Partitioning ordered hypergraphs

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Abstract

An *ordered r-graph* is an r-uniform hypergraph whose vertex set is linearly ordered. Given $2 \le k \le r$, an ordered r-graph H is *interval k-partite* if there exist at least k disjoint intervals in the ordering such that every edge of H has nonempty intersection with each of the intervals and is contained in their union.

Our main result implies that if $\alpha > k-1$, then for each $d > 0$ every n-vertex ordered r-graph with $d\,n^\alpha$ edges has for some $m \le n$ an m-vertex interval k-partite subgraph with $\Omega(d\,m^\alpha)$ edges. This is an extension to ordered r-graphs of the observation by Erdős and Kleitman that every r-graph contains an r-partite subgraph with a constant proportion of the edges. The restriction $\alpha > k-1$ is sharp. We also present applications of the main result to several extremal problems for ordered hypergraphs.

1 Introduction

We let $[n] = \{1, \dots, n\}$ and use standard asymptotic notation; in particular, given functions $f, g : \mathbb{Z}^+ \to \mathbb{R}^+$, we write $f(n) = \Omega(g(n))$ if there exists $c > 0$ such that $f(n) \ge cg(n)$ for all $n \ge 1$. We also write $f(n) = O(g(n))$ if $g(n) = \Omega(f(n))$, and write $f(n) = \Theta(g(n))$ if both, $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$. We associate a hypergraph H with its edge set and write $e(H)$ for the number of the edges and $v(H)$ for the number of the vertices in H.

An *r-graph* is a hypergraph with all edges of size r; it is *r-partite* if there is a partition of the vertex set into r parts such that every edge has exactly one vertex in each part. The following observation is due to Erdős and Kleitman:

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Proposition A. (Erdős-Kleitman [6]) *Every r-graph contains an r-partite subgraph with at least $r!/r^r$ proportion of its edges.*

In particular, any extremal problem for r-graphs can be reduced to the corresponding extremal problem where the underlying r-graph is r-partite with the loss of only a constant multiplicative factor. In this paper, we consider analogs of this result in the ordered hypergraph setting and illustrate their use on some ordered extremal hypergraph problems.

An *ordered hypergraph* is a hypergraph together with a linear ordering of its vertex set. Extremal problems on ordered hypergraphs arose from several sources, in particular, from combinatorial geometry, enumeration of permutations with forbidden subpermutations, and the study of matrices with forbidden submatrices – see for instance Anstee [1, 2], Füredi and Hajnal [11], Pach and Tardos [19], Marcus and Tardos [16], Tardos [21], Fox [8].

Let V be a linearly ordered set. An *interval* in V is a set of consecutive elements in the ordering. For $A, B \subset V$, we write $A < B$ to mean that $a < b$ for every $a \in A, b \in B$. A key definition in our work is the following:

Definition 1. *Let k be a positive integer. An ordered r-graph H is* interval k-partite *if for some $\ell \geq k$ there are intervals $I_1 < I_2 < \cdots < I_\ell$ such that every edge of H is contained in $I_1 \cup \ldots \cup I_\ell$ and has nonempty intersection with I_j for each $1 \leq j \leq \ell$.*

We allow ℓ in the definition be larger than k because the more parts we have, the more structure on H is imposed. In particular, an ordered r-graph H is *interval r-partite* if there exist intervals $I_1 < I_2 < \cdots < I_r$ in $V(G)$ such that every edge of H contains exactly one vertex from each I_i. In these terms, the Erdős–Kleitman observation, Proposition A, does not hold for ordered graphs as witnessed by the following simple example: every interval bipartite subgraph of the ordered graph with vertex set $[2n]$ and edge set $\{\{2i-1, 2i\} : 1 \leq i \leq n\}$ has at most one edge. However, Pach and Tardos [19] showed that dense ordered graphs contain relatively dense interval bipartite graphs using the following result:

Theorem B. (Pach and Tardos [19]). *Each ordered n-vertex graph G is the union of edge-disjoint subgraphs G_i for $0 \leq i \leq \lfloor \log_2 n \rfloor$ such that each G_i is a union of at most 2^i interval bipartite graphs with parts of size at most $\lceil n/2^i \rceil$.*

Our first main result is the following ordered hypergraph analog of Theorem B:

Theorem 1.1. *Let $2 \leq k \leq r \leq n$ be integers. Then every ordered n-vertex r-graph H is the union of edge-disjoint ordered r-graphs H_i for $0 \leq i \leq \lfloor \log_2 n \rfloor$ such that each H_i is a union of at most $\frac{1}{(k-1)!}\sum_{j=k}^{r}\binom{2k-2}{j} \cdot 2^{i(k-1)}$ interval k-partite r-graphs with parts of size at most $\lceil n/2^i \rceil$.*

For $k = r = 2$, Theorem 1.1 corresponds to Theorem B. Note that Theorem B easily implies the following, which appears implicitly in Pach and Tardos [19]:

Theorem C. *For each real $\alpha \geq 1$, $d > 0$ and $n > 1$, if G is an ordered n-vertex graph with $e(G) = dn^\alpha$,* *then for some $m \in [n]$, G contains an interval bipartite subgraph G' with parts of size at most m and*

$$e(G') = \begin{cases} \Omega\Big(\dfrac{dm^\alpha}{\log_2 n}\Big) & \text{if } \alpha = 1 \\ \\ \Omega(dm^\alpha) & \text{if } \alpha > 1 \end{cases} \tag{1}$$

As observed by Pach and Tardos [19], the logarithmic factor in (1) for $\alpha = 1$ is necessary: for the ordered path P with edges $\{v_i, v_{i+1}\} : 1 \le i \le 4$ such that $v_2 < v_4 < v_3 < v_1 < v_5$, extremal n-vertex ordered P-free graphs have $n \log n + O(n)$ edges, whereas an extremal n-vertex interval bipartite P-free graph has $\Theta(n)$ edges (see Füredi [10], Bienstock and Györi [3], and Tardos [22]).

Our second main result is the following generalization of Theorem C to ordered r-graphs:

Theorem 1.2. *Let $2 \le k \le r$ be fixed integers and let α be a real number with $k-1 \le \alpha \le r$. Then for every integer $n \ge r$, every ordered r-graph H with n vertices and dn^α edges has an interval k-partite subgraph H' with parts of size at most m for some $m \in [n]$ and*

$$e(H') = \begin{cases} \Omega\Big(\dfrac{dm^\alpha}{\log_2 n}\Big) & \text{if } \alpha = k-1 \\ \\ \Omega(dm^\alpha) & \text{if } \alpha > k-1 \end{cases} \tag{2}$$

The case $k = r = 2$ is Theorem C.

Remarks.

- Theorem 1.2 is sharp in that for $2 \le k < r$ and $\alpha = k - 1$, there exist n-vertex r-graphs H with $e(H) = dn^\alpha$ where every interval k-partite subgraph H' with parts of size m has $e(H') = O(dm^\alpha / \log_2 n)$, and for $\alpha < k - 1$, there exist n-vertex r-graphs H with $e(H) = dn^\alpha$ where every interval k-partite subgraph H' has $e(H') = O(dn^{\alpha - a})$ where $a = \min\{1, k - 1 - \alpha\} > 0$. We will prove this in Section 2 (see Constructions 1 and 2).

- For $\alpha > k - 1$, Theorem 1.2 guarantees that each n-vertex ordered r-graph with $\Theta(n^\alpha)$ edges has an interval k-partite subgraph with parts of size m and $\Theta(m^\alpha)$ edges for some $m \in [n]$. In sharp contrast with the Erdős–Kleitman Lemma, Proposition A, the value of m may necessarily be small relative to the number of vertices in the host r-graph: we give a construction in Section 2 (see Construction 3) where we need $m = O(n^{1-1/\alpha})$ for $\alpha > k - 1$.

- We do not optimize the constant $c = c(\alpha, k, r)$ in the bound $e(H') \ge cdm^\alpha$ for $\alpha > k - 1$ in Theorem 1.2. The proof of Theorem 1.2 gives

$$c(\alpha, k, r) \ge \frac{(k-1)!(1 - 2^{k-1-\alpha})}{\sum_{j=k}^{r} \binom{2k-2}{j}}. \tag{3}$$

In particular, $c(r,r,r) \geq (r-1)!4^{-r}$, whereas for every r-partite subgraph H' of the n-vertex complete r-graph K_n^r with parts of size m, $e(H') \leq m^r$, and so $c(r,r,r) \leq r!$.

- For each (unordered) partition π of r, one can extend Theorem 1.2 to the setting of *interval π-partite subgraphs* – here π specifies the number of vertices of an edge in each part – by replacing the range of α to $\alpha \geq f(\pi)$ where $f(\pi)$ is the maximum length of a partition that is not a refinement of π. For example, if $\pi = 1+1+\cdots+1$, then $f(\pi) = r-1$, if $\pi = 2+1+\cdots+1$, then $f(\pi) = r-2$ and if $\pi = (r-1)+1$, then $f(\pi) = \lfloor r/2 \rfloor$. This has other interesting consequences which we will explore in forthcoming work.

1.1 Applications of Theorem 1.2

We next describe how to apply Theorem 1.2 to a variety of ordered extremal problems and convex geometric extremal problems for families of r-graphs. This enables us to transfer classical extremal problems to the ordered setting via Theorem 1.2. The following definition is needed:

Definition 2. *For an r-partite r-graph F, $\mathrm{ord}(F)$ denotes the family of interval r-partite r-graphs isomorphic to F. For a family $\mathcal{F}$ of r-partite r-graphs, $\mathrm{ord}(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} \mathrm{ord}(F)$.*

A first and natural example is the case that $\mathcal{F}$ consists of the r-graph of two disjoint edges. The Erdős-Ko-Rado Theorem [7] states that for $n \geq 2r+1$, the unique extremal n-vertex r-graph without two disjoint edges consists of all r-element subsets of $[n]$ containing vertex 1, with $\binom{n-1}{r-1}$ edges. In [13], the following ordered version of the Erdős-Ko-Rado Theorem is proved:

Theorem 1.3. ([13]) *Let $r \geq 3$ and $n \geq 2r+1$. Then the maximum number of edges in an ordered n-vertex r-graph that does not contain two edges of the form $\{v_1, v_2, \ldots, v_r\}$ and $\{w_1, w_2, \ldots, w_r\}$ such that $v_1 < w_1 < v_2 < w_2 < \cdots < v_r < w_r$ is exactly $\binom{n}{r} - \binom{n-r}{r}$.*

For an ordered r-graph F, let $\mathrm{ex}_{\rightarrow}(n,F)$ denote the maximum number of edges in an n-vertex ordered r-graph that does not contain F. For a family $\mathcal{F}$ of ordered r-graphs, let $\mathrm{ex}_{\rightarrow}(n,\mathcal{F})$ denote the maximum number of edges in an n-vertex ordered r-graph that contains no members of $\mathcal{F}$. In this language Theorem 1.3 implies that for $n \geq 2r+1$,

$$\mathrm{ex}_{\rightarrow}(n, \mathrm{ord}(F)) \leq \binom{n}{r} - \binom{n-r}{r},$$

where F is the r-graph comprising two disjoint edges (in fact, it applies to a particular member of $\mathrm{ord}(F)$). Results for hypergraph matchings (i.e., for sets of disjoint edges) by Klazar and Marcus [15] show that for each interval r-partite matching M, $\mathrm{ex}_{\rightarrow}(n,M) = O(n^{r-1})$, thereby extending the celebrated Marcus-Tardos [16] theorem for matchings in ordered graphs to ordered r-graphs. We now give some further examples where classical extremal problems are transferred to the ordered setting via Theorem 1.2.

1.1.1 Simplices

A *d-dimensional r-simplex* is an r-graph of $d+1$ edges such that any d of the edges have non-empty intersection, but all $d+1$ edges have empty intersection. Denote by $\mathcal{S}_d^r$ the family of d-dimensional r-simplices. The set $\mathcal{S}_d^r$ is non-empty if $r \geq d$. The study of these abstract simplices in the context of extremal hypergraph theory was first initiated by Chvátal who posed the following conjecture.

Conjecture 1. (Chvátal [4]) *Let $r \geq d+1 \geq 3$ and $n \geq r(d+1)/d$. Then $\mathrm{ex}(n, \mathcal{S}_d^r) = \binom{n-1}{r-1}$.*

Frankl and Füredi [9] proved Conjecture 1 for large n (Keller and Lifschitz [14] improved the bounds on n) and Mubayi and Verstraëte [17] proved it for $d = 2$, which was a problem of Erdős. Very recently, Currier [5] proved the conjecture for $n \geq 2r$. We prove the following theorem.

Theorem 1.4. *For all fixed $r \geq d+1 \geq 3$,*

$$\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{S}_d^r)) = \Theta(n^{r-1}).$$

1.1.2 Expansions

Our next example is more general. If $\mathcal{F}$ is a family of $(r-1)$-graphs, let $\mathcal{F}^+$ denote the family of r-graphs F^+ obtained from each $F \in \mathcal{F}$ by adding a vertex v_e to edge $e \in F$ such that all the vertices $v_e : e \in F$ are distinct from each other and from the vertices of F. A study of extremal problems for families $\mathcal{F}^+$ is given in [18], where F^+ is referred to as an *expansion* of F. Such families lend themselves naturally to an application of Theorem 1.2:

Theorem 1.5. *Let $r \geq 3$ and $\mathcal{F}$ be a family of $(r-1)$-graphs with $\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F})) = O(n^{r-2})$. Then*

$$\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F}^+)) = O(n^{r-1}).$$

The proof of Theorem 1.5 implies that

$$\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r)) = O(n^{r-1}),$$

where $T_r = \{e, f, g\}$ is the *loose r-uniform triangle*, i.e., $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$:

Theorem 1.6. *For $r \geq 3$,*

$$\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r)) = \Theta(n^{r-1}).$$

1.1.3 Hypergraph forests

Our next application concerns hypergraph forests. The *shadow* ∂H of an r-graph H is the collection of $(r-1)$-sets contained in some edge of H. We follow Frankl and Füredi [9] for an inductive definition of trees in hypergraphs: a single edge is a tree, and given any tree T with edges $e_1, e_2, \dots, e_h$, a tree with $h+1$ edges is obtained by selecting $f \in \partial T$ and a vertex x not in T, and adding the edge $f \cup \{x\}$. A *forest* is a subgraph of a tree. By definition, each 2-uniform tree (respectively, 2-uniform forest) is a tree (respectively, forest) in the usual sense. Using Theorem 1.2, we prove the following:

Theorem 1.7. *Fix $r \geq 2$ and let F be an r-uniform forest. Then $\mathrm{ex}_{\to}(n, \mathrm{ord}(F)) = O(n^{r-1})$.*

Remarks.

- A conjecture of Pach and Tardos [19] would imply $\mathrm{ex}_{\to}(n, T) = n^{1+o(1)}$ for every 2-interval-partite tree T with at least two edges. Theorems 1.5 and 1.7 suggest that perhaps for every interval r-partite r-uniform tree T, $\mathrm{ex}_{\to}(n, T) \leq n^{r-1+o(1)}$.

- It remains an intriguing open problem to determine for which r-graph families $\mathcal{F}$

$$\mathrm{ex}(n, \mathcal{F}) = O(n^{r-1}) \qquad \Longrightarrow \qquad \mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F})) = O(n^{r-1}). \tag{4}$$

  According to Theorem 1.7, this is true for $r = 2$. Since for every r-uniform forest F, $\mathrm{ex}(n, F) = O(n^{r-1})$, Theorem 1.7 yields that the above implication is also true if $\mathcal{F}$ contains an r-uniform forest. We do not know any explicit example for $r \geq 3$ for which (4) fails, although we believe that many such examples exist. As pointed out by a referee, Theorem 1.2 implies that for each $\alpha > r-1$ and all r-graph families $\mathcal{F}$,

$$\mathrm{ex}(n, \mathcal{F}) = O(n^{\alpha}) \qquad \Longrightarrow \qquad \mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F})) = O(n^{\alpha}). \tag{5}$$

- In [12], we heavily used the $k = r-1$ case of Theorem 1.2 to prove that the extremal function of so called *crossing paths* in convex geometric hypergraphs has order n^{r-1} or $n^{r-1}\log n$.

1.1.4 Ordered Ruzsa-Szemerédi Theorem

We consider the ordered version of the famous Ruzsa-Szemerédi $(6,3)$-Theorem [20] which states that the maximum number of edges in an n-vertex 3-graph with no 6 vertices spanning 3 edges is $o(n^2)$. This is equivalent to the statement $\mathrm{ex}(n, \mathcal{F}_{RS}) = o(n^2)$ where $\mathcal{F}_{RS} = \{I_2, T_3\}$ and I_2 is the 3-graph comprising two edges sharing exactly two points.

Theorem 1.8. *Let $\mathcal{F}_{RS} = \{I_2, T_3\}$. Then $\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F}_{RS})) = o(n^2)$.*

1.1.5 Forbidden ordered intersections

Our final example addresses an r-graph problem whose answer has order of magnitude n^α where $\alpha \neq r-1$. Let $I^r(\ell)$ denote the r-graph consisting of two edges sharing exactly ℓ vertices. The study of $\mathrm{ex}(n, I^r(\ell))$ was initiated by Erdős. Frankl and Füredi [9] proved that

$$\mathrm{ex}(n, I^r(\ell)) = \Theta(n^{\max\{\ell, r-\ell-1\}}) \qquad \text{for } 0 \leq \ell \leq r-1. \tag{6}$$

We are able to prove an ordered version of this result using Theorem 1.2:

Theorem 1.9. *For $r \geq 2$ and $1 \leq \ell \leq r-1$, and $\alpha = \max\{\ell, r - \lceil(\ell+1)/2\rceil\}$*

$$\Omega(n^\alpha) = \mathrm{ex}_\to(n, \mathrm{ord}(I^r(\ell))) = \begin{cases} O(n^\alpha) & \text{if } \ell \text{ is odd} \\ O(n^\alpha \log n) & \text{if } \ell \text{ is even.} \end{cases}$$

Note that the $\ell = 0$ case is covered by Theorem 1.3 which gives $\mathrm{ex}_\to(n, \mathrm{ord}(I^r(0))) = \Theta(n^{r-1})$. A construction of a dense $\mathrm{ord}(I^r(\ell))$-free ordered r-graph is given in Construction 4. We believe the $\log n$ factor when ℓ is even can be removed, so that $\mathrm{ex}_\to(n, \mathrm{ord}(I^r(\ell))) = \Theta(n^\alpha)$.

We present four constructions in Section 2 and prove Theorem 1.2 in Section 3. Theorems 1.4–1.9 are proved in Section 4.

2 Constructions

Our first construction requires the following lemma.

Lemma 2.1. *Let H_{n_1,n_2} be the ordered bipartite graph with vertex set $[n_1+n_2]$ and parts $A = [n_1]$ and $B = \{n_1+1, \ldots, n_1+n_2\}$ such that for $i < j$, the pair ij is an edge in $H(n_1, n_2)$ iff $1 \leq i \leq n_1 < j \leq n_1+n_2$ and $j-i$ is a power of 2. Then for each $A' \subseteq A$ and $B' \subseteq B$, the number of edges in $H_{n_1,n_2}[A' \cup B']$ is at most $|A' \cup B'|$.*

Proof. Suppose for some $A' \subseteq A$ and $B' \subseteq B$, graph $H := H_{n_1,n_2}[A' \cup B']$ has more than $|A'| + |B'|$ edges. Let us assume $A' = \{a_1, a_2, \ldots, a_l\}$ and $B' = \{b_1, b_2, \ldots, b_m\}$ where

$$a_1 < a_2 < \cdots < a_l < b_1 < b_2 < \cdots < b_m.$$

For each vertex $a \in A'$, remove from H the edge $\{a, b_i\}$ where i is the minimum index for which such an edge exists. After that, for each vertex $b \in B'$, remove from H the edge $\{a_j, b\}$ where j is the maximum index for which such an edge exists. Since H has more than $|A'| + |B'|$ edges, and we removed at most $|A'| + |B'|$ edges, the remaining graph H' has an edge $\{a, b\}$ with $a \in A', b \in B'$. Now there exist vertices a' and b' such that $\{a', b\}$ and $\{a, b'\}$ are edges and $a < a' < b' < b$. However, it is not possible for $b-a$, $b'-a$ and $b-a'$ all to be powers of 2. $\square$

In all our constructions, the extra parameter d does not need to be a constant, it may depend on other parameters. Our first construction shows that the logarithmic factor in the first bound in Theorem 1.2 is necessary.

Construction 1: *An n-vertex ordered r-graph $H_r(n)$ with dn^{k-1} edges such that for every $m \le n$*

$$e(H') = O\Big(\frac{dm^{k-1}}{\log n}\Big) \tag{7}$$

for every interval k-partite $H' \subset H_r(n)$ with parts of size m.

The vertex set of $H_r(n)$ is $[n]$ ordered as $1 < 2 < \cdots < n$. The edges of $H_r(n)$ are the sets $\{v_1, v_2, \dots, v_r\}$ with $v_1 < v_2 < \cdots < v_r$ such that the difference $v_{i+1} - v_i$ is a power of 2 for $i = 1, \dots, r-k+1$. Then $e(H_r(n)) = \Theta(n^{k-1}(\log n)^{r-k+1}) = \Theta(dn^{k-1})$ where $d = (\log n)^{r-k+1}$. Let H' be any interval k-partite subgraph of $H_r(n)$, with ordered parts $I_1 < I_2 < \ldots < I_k$ each of size m, and G' be the bipartite graph whose edges are pairs $\{v, w\} \subset e = \{v_1, v_2, \dots, v_r\} \in H'$ where v is the largest vertex in I_1 and w is the smallest vertex in I_2. Note, crucially, that $w - v$ must be a power of 2, since otherwise the $r-k+2$ smallest vertices of e lie in I_1, which means that some I_j is empty. Lemma 2.1 now yields that $e(G') \le v(G') \le 2m$. But then $e(H') = O(m^{k-1}(\log n)^{r-k})$, so $e(H') = O(dm^{k-1}/\log n)$.

Our next construction shows that the bound $\alpha \ge k-1$ in Theorem 1.2 cannot be improved.

Construction 2: *An rn-vertex r-graph $H_n^r(k)$ with dn^α edges such that for $1 \le \alpha < k-1 \le r-1$ and $a = \min\{1, k-1-\alpha\} > 0$,*

$$e(H') = O(dm^\alpha/n^a) \tag{8}$$

for every interval k-partite $H' \subset H_n^r(k)$ with parts of size m.

For $k \le r$ and $1 \le j \le n$, let $I_j = \{(r-k+2)j-(r-k+1), (r-k+2)j-(r-k+1)+1, \dots, (r-k+2)j\}$, so that $|I_j| = r-k+2$, and let $H_n^r(k)$ be the ordered r-graph with vertex set $[rn]$ and edge set

$$\{I_j \cup \{a_{r-k+3}, a_{r-k+4}, \dots, a_r\} : 1 \le j \le n, \text{ and } (\ell-1)n < a_\ell \le \ell n \text{ for } r-k+3 \le \ell \le r\}.$$

By definition, $e(H_n^r(k)) = n^{k-1} = dn^\alpha$ where $d = n^{k-1-\alpha}$. On the other hand, let H' be an interval k-partite subgraph of $H_n^r(k)$ with parts of size m. Then there exists j such that every edge of H' contains I_j. In particular,

$$e(H') = O(m^{k-2}) = O\Big(dm^\alpha \cdot \frac{m^{k-2-\alpha}}{n^{k-1-\alpha}}\Big).$$

If $\alpha \le k-2$, then $m^{k-2-\alpha}/n^{k-1-\alpha} = O(1/n)$. If $\alpha > k-2$, then $m^{k-2-\alpha}/n^{k-1-\alpha} = O(n^{\alpha-k+1})$. This implies (8).

The next construction shows that the interval k-partite subgraph guaranteed by Theorem 1.2 may have few vertices.

Construction 3: *Let r be fixed and let $k-1<\alpha\le r$. We give an ordered n-vertex r-graph $H(n,r)$ with dn^α edges such that for every interval k-partite subgraph H' of $H(n,r)$ with parts of size m and $e(H')=\Omega(dm^\alpha)$,*

$$m=O(n^{1-1/\alpha}).$$

Consider the ordered r-graph $H=H(n,r)$ with vertex set $[n]$ and edge set $\{\{i,i+1,\dots,i+r-1\}: 1\le i\le n-r+1\}$. Then

$$e(H)=n-r+1=dn^\alpha$$

where $d=\Theta(n^{1-\alpha})$. On the other hand, if H' is an interval k-partite subgraph with parts of size m, then H' cannot contain two disjoint edges, so $e(H')\le r$. So if $e(H')=\Omega(dm^\alpha)$, then $dm^\alpha=O(r)$ so $m^\alpha=O(1/d)=O(n^{\alpha-1})$.

Our final construction provides a lower bound on $\mathrm{ex}_{\to}(n,\mathrm{ord}(I^r(\ell))$ for Theorem 1.9:

Construction 4: *For $0\le\ell\le r-1$ and $\alpha=\max\{\ell,r-\lceil(\ell+1)/2\rceil\}$, we give an ordered $3n$-vertex r-graph $H(n,r,\ell)$ with $\Omega(n^\alpha)$ edges not containing an interval r-partite r-graph consisting of two edges with intersection size ℓ.*

Consider first the case $\alpha=\ell$. An easy application of the probabilistic method implies that there exists a $3n$-vertex r-graph $G(n,r,\ell)$ with $\Omega(n^\ell)$ edges in which every ℓ vertices lie in at most one edge. Let $H(n,r,\ell)$ be such $G(n,r,\ell)$ with any ordering of the vertices. Since $G(n,r,\ell)$ is $I^r(\ell)$-free, $H(n,r,\ell)$ is $\mathrm{ord}(I^r(\ell))$-free.

If $\alpha=r-\lceil(\ell+1)/2\rceil>\ell$, define $H(n,r,\ell)$ as follows. The vertex set of $H(n,r,\ell)$ is $[3n]$. Let M be the set of pairs $\{2i-1,2i\}$ for $1\le i\le n$ The edges of $H(n,r,\ell)$ consist of $\lceil(\ell+1)/2\rceil$ pairs from M and $r-2\lceil(\ell+1)/2\rceil$ vertices from $\{2n+1,2n+2,\dots,3n\}$. Then

$$e(H(n,r,\ell))=\binom{n}{\lceil(\ell+1)/2\rceil}\cdot\binom{n}{r-2\lceil(\ell+1)/2\rceil}=\Theta(n^\alpha).$$

Suppose $H(n,r,\ell)$ contains an interval r-partite 2-edge r-graph $G(r,\ell)$ with edge set $\{e,f\}$ and $|e\cap f|=\ell$. Let $(W_1,\dots,W_r)$ be an interval r-partition of $e\cup f$. Let $e=(a_1,\dots,a_r)$ and $f=(b_1,\dots,b_r)$ be such that $\{a_j,b_j\}\subseteq W_j$ for $j=1,\dots,r$. By the definition of M, $(a_1,a_2)=(2i-1,2i)$ for some $1\le i\le n$. In order to have $G(r,\ell)$ interval r-partite, a_1 is the rightmost vertex in W_1 and a_2 is the leftmost vertex in W_2. Similarly, a_3 is the rightmost vertex in W_3 and a_4 is the leftmost vertex in W_4, and so on. But for the same reasons the same must hold for f. Thus, $e\cap[2n]=f\cap[2n]$. But $|e\cap[2n]|=2\lceil(\ell+1)/2\rceil\ge\ell+1$, contradicting the condition $|e\cap f|=\ell$.

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $g = \lfloor \log_2 n \rfloor$, so that $2^g \le n < 2^{g+1}$. For $0 \le i \le g$, let $\mathcal{I}_i$ be the partition of $V(H)$ into intervals of length 2^{g-i} plus one interval of length at most 2^{g-i} containing the vertex n. Note that $\mathcal{I}_g$ is the partition into singletons, so for each $e \in H$, there exists a minimum $i(e)$ such that e intersects at least k intervals in $\mathcal{I}_{i(e)}$. For $0 \le i \le g$, let

$$H_i = \{e \in H : i(e) = i\}$$

so that $H = \bigsqcup_{i=0}^{g} H_i$ – the H_i are edge-disjoint. Since each part in every $\mathcal{I}_i$ has size at most $2^{g-i} \le \lceil n/2^i \rceil$, Theorem 1.1 follows from the following claim:

For $0 \le i \le g$, H_i is a union of $t_i \le \sum_{\ell=k}^{r} \binom{2k-2}{\ell} \cdot \frac{2^{i(k-1)}}{(k-1)!}$ interval k-partite hypergraphs $H_{ij} : 1 \le j \le t_i$ with parts from $\mathcal{I}_i$.

The claim is trivial for $i = 0$, since H_0 is empty unless $k = 2$ and $n > 2^g$, in which case H_0 is k-partite and $t_0 = 1$. To see the claim for $i \ge 1$, note that for $e \in H_i$, there are $s \le k-1$ intervals $I_1, I_2, \dots, I_s \in \mathcal{I}_{i-1}$ such that $e = \bigcup_{\ell=1}^{s} (e \cap I_\ell) \subset \bigcup_{\ell=1}^{s} I_\ell$, by the definition of $i(e) = i$.

If $|\mathcal{I}_{i-1}| \ge k-1$, then let $I_{s+1}, \dots, I_{k-1} \in \mathcal{I}_{i-1}$ be $k-s-1$ new intervals chosen arbitrarily, so that $I_1, \dots, I_{k-1} \in \mathcal{I}_{i-1}$. There are at most $\binom{|\mathcal{I}_{i-1}|}{k-1} \le \frac{|\mathcal{I}_{i-1}|^{k-1}}{(k-1)!} \le \frac{2^{i(k-1)}}{(k-1)!}$ choices for these $k-1$ intervals, and then at most $\sum_{\ell=k}^{r} \binom{2k-2}{\ell}$ choices for the intervals from $\mathcal{I}_i$ contained in $\bigcup_{\ell=1}^{s} I_\ell$ and intersecting e.

If $|\mathcal{I}_{i-1}| \le k-2$, then $s \le |\mathcal{I}_{i-1}| \le k-2$ and we add $|\mathcal{I}_{i-1}| - s$ new intervals from $\mathcal{I}_{i-1}$ arbitrarily to $I_1, I_2, \dots, I_s$, thereby obtaining all intervals of $\mathcal{I}_{i-1}$. There are at most $\sum_{j=k}^{r} \binom{2|\mathcal{I}_{i-1}|}{j} \le \sum_{j=k}^{r} \binom{2k-4}{j}$ choices for the intervals from $\mathcal{I}_i$ contained in $[n]$ and intersecting e. Since e intersects at least k intervals in $\mathcal{I}_{i(e)}$, $k \le 2s \le 2^{i+1}$, and $2^{i(k-1)} \ge \left(\frac{k}{2}\right)^{k-1} \ge (k-1)!$; so the claim holds again. □

Proof of Theorem 1.2. We derive Theorem 1.2 from Theorem 1.1, using the notation of its proof. Let H be an n-vertex ordered r-graph with dn^α edges. Let $C = \sum_{j=k}^{r} \binom{2k-2}{j}$ and $m_i = 2^{g-i}$. We prove that some H_{ij} has

$$e(H_{ij}) \ge \begin{cases} \dfrac{1}{C} \cdot \dfrac{dm_i^\alpha}{1 + \log_2 n} & \text{if } \alpha = k-1 \\[2ex] \dfrac{1 - 2^{k-1-\alpha}}{C} \cdot dm_i^\alpha & \text{if } \alpha > k-1 \end{cases}$$

Note that H_{ij} is k-partite and the parts of H_{ij} have size at most m_i, so the above statements imply Theorem 1.2. Suppose, for a contradiction, that no H_{ij} satisfies the above bounds.

Case 1. $\alpha = k-1$. Then recalling $2^g \le n$ and $t_i \le C \cdot 2^{i(k-1)}$,

$$\begin{aligned} e(H) &= \sum_{i=0}^{g} e(H_i) \le \sum_{i=0}^{g}\sum_{j=1}^{t_i} e(H_{ij}) < \sum_{i=0}^{g}\sum_{j=1}^{t_i} \frac{1}{C}\cdot\frac{dm_i^\alpha}{1+\log_2 n} \\ &= \sum_{i=0}^{g}\sum_{j=1}^{t_i} \frac{d\cdot 2^{(k-1)(g-i)}}{C(1+\log_2 n)} \\ &= \frac{d}{C(1+\log_2 n)}\sum_{i=0}^{g} t_i \cdot 2^{(k-1)(g-i)} \\ &\le dn^{k-1}\sum_{i=0}^{g}\frac{1}{1+\log_2 n} = (g+1)\cdot\frac{d\cdot n^{k-1}}{1+\log_2 n}. \end{aligned}$$

Since $g \le \log_2 n$, $e(H) < d\cdot n^{k-1}$, a contradiction.

Case 2. $\alpha > k-1$. Let $c = (1-2^{k-1-\alpha})/C$. Then using $t_i \le C\cdot 2^{i(k-1)}$,

$$\begin{aligned} e(H) &\le \sum_{i=0}^{g}\sum_{j=1}^{t_i} e(H_{ij}) \\ &< \sum_{i=0}^{g}\sum_{j=1}^{t_i} c\cdot dm_i^\alpha \\ &\le \sum_{i=0}^{g} C\cdot 2^{i(k-1)}\cdot c\cdot d2^{\alpha(g-i)} \\ &\le dn^\alpha\cdot(1-2^{k-1-\alpha})\cdot\sum_{i=0}^{g} 2^{i(k-1-\alpha)}. \end{aligned}$$

Since $\alpha > k-1$, the geometric series sum is less than $1/(1-2^{k-1-\alpha})$, and $e(H) < dn^\alpha$. This contradiction completes the proof. □

4 Proofs of Theorems 1.4 – 1.9

Let P_k^r denote the r-uniform tight path, which has vertex set $V = \{v_0,\dots,v_{k+r-2}\}$ and edge set $\{\{v_i,v_{i+1},\dots,v_{i+r-1}\} : 0 \le i \le k-1\}$. Then $\mathrm{ord}(P_k^r)$ contains the ordered r-graph ZP_k^r with edges $\{v_i,v_{i+1},\dots,v_{i+r-1}\}$ for $0 \le i < k$ with a partition of V into r intervals $X_0 < X_1 < \dots < X_{r-1}$ such that vertices $v_i < v_{i+r} < v_{i+2r} < \dots$ are in X_i if i is even and $v_i > v_{i+r} > v_{i+2r} > \dots$ in X_i if i is odd. Extremal problems for ZP_k^r are studied in [12], where the following theorem is (implicitly) proved:

Theorem 4.1. *For $k, r \ge 2$,*

$$\mathrm{ex}_\to(n, ZP_k^r) \le (k-1)\binom{n}{r-1}.$$

In particular, this theorem gives the same upper bounds for the extremal function for $\mathrm{ord}(P_k^r)$, because

$ZP_k^r \in \mathrm{ord}(P_k^r)$. In [12, 13] we also obtain ordered versions of the Erdős-Ko-Rado Theorem by taking every rth edge of P_k^r.

Definition 3. *An ordered r-graph H with vertex set V is a* $(1, r-1)$-graph *if there exist intervals $X < Y$ or $X > Y$ in V such that every edge of H has exactly one vertex in X and $r-1$ vertices in Y.*

Note that an interval $(r-1)$-partite r-graph contains a $(1, r-1)$-graph with at least half the edges: if $I_1 < I_2 < \cdots < I_{r-1}$ are intervals intersecting every edge in the r-graph, then some $X \in \{I_1, I_{r-1}\}$ contains exactly one vertex from at least half of the edges of the r-graph.

Proof of Theorem 1.4. A *strong d-dimensional r-simplex* $\hat{\mathcal{S}}_d^r$ is an r-graph consisting of $d+2$ edges such that we may order the edges so that the first $d+1$ edges form a d-dimensional simplex (see the definition in Section 1.1.1), and the last edge contains at least one vertex from the intersection of every d-tuple of the edges of the d-dimensional simplex. For example, a strong 1-dimensional simplex comprises three edges e, f, g such that $e \cap f = \emptyset$ (so e and f form a 1-dimensional simplex), and both $e \cap g$ and $f \cap g$ are nonempty. It is convenient to assume such an ordering of the edges of a strong simplex is given. We introduce strong simplices for the purpose of doing a simple induction on d: we show that

$$\mathrm{ex}_{\rightarrow}(n, \mathrm{ord}(\hat{\mathcal{S}}_d^r)) \le r^{10dr} n^{r-1}.$$

The base case $d = 1$ follows easily from Theorem 4.1: if H is an ordered r-graph with more than $r^{10r}\binom{n}{r-1}$ edges, then $ZP_{r+1}^r \subset H$, and any three edges of ZP_{r+1}^r that include the first and last edge form a strong 1-dimensional simplex. Now suppose we have proved the theorem for strong $(d-1)$-dimensional simplices for some $d \ge 2$, and let H be an n-vertex ordered r-graph with more than $r^{10dr} n^{r-1}$ edges. Applying Theorem 1.2 with $k = \alpha = r-1$ and the bound on $c(r-1, r-1, r)$ in (3), we find an interval $(r-1)$-partite subgraph G of H with parts of size at most m and

$$e(G) \ge c(r-1, r-1, r) \cdot r^{10dr} m^{r-1}$$

for some $m > 0$ with intervals X and $Y = V(G) - X$ as the parts of G. By (3), it is straightforward to check that $c(r-1, r-1, r) > r^{-4r}$, and therefore

$$e(G) > r^{-4r} r^{10dr} m^{r-1} > 2r^{10(d-1)r} m^{r-1}. \tag{9}$$

We remove from G each edge containing an $(r-1)$-set in Y which is contained in at most two edges of G. This way, we delete at most $2\binom{m}{r-1}$ edges. Since $r \ge 3$, this is less than m^{r-1}, and hence the remaining $(1, r-1)$-subgraph G' of G has at least $r^{10(d-1)r} m^{r-1}$ edges. By averaging, some vertex $x \in X$ is contained in at least

$$r^{1+10(d-1)r} m^{r-2} \ge r^{10(d-1)(r-1)} m^{r-2}$$

edges of G'. By induction, $\{e \backslash \{x\} : e \in G'\}$ – the *link hypergraph* of x in G' – contains a strong $(d-1)$-dimensional $(r-1)$-simplex F, say with edges $e_1, e_2, \ldots, e_d, f$, with $e_1, e_2, \ldots, e_d$ forming a

$(d-1)$-dimensional simplex. Since f is contained in at least 3 edges of G, there exists $y \in X\backslash\{x\}$ such that $f \cup \{y\} \in G$. Then $e_1 \cup \{x\}, e_2 \cup \{x\}, \dots, e_d \cup \{x\}, f \cup \{y\}$ is a d-dimensional simplex in H, and together with $f \cup \{x\}$, we have a strong simplex in H. This proves the theorem. □

Proof of Theorem 1.5. Let M denote the largest number of edges in an r-graph in $\mathcal{F}^+$ and suppose $\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F})) \leq cn^{r-2}$ for all $n > 1$. We will prove that $\mathrm{ex}_{\to}(n, \mathrm{ord}(\mathcal{F}^+)) \leq c'n^{r-1}$ where $c' = \frac{1}{2}(M + c)r^{10r}$. Suppose that H is an ordered n-vertex r-graph with more than $c'n^{r-1}$ edges. Applying Theorem 1.2 with $k = \alpha = r - 1$, we find an m-vertex $(1, r-1)$-subgraph G of H with at least $r^{-4r}c'm^{r-1} > 2c'r^{-10r}m^{r-1}$ edges as in (9), with parts X and Y, such that every edge has one vertex in X. For each $(r-1)$-set in Y contained in at most $M - 1$ edges of G, remove all edges of G containing that $(r-1)$-set. The number of edges that we removed is at most Mm^{r-1}, so the remaining r-graph $G' \subset G$ has more than

$$(2c'r^{-10r} - M)m^{r-1} = c\,m^{r-1}$$

edges. By averaging, there exists a vertex $x \in X$ whose *link hypergraph* $G'' = \{e\backslash\{x\} : e \in G'\}$ has more than cm^{r-2} edges. Then G'' contains a member F of $\mathrm{ord}(\mathcal{F})$. Since every edge of F is contained in at least M edges of G, we can expand the edges of F to distinct vertices of X to obtain a copy of F^+ in H. □

Proof of Theorem 1.6. The proof of Theorem 1.5 gives a statement which is slightly stronger than the statement of Theorem 1.5: if $\mathcal{F}$ is a family of $(r-1)$-graphs such that $\mathrm{ex}_{\to}(\mathrm{ord}(\mathcal{F}) \leq cn^{r-2}$, then for any $M \geq 1$ in any n-vertex ordered r-graph H with more than $\frac{1}{2}(M + c)r^{10r}n^{r-1}$ edges, we find intervals $I_1 < I_2 < \cdots < I_{r-1}$ and an interval $X < I_1$ or $X > I_{r-1}$ with the following structure:

(i) a copy $F_0 \subset H$ of F has intervals $I_1 < I_2 < \cdots < I_{r-1}$,
(ii) for some $v \in X$, F_0 is contained in $\{e\backslash\{v\} : e \in H\}$,
(iii) for every $e \in F_0$, there exist M vertices $x \in X$ such that $e \cup \{x\} \in H$.

Now we prove $\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r)) = \Theta(n^{r-1})$ for $r \geq 3$. First note that any ordered r-graph on n vertices with transversal number 1 has $\binom{n-1}{r-1}$ and no subgraph in $\mathrm{ord}(T_r)$. Therefore $\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r)) = \Omega(n^{r-1})$. For an upper bound on $\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r))$, fix $r \geq 3$, and Let H be an ordered n-vertex r-graph with more than $2r^{10r}n^{r-1}$ edges. Let F denote the $(r-1)$-graph consisting of three edges $e = \{v_1, v_2, \dots, v_{r-1}\}, f = \{v_{r-1}, v_r, \dots, v_{2r-3}\}, g = \{v_{2r-3}, v_{2r-2}, \dots, v_{3r-5}\}$ – this is the *loose path* with three edges. Then F is contained in the hypergraph P consisting of all edges $\{v_i, v_{i+1}, \dots, v_{i+r-2}\}$ for $1 \leq i \leq 2r-3$. Since $ZP_{2r-3}^{r-1} \in \mathrm{ord}(P)$, Theorem 4.1 gives:

$$\mathrm{ex}_{\to}(n, \mathrm{ord}(F)) \leq \mathrm{ex}_{\to}(n, \mathrm{ord}(P)) \leq (2r-4)\binom{n}{r-2} < 2n^{r-2}.$$

Taking $c = M = 2$, since $e(H) > 2r^{10r}n^{r-1} \geq \frac{1}{2}(M + c)r^{10r}n^{r-1}$, we find the structure prescribed by (i) – (iii) in H. Now since $e, g \in F_0$, by (iii) there exists a vertex $v \in X$ such that $e \cup \{v\}, g \cup \{v\} \in H$. Since $M = 2$, there exists $x \in X\backslash\{v\}$ such that $f \cup \{x\} \in H$. The three edges $e \cup \{v\}, f \cup \{x\}, g \cup \{v\}$ form an r-graph in $\mathrm{ord}(T_r)$ contained in H. We conclude $\mathrm{ex}_{\to}(n, \mathrm{ord}(T_r)) \leq 2r^{10r}n^{r-1}$. □

Proof of Theorem 1.7. We first present an easy proof for $r = 2$, and then a significantly more involved general proof.

Case 1: $r = 2$. Suppose that F is a forest with k edges. By adding edges, we may assume that F is a tree. We prove by induction on k that $\mathrm{ex}_{\to}(n, \mathrm{ord}(F)) \le 2k^2n$. Let H be an ordered n-vertex graph with more than $2k^2n$ edges and let F' be a tree obtained from F by deleting a leaf y. Let $x \in V(F')$ be the neighbor of y. For each vertex v of H, mark the edges from v to the k smallest neighbors of v and the edges to the k largest neighbors of v. Note that if v has fewer than k smaller neighbors then we marked all edges between v and those neighbors, and similarly for larger neighbors. We marked at most $2kn$ edges so the resulting unmarked graph $H' \subset H$ has more than $2k^2n - 2kn \ge 2(k-1)^2n$ edges. By induction, H' contains an interval 2-partite subgraph K' isomorphic to F', with parts $A < B$. Suppose that v is the vertex of K' that plays the role x in F', and assume first that $v \in A$. Then there is a vertex $w \in B$ with $\{v, w\} \in K'$, so by construction of H', there is another vertex $w' > w$ such that $\{v, w'\} \in H$ and $w' \notin V(K')$. Adding edge $\{v, w'\}$ to K' gives a copy K of the 2-interval-partite graph F (w' plays the role of y). The same argument applies if $v \in B$. □

Case 2: $r \ge 3$. By Theorem 1.2 with $\alpha = r - 1 = k$, it is enough to prove Theorem 1.7 for interval $(r-1)$-partite r-graphs. Let H be an interval $(r-1)$-partite r-graph with n vertices and a partition of $V(H)$ into intervals $X_1 < X_2 < \cdots < X_{r-1}$ where for some i, and every $e \in H$, $|e \cap X_i| = 2$ and $|e \cap X_j| = 1$ for $j \neq i$. It is easy to check that every forest F is contained in a tight tree T with the same set of vertices. We show by induction on $t = v(T) \ge r$ that if $e(H) > 2t^2\binom{n}{r-1}$, then H contains a member of $\mathrm{ord}(T)$. If $t = r$, then T has one edge and clearly $e(H) = 0$ if H is $\mathrm{ord}(T)$-free. Suppose the statement is true for all tight trees with fewer than t vertices, and let T be a tight tree with t vertices. Let H be an n-vertex interval $(r-1)$-partite r-graph with more than $2t^2\binom{n}{r-1}$ edges. For each $f \in \partial H$, let $S(f)$ and $L(f)$ denote the set of the t smallest and t largest vertices $x \in V(H)$ such that $f \cup \{x\} \in H$. Then we remove all edges $f \cup \{x\}$ from H such that $x \in S(f) \cup L(f)$. We obtain a new ordered interval $(r-1)$-partite r-graph H' with parts $X_1 < X_2 < \cdots < X_{r-1}$.

Fix a leaf y of T. Let e be the edge of T containing y, $T' = T - \{y\}$ and $g = e - \{y\}$. By induction, H' contains a member of $\mathrm{ord}(T')$, since

$$e(H') > 2t^2\binom{n}{r-1} - 2t\binom{n}{r-1} > 2(t-1)^2\binom{n}{r-1}.$$

Let this member of $\mathrm{ord}(T')$ be denoted by S, and have parts $A_0 < A_1 < \cdots < A_{r-1}$, where $A_{i-1}, A_i \subseteq X_i$ and $A_j \subseteq X_j$ for $j \neq i$. Let g' be the image of g in S. Since $f \in \partial S \subset \partial H'$, $f \cap A_j = \emptyset$ for some $j \le r$. If $j \notin \{i, i-1\}$, then $S(f) \cup L(f) \subset X_j$, and since $|S(f) \cup L(f)| > t$ and $|V(T) \cap X_j| < t$, there exists $x \in X_j \backslash V(T)$ such that $f \cup \{x\} \in H$ together with S forms a copy of T in H, with interval coloring $A'_0 < A'_1 < \ldots, < A'_{r-1}$ where $A'_h = A_h$ for $h \neq j$ and $A'_j = A_j \cup \{x\}$. If $j = i$, then $f \cup \{z\} \in S$ for some $z \in A_i$. For every $x \in L(f)$, we have $x > z$ and $x \in X_i$. Since $|L(f)| = t$, there exists $x \in L(f)$ such that $x > z$ and $x \notin V(S)$. Now $f \cup \{x\} \in H$ together with S is a copy of an element of $\mathrm{ord}(T)$ in H, with interval r-coloring $A'_0 < A'_1 < \ldots, < A'_{r-1}$ where $A'_h = A_h$ for $h \neq i$ and $A'_i = A_i \cup \{x\}$. Finally, if $j = i - 1$, then $f \cup \{z\} \in S$ for some $z \in A_0$. For every $x \in S(f)$, we have $x < z$ and $x \in X_{i-1}$. Since $|S(f)| = t$, there exists $x \in S(f)$ such that $x < z$ and $x \notin V(S)$. Now $f \cup \{x\} \in H$ together with S is a copy of an element of $\mathrm{ord}(T)$ in H, with interval r-coloring $A'_0 < A'_1 < \ldots, < A'_{r-1}$ where $A'_h = A_h$ for $h \neq i-1$ and $A'_{i-1} = A_{i-1} \cup \{x\}$. This completes the proof. □

Proof of Theorem 1.8. By Theorem 1.2 with $k = \alpha = 2$, it is enough to prove Theorem 1.8 for interval 2-partite 3-graphs. Suppose that $\epsilon > 0$ and n_0 is sufficiently large. Let H be an n-vertex ordered interval 2-partite 3-graph with at least ϵn^2 edges ($n > n_0$) containing no member of $\mathrm{ord}(I_2)$ and $A < B$ be intervals where every edge of H has exactly one vertex in A. Let G be the graph with vertex set B and edge set $\{yz : \exists x \in A, xyz \in H\}$. Since H contains no member of $\mathrm{ord}(I_2)$, $e(G) = e(H) \geq \epsilon n^2$. By Theorem C, there is an interval 2-partite subgraph $G' \subset G$ with at least δn^2 edges, for some δ depending only on ϵ. Consequently, there is an interval 3-partite subgraph $H' \subset H$ with δn^2 edges and we apply the Ruzsa-Szemerédi Theorem to H' to obtain a copy of some member of $\mathrm{ord}(T_3)$. □

Proof of Theorem 1.9. We use the result of Frankl and Füredi [9] stating that for $0 \leq \ell \leq r-1$ and some constant $C(r,\ell) > 0$,

$$\mathrm{ex}(n, I^r(\ell)) < C(r,\ell) \cdot n^{\max\{\ell, r-\ell-1\}}. \tag{10}$$

Construction 4 gives a lower bound of order n^α for $\mathrm{ex}_\to(n, \mathrm{ord}(I^r(\ell)))$, so it remains to prove the upper bound in Theorem 1.9. We first prove the upper bound when ℓ is odd.

Recall $\alpha = \max\{\ell, r - (\ell+1)/2\}$, and let $k = \alpha$, $\ell' = \ell - r + k \geq 0$. Let H be an ordered n-vertex r-graph with $C(k,\ell')(kn)^\alpha/c$ edges, where c is the implicit constant in the second inequality of Theorem 1.2, namely (3). We aim to show that H contains a member of $\mathrm{ord}(I^r(\ell))$. By Theorem 1.2 with $k = \alpha$, there is for some $m \in [n]$ an interval k-partite subgraph H' of H with $e(H') \geq C(k,\ell')(km)^\alpha$ and parts of size at most m. For each edge $e \in H'$,

$$\sum_{j=1}^{k} (|e \cap I_j| - 1) = r - k.$$

Let $f(e)$ be the set of the first $|e \cap I_j| - 1$ elements of $e \cap I_j$ for $1 \leq j \leq k$, so that $|f(e)| = r - k$. By the pigeonhole principle, there exists a set S of size $r-k$ such that $f(e) = S$ for at least $|H'|/m^{r-k} \geq C(k,\ell')(km)^{\alpha-r+k}$ edges $e \in H'$. Let $H'' = \{e \backslash S : S \subset e \in H'\}$, so H'' is an ordered k-uniform k-partite hypergraph with $N = v(H'') \leq km$ and $e(H'') \geq C(k,\ell')N^{\alpha-r+k}$. Since $2\alpha = \max\{2\ell, 2r-\ell-1\} \geq 2r-\ell-1$,

$$\begin{aligned}
\max\{k-\ell'-1, \ell'\} &= \max\{r-\ell-1, \ell-r+k\} \\
&= \max\{2r-\ell-1-k, \ell\} - r + k \\
&\leq \max\{2\alpha - k, \ell\} - r + k \;=\; \max\{\alpha, \ell\} - r + k \;=\; \alpha - r + k.
\end{aligned}$$

It follows from (10) that $e(H'') \geq C(k,\ell')N^{\alpha-r+k} > \mathrm{ex}(N, I^k(\ell'))$. Therefore there exist $f, g \in H''$ with $|f \cap g| = \ell'$. Since H'' is k-partite, $\{f, g\} \in \mathrm{ord}(I^k(\ell'))$ and now $\{f \cup S, g \cup S\} \in \mathrm{ord}(I^r(\ell))$. We conclude

$$\mathrm{ex}_{\rightarrow}(n, \mathrm{ord}(I^r(\ell))) < \frac{C(k,\ell')}{c(\alpha,k,r)}(kn)^{\alpha}.$$

This completes the proof of Theorem 1.9 when ℓ is odd.

When $\ell \geq 2$ is even, $\alpha = \max\{\ell, r - (\ell+2)/2\}$. Let $k = \alpha + 1$, $\ell' = \ell - r + k \geq 0$, and let H be an ordered n-vertex r-graph with $C(k,\ell')(kn)^{\alpha}(1 + \log_2 n)/c$ edges where c is the implicit constant in the first inequality of Theorem 1.2. Then for some $m \in [n]$ there is an interval k-partite subgraph H' of H with $e(H') \geq C(k,\ell')(km)^{\alpha}$ and parts of size at most m. Define the interval k-partite k-graph $H'' \subseteq H'$ as above. Since ℓ is even, $2\alpha = \max\{2\ell, 2r - \ell - 2\} \geq 2r - \ell - 2$, and therefore

$$\begin{aligned}
\max\{k - \ell' - 1, \ell'\} &= \max\{r - \ell - 1, \ell - r + k\} \\
&= \max\{2r - \ell - 1 - k, \ell\} - r + k \\
&\leq \max\{2\alpha - k + 1, \ell\} - r + k \;=\; \max\{\alpha, \ell\} - r + k \;=\; \alpha - r + k.
\end{aligned}$$

In the last line we used $k = \alpha + 1$. It follows from (10) that H'' contains a member of $I^k(\ell')$ and then H contains a member of $I^r(\ell)$. This completes the proof of Theorem 1.9 when ℓ is even. $\square$

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