \log_2 n} = (g + 1) \cdot \frac{d \cdot n^{k-1}}{1 + \log_2 n}.$$ 

Since $g \leq \log_2 n$, $e(H) < d \cdot n^{k-1}$, a contradiction.
Case 2. \( \alpha > k - 1 \). Let \( c = (1 - 2^{k-1-\alpha})/C \). Then using \( t_i \leq C \cdot 2^i(k-1) \),

\[
e(H) \leq \sum_{i=0}^{g} \sum_{j=1}^{t_i} e(H_{ij}) < \sum_{i=0}^{g} \sum_{j=1}^{t_i} c \cdot d^i \alpha_i \leq \sum_{i=0}^{g} C \cdot 2^{i(k-1)} \cdot c \cdot d_2 \alpha (g-i) \leq dn^\alpha \cdot \sum_{i=0}^{g} 2^{i(k-1-\alpha)}.
\]

Since \( \alpha > k - 1 \), the geometric series sum is less than \( 1/(1 - 2^{k-1-\alpha}) \), and \( e(H) < dn^\alpha \). This contradiction completes the proof.

### 4 Proofs of Theorems 1.4 – 1.8

Let \( P^r_k \) denote the \( r \)-uniform tight path, which has vertex set \( V = \{v_0, \ldots, v_{k+r-2}\} \) and edge set \( \{\{v_i, v_{i+1}, \ldots, v_{i+r-1}\} : 0 \leq i \leq k-1\} \). Then \( \text{ord}(P^r_k) \) contains the ordered \( r \)-graph \( ZP^r_k \) with edges \( \{v_i, v_{i+1}, \ldots, v_{i+r-1}\} \) for \( 0 \leq i < k \) with a partition of \( V \) into \( r \) intervals \( X_0 < X_1 < \cdots < X_{r-1} \) such that vertices \( v_i < v_{i+r} < v_{i+2r} < \cdots \) are in \( X_i \) if \( i \) is even and \( v_i > v_{i+r} > v_{i+2r} > \cdots \) in \( X_i \) if \( i \) is odd.

Extremal problems for \( ZP^r_k \) are studied in [12], where the following theorem is (implicitly) proved:

**Theorem 4.1.** For \( k, r \geq 2 \),

\[
\text{ex} \rightarrow (n, ZP^r_k) \leq (k-1) \binom{n}{r-1}.
\]

In particular, this theorem gives the same upper bounds for the extremal function for \( \text{ord}(P^r_k) \), as \( ZP^r_k \in \text{ord}(P^r_k) \). In [12, 13] we also obtain ordered versions of the Erdős-Ko-Rado Theorem by taking every \( r \)-th edge of \( P^r_k \).

An ordered \( r \)-graph \( H \) with vertex set \( V \) is a \( (1, r-1) \)-graph if there exists an interval \( X \subset V \) such that every edge of \( H \) has exactly one vertex in \( X \). Finally, an interval \( (r-1) \)-partite \( r \)-graph is a \( (1, r-1) \)-graph simply by combining parts.

**Proof of Theorem 1.4.** A strong \( d \)-dimensional \( r \)-simplex \( \hat{S}_d^r \) is an \( r \)-graph consisting of \( d + 2 \) edges such that we may order the edges so that the first \( d + 1 \) edges form a \( d \)-simplex (see the definition in Section 1.3), and the last edge contains at least one vertex from the intersection of every \( d \)-tuple of the edges of the \( d \)-simplex. For example, a strong 1-simplex comprises three edges \( e, f, g \) such that \( e \cap f = \emptyset \) (so \( e \) and \( f \) form a 1-simplex), and both \( e \cap g \) and \( f \cap g \) are nonempty. It is convenient to assume such an ordering of the edges of a strong simplex is given. We introduce strong simplices for
The purpose of doing a simple induction on $d$: we show that
\[ \text{ex}_{\alpha}(n, \text{ord}({\mathcal S}_d^r)) \leq r^{10dr}n^{r-1}. \]

The base case $d = 1$ follows easily from Theorem 4.1 if $H$ is an ordered $r$-graph with more than $r^{10r}(\frac{n}{r-1})$ edges, then $ZP_{r+1}^r \subset H$, and any three edges of $ZP_{r+1}^r$ that include the first and last edge form a strong 1-simplex. Now suppose we have proved the theorem for strong $d-1$ simplices for some $d \geq 2$, and let $H$ be an $n$-vertex ordered $r$-graph with more than $r^{10dr}n^{r-1}$ edges. Applying Theorem 1.2 with $k = \alpha = r - 1$ we find an interval $(r-1)$-partite subgraph $G$ of $H$ with parts of size at most $m$ and
\[ e(G) \geq c(r-1, r-1, r) \cdot r^{10dr}m^{r-1} \]
for some $m > 0$ with intervals $X$ and $Y = V(G) - X$ as the parts of $G$. By (8), it is straightforward to check $c(r-1, r-1, r) > r^{-4r}$, and therefore
\[ e(G) > r^{-4r}r^{10dr}m^{r-1} > 2r^{10(d-1)}r m^{r-1}. \]
(8)

We remove from $G$ each edge containing an $(r-1)$-set in $Y$ which is contained in at most two edges of $G$. This way, we delete at most $2(\frac{m}{r-1})$ edges. Since $r \geq 3$, this is less than $m^{r-1}$, and hence the remaining $(1, r-1)$-subgraph $G'$ of $G$ has at least $r^{10(d-1)}m^{r-1}$ edges. By averaging, some vertex $x \in X$ is contained in at least
\[ r^{1+10(d-1)}r m^{r-2} \geq r^{10(d-1)}(r-1) m^{r-2} \]
edges of $G'$. By induction, the link of $x$ in $G'$ contains a strong $(d-1)$-dimensional simplex $F$, say with edges $e_1, e_2, \ldots, e_d, f$, with $e_1, e_2, \ldots, e_d$ forming a $(d-1)$-dimensional simplex. Since $f$ is contained in at least 3 edges of $G$, there exists $y \neq x$ such that $f \cup \{y\} \in G$. Then $e_1 \cup \{x\}, e_2 \cup \{x\}, \ldots, e_d \cup \{x\}, f \cup \{y\}$ is a $d$-dimensional simplex in $H$, and together with $f \cup \{x\}$, we have a strong simplex in $H$. This proves the theorem.

\[ \square \]

**Proof of Theorem 1.5.** Let $M$ denote the largest number of edges in an $r$-graph in ${\mathcal F}^+$ and suppose \( \text{ex}_{\alpha}(n, \text{ord}({\mathcal F})) \leq cn^{r-2} \) for all $n > 1$. We will prove that \( \text{ex}_{\alpha}(n, \text{ord}({\mathcal F}^+)) \leq c'n^{r-1} \) where $c' = (\frac{M + c}{2})^{10r}$. Suppose that $H$ is an $n$-vertex $r$-graph with more than $c'n^{r-1}$ edges. Applying Theorem 1.2 with $k = \alpha = r - 1$, we find an $m$-vertex $(1, r-1)$-subgraph $G$ of $H$ with at least $r^{-4r}c'm^{r-1} > 2c'r^{-10r}m^{r-1}$ edges as in (8), with parts $X$ and $Y$, such that every edge has one vertex in $X$. For each $(r-1)$-set in $Y$ contained in at most $M - 1$ edges of $G$, remove all edges of $G$ containing that $(r-1)$-set. The number of edges that we removed is at most $Mm^{r-1}$, so the remaining $r$-graph $G' \subset G$ has more than
\[ (2c'r^{-10r} - M)m^{r-1} = cm^{r-1} \]
edges. By averaging, there exists a vertex $x \in X$ whose link $(r-1)$-graph $G''$ has more than $cm^{r-2}$ edges. Then $G''$ contains a member $F$ of $\text{ord}({\mathcal F})$. Since every edge of $F$ is contained in at least $M$
edges of $G$, we can expand the edges of $F$ to distinct vertices of $X$ to obtain a copy of $F^+$ in $H$. 

**Proof of Theorem 1.6.** We first present an easy proof for $r = 2$, and then a significantly more involved general proof.

**Case 1:** $r = 2$. Suppose that $F$ is a forest with $k$ edges. By adding edges, we may assume that $F$ is a tree. We prove by induction on $k$ that $\text{ex}_\sim(n, \text{ord}(F)) \leq 2k^2n$. Let $H$ be an ordered $n$-vertex graph with more than $2k^2n$ edges and let $F'$ be a tree obtained from $F$ by deleting a leaf $y$. Let $x \in V(F')$ be the neighbor of $y$. For each vertex $v$ of $H$, mark the $k$ smallest neighbors of $v$ and the $k$ largest neighbors of $v$. Note that if $v$ has fewer than $k$ smaller neighbors then we mark them all, and similarly for larger neighbors. We marked at most $2kn$ edges so the resulting unmarked graph $H' \subset H$ has more than $2k^2n - 2kn \geq 2(k - 1)^2n$ edges. By induction, $H'$ contains an interval 2-partite subgraph $K'$ isomorphic to $F'$, with parts $A < B$. Suppose that $v$ is the vertex of $K'$ that plays the role $x$ in $F'$, and assume first that $v \in A$. Then there is a vertex $w \in B$ with $\{v, w\} \in K'$, so by construction of $H'$, there is another vertex $w' > w$ such that the marked edge $\{v, w'\} \in H$ and $w' \notin V(K')$. Adding edge $\{v, w'\}$ to $K'$ gives a copy $K$ of the 2-interval-partite graph $F$ ($w'$ plays the role of $y$). The same argument applies if $v \in B$.

**Case 2:** $r \geq 3$. By Theorem 1.2 with $\alpha = r - 1 = k$, it is enough to prove Theorem 1.6 for interval $(r - 1)$-partite $r$-graphs. Let $H$ be an interval $(r - 1)$-partite $r$-graph with $n$ vertices and a partition of $V(H)$ into intervals $X_1 < X_2 < \cdots < X_{r-1}$ where for some $i$, and every $e \in H$, $|e \cap X_{i}| = 2$ and $|e \cap X_{j}| = 1$ for $j \neq i$. It is easy to check that every forest $F$ is contained in a tight tree $T$ with the same set of vertices. We show by induction on $t = v(T) \geq r$ that if $e(H) > 2t^2\binom{n}{r-1}$, then $H$ contains a member of ord$(T)$. If $t = r$, then $T$ has one edge and clearly $e(H) = 1$ if $H$ is an ord$(T)$-free. Suppose the statement is true for all tight trees with fewer than $t$ vertices, and let $T$ be a tight tree with $t$ vertices. Let $H$ be an $n$-vertex interval $(r - 1)$-partite $r$-graph with more than $2t^2\binom{n}{r-1}$ edges. For each $f \in \partial H$, let $S(f)$ and $L(f)$ denote the set of the $t$ smallest and $t$ largest vertices $x \in V(H)$ such that $f \cup \{x\} \in H$. Then we remove all edges $f \cup \{x\}$ from $H$ such that $x \in S(f) \cup L(f)$. We obtain a new ordered interval $(r - 1)$-partite $r$-graph $H'$ with parts $X_1 < X_2 < \cdots < X_{r-1}$. Let $T' = T - \{y\}$ where $y$ is a leaf of $T$, and $f \cup \{y\} \in T$. By induction, $H'$ contains a member of ord$(T')$, since

$$e(H') > 2t^2\binom{n}{r-1} - 2t\binom{n}{r-1} > 2(t - 1)^2\binom{n}{r-1}.$$ 

Let this member of ord$(T')$ be denoted by $S$, and have parts $A_0 < A_1 < \cdots < A_{r-1}$, where $A_i = X_i$ and $A_j \subseteq X_j$ for $j \neq i$. Since $f \in \partial S \cap \partial H'$, $f \cap A_j = \emptyset$ for some $j \leq r$. If $j \notin \{i, i-1\}$, then $S(f) \cup L(f) \subseteq X_j$, and since $|S(f) \cup L(f)| > t$ and $|V(T) \cap X_j| < t$, there exists $x \in X_j \setminus V(T)$ such that $f \cup \{x\} \in H$ together with $S$ forms a copy of $T$ in $H$, with interval coloring $A'_0 < A'_1 < \cdots < A'_{r-1}$ where $A'_h = A_h$ for $h \neq j$ and $A'_j = A_j \cup \{x\}$. If $j = i$, then $f \cup \{z\} \in S$ for some $z \in A_i$. For every $x \in L(f)$, we have $x > z$ and $x \in X_i$. Since $|L(f)| = t$, there exists $x \in L(f)$ such that $x > z$ and $x \notin V(S)$. Now $f \cup \{z\} \in H$ together with $S$ is a copy of an element of ord$(T)$ in $H$, with interval $r$-coloring $A'_0 < A'_1 < \cdots < A'_{r-1}$ where $A'_h = A_h$ for $h \neq i$ and $A'_i = A_i \cup \{x\}$. Finally, if $j = i - 1$,
then $f \cup \{z\} \in S$ for some $z \in A_0$. For every $x \in S(f)$, we have $x < z$ and $x \in X_{i-1}$. Since $|S(f)| = t$, there exists $x \in S(f)$ such that $x < z$ and $x \notin V(S)$. Now $f \cup \{x\} \in H$ together with $S$ is a copy of an element of $\text{ord}(T)$ in $H$, with interval $r$-coloring $A_0' < A_1' < \ldots < A_{r-1}'$ where $A_h' = A_h$ for $h \neq i - 1$ and $A_{i-1}' = A_{i-1} \cup \{x\}$. This completes the proof. 

**Proof of Theorem 1.7** By Theorem 1.2 with $k = \alpha = 2$, it is enough to prove Theorem 1.7 for interval $2$-partite $3$-graphs. Suppose that $\epsilon > 0$ and $n_0$ is sufficiently large. Let $H$ be an $n$-vertex ordered interval $2$-partite $3$-graph with at least $\epsilon n^2$ edges ($n > n_0$) containing no member of $\text{ord}(I_2)$ and $A < B$ be intervals where every edge of $H$ has exactly one vertex in $A$. Let $G$ be the graph with vertex set $V(H) \cap B$ and edge set $\{yz : \exists x \in A, xyz \in H\}$. Since $H$ contains no member of $\text{ord}(I_2)$, $e(G) = e(H) \geq \epsilon n^2$. By Theorem B, there is an interval $2$-partite subgraph $G' \subset G$ with at least $\delta n^2$ edges, for some $\delta$ depending only on $\epsilon$. Consequently, there is an interval $3$-partite subgraph $H' \subset H$ with $\delta n^2$ edges and we apply the Ruzsa-Szemerédi Theorem to $H'$ to obtain a copy of some member of $\text{ord}(T_3)$.

**Proof of Theorem 1.8.** We use the result of Frankl and Füredi [9] stating that for $0 \leq \ell \leq r - 1$ and some constant $C(r, \ell) > 0$,

$$\text{ex}(n, I'(\ell)) < C(r, \ell) \cdot n^{\max\{\ell, r-\ell-1\}}. \quad (9)$$

Construction 4 gives a lower bound of order $n^\alpha$ for $\text{ex}_\rightarrow(n, \text{ord}(I'(\ell)))$, so it remains to prove the upper bound in Theorem 1.8. We first prove the upper bound when $\ell$ is odd.

Recall $\alpha = \max\{\ell, r - (\ell + 1)/2\}$, and let $k = \alpha$, $\ell' = \ell - r + k \geq 0$. Let $H$ be an ordered $n$-vertex $r$-graph with $C(k, \ell')(kn)^\alpha/c$ edges, where $c$ is the implicit constant in the second inequality of Theorem 1.2, namely (3). We aim to show $H$ contains a member of $\text{ord}(I'(\ell))$. By Theorem 1.2 with $k = \alpha$, there is for some $m \in [n]$ an interval $k$-partite subgraph $H'$ of $H$ with $e(H') \geq C(k, \ell')(km)^\alpha$ and parts of size at most $m$. For each edge $e \in H'$,

$$\sum_{j=1}^{k} (|e \cap I_j| - 1) = r - k.$$

Let $f(e)$ be the set of the first $|e \cap I_j| - 1$ elements of $e \cap I_j$ for $1 \leq j \leq k$, so that $|f(e)| = r - k$. By the pigeonhole principle, there exists a set $S$ of size $r - k$ such that $f(e) = S$ for at least $|H'|/m^{r-k} \geq C(k, \ell')(km)^{\alpha-r+k}$ edges $e \in H'$. Let $H'' = \{e \setminus S : S \subset e \in H'\}$, so $H''$ is an ordered $k$-uniform $k$-partite hypergraph with $N = v(H'') \leq km$ and $e(H'') \geq C(k, \ell')N^{\alpha-r+k}$. Since $2\alpha = \max\{2\ell, 2r - \ell - 1\} \geq 2r - \ell - 1$,

$$\max\{k - \ell' - 1, \ell'\} = \max\{r - \ell - 1, \ell - r + k\}$$

$$= \max\{2r - \ell - 1 - k, \ell\} - r + k$$

$$\leq \max\{2\alpha - k, \ell\} - r + k = \max\{\alpha, \ell\} - r + k = \alpha - r + k.$$
It follows from (9) that $e(H'') \geq C(k, \ell')N^{\alpha-r+k} > \text{ex}(N, I^k(\ell'))$. Therefore there exist $f, g \in H''$ with $|f \cap g| = \ell'$. Since $H''$ is $k$-partite, $\{f, g\} \in \text{ord}(I^k(\ell'))$ and now $\{f \cup S, g \cup S\} \in \text{ord}(I^r(\ell))$. We conclude

$$\text{ex} \rightarrow (n, \text{ord}(I^r(\ell))) < \frac{C(k, \ell')}{c(\alpha, k, r)(kn)^{\alpha}}.$$ 

This completes the proof of Theorem 1.8 when $\ell$ is odd.

When $\ell \geq 2$ is even, $\alpha = \max\{\ell, r - (\ell + 2)/2\}$. Let $k = \alpha + 1$, $\ell' = \ell - r + k \geq 0$, and let $H$ be an ordered $n$-vertex $r$-graph with $C(k, \ell')(km)^{\alpha}(1 + \log_2 n)/c$ edges where $c$ is the implicit constant in the first inequality of Theorem 1.2. Then for some $m \in [n]$ there is an interval $k$-partite subgraph $H'$ of $H$ with $e(H') \geq C(k, \ell')(km)^{\alpha}$ and parts of size at most $m$. Define the interval $k$-partite $k$-graph $H'' \subseteq H'$ as above. Since $\ell$ is even, $2\alpha = \max\{2\ell, 2r - \ell - 2\} \geq 2r - \ell - 2$, and therefore

$$\max\{k - \ell' - 1, \ell'\} = \max\{r - \ell - 1, \ell - r + k\} = \max\{2r - \ell - 1 - k, \ell\} - r + k \leq \max\{2\alpha - k + 1, \ell\} - r + k = \max\{\alpha, \ell\} - r + k = \alpha - r + k.$$ 

In the last line we used $k = \alpha + 1$. It follows from (9) that $H''$ contains a member of $I^k(\ell')$ and then $H$ contains a member of $I^r(\ell)$. This completes the proof of Theorem 1.8 when $\ell$ is even. \qed

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