The chromatic spectrum of mixed hypergraphs

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Abstract

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set*, and each of $\mathcal{C}$, $\mathcal{D}$ is a list of subsets of X. A *strict k-coloring* of $\mathcal{H}$ is a surjection $c : X \to \{1, \ldots, k\}$ such that each member of $\mathcal{C}$ has two vertices assigned a common value and each member of $\mathcal{D}$ has two vertices assigned distinct values. The *feasible set* of H is $\{k\colon\ H \text{ has a strict } k\text{-coloring}\}$.

Among other results, we prove that a finite set of positive integers is the feasible set of some mixed hypergraph if and only if it omits the number 1 or is an interval starting with 1. For the set $\{s, t\}$ with $2 \le s \le t - 2$, the smallest realization has $2t - s$ vertices. When every member of $\mathcal{C} \cup \mathcal{D}$ is a single interval in an underlying linear order on the vertices, the feasible set is also a single interval of integers.

1 Introduction

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is a set, called the *vertex set*, and $\mathcal{C}$, $\mathcal{D}$ are families of subsets of X, with each subset having at least two elements. A *proper k-coloring* of a mixed hypergraph is a function from the vertex set to a set of k *colors* so that each $\mathcal{C}$-edge has two vertices with a common color and each $\mathcal{D}$-edge has two vertices with distinct colors. A mixed hypergraph is *k-colorable* if it has a proper coloring with at most k colors. A *strict k-coloring* is a proper k-coloring using all k colors. The minimum number of colors in a strict coloring of a mixed hypergraph $\mathcal{H}$ is its *lower chromatic number* $\chi(\mathcal{H})$; the maximum number is its *upper chromatic number* $\bar{\chi}(\mathcal{H})$.

Introduced by Voloshin [13], the theory of mixed hypergraphs is growing rapidly. It has many potential applications, as mixed hypergraphs can be used to encode various partitioning constraints. They have been used to model problems in such areas as list-coloring of graphs [10], integer programming [10, 5], coloring of block designs [8, 6, 7, 9], and a variety of applied areas [14].

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A variant of mixed hypergraph coloring was investigated by Ahlswede [1]. He showed that questions about colorings in which the vertices of every edge received a certain percentage of distinct colors were equivalent to several multi-user source coding problems. A variant of a canonical Ramsey problem on edge-colorings studied in [4] and [3] can also be phrased using coloring of mixed hypergraphs.

We will assume throughout that the vertex set X is finite, and let $|X| = n$. For each k, let r_k be the number of partitions of the vertex set into k nonempty parts (color classes) such that the coloring constraint is satisfied on each edge. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n)$ is the *chromatic spectrum* of $\mathcal{H}$, introduced in [12]. The set of values k such that $\mathcal{H}$ has a strict k-coloring is the *feasible set* of $\mathcal{H}$, written $S(\mathcal{H})$; this is the set of indices i such that $r_i > 0$.

A mixed hypergraph $\mathcal{H}$ has lower chromatic number 1 if and only if $\mathcal{H}$ has no $\mathcal{D}$-edges. In this case, color classes in a proper coloring can be combined to form a proper coloring using fewer colors, and thus $S(\mathcal{H}) = \{1, \ldots, \bar{\chi}(\mathcal{H})\}$. Similarly, $\bar{\chi}(\mathcal{H}) = n$ if and only if $\mathcal{H}$ has no $\mathcal{C}$-edges. In this case, color classes in a proper coloring can be partitioned to form a proper coloring using more colors, and thus $S(\mathcal{H}) = \{\chi(\mathcal{H}), \ldots, n\}$.

For spectra of mixed hypergraphs studied earlier, the feasible set was always the full interval from $\chi(\mathcal{H})$ to $\bar{\chi}(\mathcal{H})$. At many conferences, the fourth author asked whether this holds for all mixed hypergraphs. We settle this in the negative. A mixed hypergraph has a *gap at k* if its feasible set contains elements larger and smaller than k but omits k. In Section 2, we construct for $2 \le s \le t-2$ a mixed hypergraph $\mathcal{H}_{s,t}$ with feasible set $\{s, t\}$. Furthermore, we prove that $\mathcal{H}_{s,t}$ has the fewest vertices among all s-colorable mixed hypergraphs that have a gap at $t-1$; this minimum number of vertices is $2t - s$.

This raises the question of which sets of positive integers are feasible sets of mixed hypergraphs. We solve this for finite sets in Section 3, where we prove that a finite set of positive integers is a feasible set if and only if it is an initial interval $\{1, \ldots, t\}$ or does not contain the element 1.

Our proof is constructive, producing mixed hypergraphs in which the size of the vertex set is exponential in the number of elements below the highest gap in the feasible set. The question of finding the minimum number of vertices in a mixed hypergraph with feasible set S of size at least 3 remains open. We present a second construction that produces smaller mixed hypergraphs when S is composed of few intervals.

In Section 4 we consider special families of mixed hypergraphs. We prove that gaps can arise even when $\mathcal{C} = \mathcal{D}$ and all the edges have the same size. We also show that gaps cannot arise when each member of $\mathcal{C}$ and $\mathcal{D}$ is an interval in an underlying linear order on the vertices.

2 The Smallest Mixed Hypergraphs with Gaps

We begin with an explicit construction of a mixed hypergraph with $2t - 2$ vertices and feasible set $\{2, t\}$. Let K_n denote the mixed hypergraph with n vertices in which $\mathcal{C} = \emptyset$ and $\mathcal{D}$ is the set of all pairs of vertices. Trivially, $S(K_n) = \{n\}$.

We first describe the construction informally. Beginning with K_t, we expand $t - 2$ of the vertices into pairs, leaving two special vertices unexpanded. The $\mathcal{D}$-edge consisting of the two special vertices remains, and the other $\mathcal{D}$-edges expand into $\mathcal{D}$-edges of size 3 or 4 (special vertex plus pair, or union of two pairs). We add as $\mathcal{C}$-edges all triples consisting of three vertices arising from two original vertices (special vertex plus pair, or three vertices from two pairs).

This describes the construction completely, but we present it more formally to facilitate proofs. The smallest instance is for $t = 4$; from K_4 we produce a 6-vertex mixed hypergraph with spectrum $\{2, 4\}$. We use the notation $[m] = \{1, \ldots, m\}$.

Construction 1 We define a hypergraph $\mathcal{H}_{2,t}$ with vertex set $\{x_1, x_2, a_1, \ldots, a_{t-2}, b_1, \ldots, b_{t-2}\}$. Let T be the set of all triples of the form $x_r a_i b_i$, for $r \in \{1, 2\}$ and $i \in [t-2]$. Let U be the set of quadruples of the form $a_i b_i a_j b_j$ for $i, j \in [t-2]$. Let W be the union, over $i, j \in [t-2]$, of the sets of four triples contained in $\{a_i, b_i, a_j, b_j\}$. The $\mathcal{C}$-edges in $\mathcal{H}_{2,t}$ are $T \cup W$. The $\mathcal{D}$-edges are $T \cup U \cup \{x_1 x_2\}$. □

Lemma 1 *The feasible set of the hypergraph $\mathcal{H}_{2,t}$ in Construction 1 is $\{2, t\}$.*

Proof. Let c be an arbitrary proper coloring of $\mathcal{H}_{2,t}$. If $c(a_i) \neq c(b_i)$, then the $\mathcal{C}$-edges in T and W that contain a_i and b_i force all other vertices to have the same color as a_i or b_i. Thus in this case there are at most two colors. The existence of $\mathcal{D}$-edges prevents a proper 1-coloring, and for $j \in [t-2]$, setting all $c(a_j) = c(x_1) = 1$ and $c(b_j) = c(x_2) = 2$ completes a proper 2-coloring.

Hence we may assume that $c(a_i) = c(b_i)$ for all $i \in [t-2]$. Now the $\mathcal{D}$-edges in U force these colors to be distinct for all i, and the $\mathcal{D}$-edges in T along with $x_1 x_2$ require additional colors for x_1 and x_2. This completely forces the coloring, which uses t colors and is proper. □

In order to extend this construction to lower chromatic number s, we use a simple lemma about combining feasible sets. The *join* of two mixed hypergraphs $(X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $(X_2, \mathcal{C}_2, \mathcal{D}_2)$ with disjoint vertex sets is the mixed hypergraph $(X, \mathcal{C}, \mathcal{D})$ defined by $X = X_1 \cup X_2$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, and $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup R$, where R is the set of pairs consisting of one vertex from X_1 and one from X_2.

Lemma 2 *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are mixed hypergraphs, then the feasible set of the join of $\mathcal{H}_1$ and $\mathcal{H}_2$ is $\{i + j\colon\ i \in S(\mathcal{H}_1), j \in S(\mathcal{H}_2)\}$.*

Proof. The $\mathcal{D}$-edges added between the vertex sets of $\mathcal{H}_1$ and $\mathcal{H}_2$ prohibit colors from appearing in both sets. Thus the proper colorings of the join are precisely the colorings that consist of proper colorings of $\mathcal{H}_1$ and $\mathcal{H}_2$ using disjoint sets of colors. □

In particular, taking the join of a mixed hypergraph $\mathcal{H}$ with the mixed hypergraph K_a has the effect of "shifting" the feasible set of $\mathcal{H}$ to the right by a units, adding a to each element of $S(\mathcal{H})$.

Theorem 1 *If $\mathcal{H}$ is an s-colorable mixed hypergraph with a gap at $t-1$, then $n \geq 2t - s$, and for $2 \leq s \leq t-2$, this bound is sharp.*

Proof. Consider a proper coloring of $\mathcal{H}$ using k colors, where k is the smallest element of the feasible set larger than $t-1$. If $n < 2t-s$, then using at least t colors requires having at least $s+1$ color classes of size 1. Two such color classes can be combined to obtain a proper coloring using $k-1$ colors unless they form a $\mathcal{D}$-edge of size 2. Since $k-1$ is not in the feasible set, $\mathcal{H}$ contains K_{s+1}. Now $\mathcal{H}$ is not s-colorable; the contradiction yields $n \ge 2t-s$.

For $s = 2$, Lemma 1 shows that Construction 1 achieves the bound. For $s > 2$, we define $\mathcal{H}_{s,t}$ to be the join of K_{s-2} and $\mathcal{H}_{2,t-s+2}$. By Lemma 2, the feasible set of $\mathcal{H}_{s,t}$ is $\{s-2\} + \{2, t-s+2\} = \{s,t\}$. The number of vertices in $\mathcal{H}_{s,t}$ is $s-2+2(t-s+2)-2 = 2t-s$. □

Corollary 1 *The minimum number of vertices in a mixed hypergraph with a gap in its feasible set is 6, achieved by* $\mathcal{H}_{2,4}$.

Proof. Every mixed hypergraph with a gap in its feasible set is s-colorable with a gap at $t-1$, for some s,t with $t-1 > s \ge 2$. Thus $t \ge 4$ and $t-s \ge 2$. By Theorem 1, $n \ge t + (t-s) \ge 6$. □

A closer analysis allows one $\mathcal{C}$-edge in the 6-vertex example to be dropped without changing the spectrum. Thus 6 $\mathcal{D}$-edges and 7 $\mathcal{C}$-edges suffice.

3 The Family of Feasible Sets

We determine which finite sets are feasible sets of mixed hypergraphs. The n-vertex *trivial* mixed hypergraph $(X, \emptyset, \emptyset)$ has feasible set $\{1,\ldots,n\}$. This gives us all intervals containing 1, and we have observed in the introduction that these are the only feasible sets containing 1.

We construct mixed hypergraphs realizing all other feasible sets using the trivial mixed hypergraphs, the join operation of Lemma 2, and one additional operation. This operation is similar to the construction of $\mathcal{H}_{2,t}$ from K_t. In Construction 1, we avoided expanding two of the vertices in order to create few vertices. Here our constructions will already be exponential in the size of the feasible set, so we prefer the simplicity gained by expanding all vertices into pairs. The new wrinkle in the construction is that we may have $\mathcal{C}$-edges in the mixed hypergraph being expanded.

Construction 2 Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph. We construct a mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ with $X' = \bigcup_{v\in X}\{v^-, v^+\}$. For each $D \in \mathcal{D}$, we add $D' = \bigcup_{v\in D}\{v^-, v^+\}$ to $\mathcal{D}'$. For each $C \in \mathcal{C}$, we add $C' = \{v^- :\ v \in C\}$ to $\mathcal{C}'$. Finally, for each ordered pair $u, v \in X$, we add the triples $\{v^-, v^+, u^-\}$ and $\{v^-, v^+, u^+\}$ to $\mathcal{C}'$. □

The application of Construction 2 may be called *doubling.* It has the effect of appending the element 2 to the feasible set. This is what Construction 1 did to K_t, and the analysis here generalizes Lemma 1.

Lemma 3 *Let* $\mathcal{H}$ *be a mixed hypergraph with feasible set* S*. If* $\chi(\mathcal{H}) \ge 2$*, then the mixed hypergraph* $\mathcal{H}'$ *obtained from* $\mathcal{H}$ *via Construction 2 has feasible set* $S \cup \{2\}$.

Proof. Let c be an arbitrary coloring of $\mathcal{H}'$. If $c(v^-) \neq c(v^+)$ for some vertex v of $\mathcal{H}$, then the $\mathcal{C}$-edges that are triples containing v^-, v^+ force all other vertices to have color $c(v^-)$ or $c(v^+)$. Thus such a coloring uses exactly two colors. We obtain a strict 2-coloring by setting $c(u^-) = c(v^-)$ and $c(u^+) = c(v^+)$ for all u. Since each member of $\mathcal{D}'$ consists of full pairs, the constraints on $\mathcal{D}$-edges are satisified. Also each member of $\mathcal{C}'$ contains two vertices with superscripts of the same type.

It remains to consider colorings with $c(v^-) = c(v^+)$ for each vertex v of $\mathcal{H}$. Let $\tilde{c}$ be the coloring of $\mathcal{H}$ defined by $\tilde{c}(v) = c(v^-)$. For each member of $\mathcal{D}'$, the coloring constraint is satisfied by c if and only if $\tilde{c}$ satisfies the constraint for the corresponding member of $\mathcal{D}$. The same statement holds for members of $\mathcal{C}'$ that arise from members of $\mathcal{C}$. By construction, the new triples in $\mathcal{C}'$ are automatically satisfied. Thus c is a proper coloring of $\mathcal{H}'$ if and only if $\tilde{c}$ is a proper coloring of $\mathcal{H}$. Note that $\tilde{c}$ uses the same number of colors as c.

Similarly, we can extend each proper coloring of $\mathcal{H}$ to a proper coloring of $\mathcal{H}'$ using the same number of colors, by copying the color of each vertex v onto both v^- and v^+. This implies that an integer greater than 2 is feasible for $\mathcal{H}$ if and only if it is feasible for $\mathcal{H}'$. □

The proof establishes a bijection between strict colorings of $\mathcal{H}$ with at least three colors and strict colorings of $\mathcal{H}'$ with at least three colors.

Using shiftings (joins with cliques) and doublings, we can produce all feasible sets.

Theorem 2 *A finite set of positive integers is the feasible set for some mixed hypergraph if and only if it omits the number 1 or is an interval containing 1.*

Proof. It remains only to consider the sets not containing 1. We produce a mixed hypergraph $\mathcal{H}(T)$ with feasible set T. We use induction on the size of the set T, and within each size we use induction on the smallest element t of T. For $T = \{t\}$, we set $\mathcal{H}(T) = K_t$.

For $|T| > 1$ and $t = 2$, we let $\mathcal{H}(T)$ be the mixed hypergraph obtained by applying Construction 2 to $\mathcal{H}(T - \{2\})$. Lemma 3 implies that this works.

For $|T| > 1$ and $t > 2$, we let $\mathcal{H}(T)$ be the join of K_{t-2} with the mixed hypergraph $\mathcal{H}(T')$, where T' is obtained from T by subtracting $t - 2$ from each element. Lemma 2 implies that this works. □

By modifying Construction 2 slightly, we can characterize the feasible sets realizable by mixed hypergraphs having only one feasible partition for each feasible number of colors.

Theorem 3 *A finite set S of positive integers is the feasible set of some mixed hypergraph whose spectrum has each $r_i \in \{0, 1\}$ if and only if $1 \notin S$ or $\max(S) \leq 2$.*

Proof. A mixed hypergraph has 1 in its feasible set if and only if it has no $\mathcal{D}$-edges. In such a mixed hypergraph, combining any two color classes in a strict coloring yields a strict coloring with fewer colors. If there is a strict coloring with k colors, then combining two color classes yields $\binom{k}{2}$ distinct partitions for strict colorings with $k - 1$ colors. Thus the condition is necessary. Among mixed hypergraphs with no $\mathcal{D}$-edges, the sets $\{1\}$ and $\{1, 2\}$ are realized with two vertices by having one $\mathcal{C}$-edge or no $\mathcal{C}$-edges.

For sets not containing 1, we use induction to prove a stronger result. We say that a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ *properly realizes* a set S if S is the feasible set of $\mathcal{H}$, each entry in the spectrum of $\mathcal{H}$ is 0 or 1, and in every proper coloring of $\mathcal{H}$, every color class has fewer than $|X|/2$ vertices, except for the unique half/half partition that occurs when $2 \in S$. For a finite set S of integers greater than 1, let $m(S) = 2\max(S) - \min(S)$. We use induction on $m(S)$ to construct a mixed hypergraph that properly realizes S.

The smallest value of $m(S)$ is 2, which occurs only when $S = \{2\}$. This set is properly realized by K_2. For $m(S) > 2$, we have two cases, depending on whether $2 \in S$.

If $2 \notin S$, then we properly realize S by taking the join of K_1 with a mixed hypergraph $\mathcal{H}$ that properly realizes $T = \{s - 1\colon\ s \in S\}$. Note that $m(T) = m(S) - 1$. The feasible partitions of the resulting $\mathcal{H}'$ consist of those of $\mathcal{H}$ with the new element added as a singleton class. Thus the limit on the size of color classes still holds.

If $2 \in S$, then we modify the application of the doubling construction (Construction 2) to a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ that properly realizes $T = S - \{2\}$. Note that $m(T) < m(S)$, since $\min(T) > \min(S)$. Let $n = |X|$. To the doubled hypergraph $\mathcal{H}'$, we add $\binom{2n}{n} - 2$ further $\mathcal{D}$-edges. We add all n sets of the vertices except $\{v^-\colon\ v \in X\}$ and $\{v^+\colon\ v \in X\}$.

In analyzing the proper colorings of $\mathcal{H}'$, we have the same two cases as in the proof of Lemma 3. When $c(v^-) = c(v^+)$ for every $v \in X$, all the new $\mathcal{D}$-edges are satisfied if and only if no class contains half of the pairs. Thus the coloring c is proper if and only if the coloring $\tilde{c}$ of $\mathcal{H}$ obtained by setting $\tilde{c}(v) = c(v^-)$ is proper, since all proper colorings of $\mathcal{H}$ have no class with half the vertices. Thus $r_i(\mathcal{H}') = r_i(\mathcal{H})$ for $i > 2$, and also $\mathcal{H}'$ has no proper 2-colorings of this type.

It remains to consider colorings with $c(v^-) \neq c(v^+)$ for some v. As in the proof of Lemma 3, all other vertices must have color $c(v^-)$ or $c(v^+)$. The added $\mathcal{D}$-edges of size n permit only the partition with one class $\{u^-\colon\ u \in X\}$ and the other class $\{u^+\colon\ u \in X\}$. □

For a set S of size k, our constructions produce mixed hypergraphs with feasible set S that have more than 2^k vertices. We obtain quick proofs of realizability and realizability with 0,1-spectrum. For realizability in general, this number of vertices is far from minimal. Our next construction usually yields smaller realizations. It enables us to take unions of feasible sets.

Construction 3 Let $\mathcal{H}_1 = (X_1, \mathcal{C}_1, \mathcal{D}_1)$ and $\mathcal{H}_2 = (X_2, \mathcal{C}_2, \mathcal{D}_2)$ be mixed hypergraphs. Let G be the complete bipartite graph with bipartition X_1, X_2. We construct a mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ with $X' = E(G)$; we define $\mathcal{C}'$ and $\mathcal{D}'$ as follows after distinguishing two vertices $x^* \in X_1$ and $y^* \in X_2$.

For $C \in \mathcal{C}_1$, let $C' = \{xy^*\colon\ x \in C\}$.

For $C \in \mathcal{C}_2$, let $C' = \{x^*y\colon\ y \in C\}$.

For $D \in \mathcal{D}_1$, let $D' = \{xy\colon\ x \in D, y \in X_2\}$.

For $D \in \mathcal{D}_2$, let $D' = \{xy\colon\ x \in X_1, y \in D\}$.

Let $\mathcal{C}'_0$ consist of the edge sets of paths of length three in G.

Let $\mathcal{C}' = \mathcal{C}'_0 \cup \{C'\colon\ C \in \mathcal{C}_1 \cup \mathcal{C}_2\}$.

Let $\mathcal{D}' = \{D'\colon\ D \in \mathcal{D}_1 \cup \mathcal{D}_2\}$. □

In a graph G, a *copy* of F is a subgraph of G isomorphic to F. In an edge-coloring of G, we say that (a copy of) F is *polychromatic* if its edges all receive distinct colors.

Lemma 4 *Let c be a coloring of the edges of a complete bipartite graph G. If c uses at least three colors and has no polychromatic P_4, then for one of the two partite sets, each vertex is incident to edges of only one color.*

Proof. Since more than one color is used, some vertex x of G is incident to edges with at least two colors. If some color i is missing at x, then every edge in G with color i forms a polychromatic P_4 with some two edges incident to x. Thus x is incident to edges of all colors. If any vertex y in the partite set not containing x is incident to edges with two colors, then there is a polychromatic P_4 with yx as its central edge. Thus each vertex in the partite set not containing x is incident to edges of only one color, and $c(e)$ is determined by the endpoint of e in that partite set. $\square$

Lemma 5 *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are mixed hypergraphs such that $1 \notin S(\mathcal{H}_1) \cup S(\mathcal{H}_2)$, then the mixed hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}_1$ and $\mathcal{H}_2$ via Construction 3 has feasible set $S(\mathcal{H}_1) \cup S(\mathcal{H}_2) \cup \{2\}$.*

Proof. Since $1 \notin S(\mathcal{H}_1) \cup S(\mathcal{H}_2)$, both $\mathcal{H}_1$ and $\mathcal{H}_2$ have a $\mathcal{D}$-edge, and thus also $\mathcal{H}'$ has a $\mathcal{D}$-edge and is not 1-colorable. We obtain a strict 2-coloring of $\mathcal{H}'$ by letting all edges incident to x^* or y^* have color 1 and letting all other edges have color 2.

Hence we need only consider colorings of $\mathcal{H}'$ using at least three colors. By Lemma 4, each such proper coloring c is monochromatic at vertices of one partite set. By symmetry, we may assume that it is monochromatic at vertices of X_1. Now edges of the form C' such that $C \in \mathcal{C}_2$ and D' such that $D \in \mathcal{D}_2$ are all satisfied. We extract a coloring $\tilde{c}$ of X_1 by letting $\tilde{c}(x)$ be the color in c of the edges incident to x. Now $\tilde{c}$ is a proper coloring of $\mathcal{H}_1$ if and only if c is a proper coloring of $\mathcal{H}'$, and the number of colors used is the same. $\square$

Let S be a finite set of integers greater than 1. When S consists of m intervals of consecutive integers, we describe S by the numbers $(g_1, l_1, \ldots, g_m, l_m)$, where $g_i + 1$ is the number of integers skipped to reach the ith interval, and $l_i - 1$ is the number of integers in the ith interval. Thus $\{3, 5, 6, 9, 10\}$ is described by $(1, 2, 0, 3, 1, 3)$. The shift of one unit from gap measure to length measure simplifies the formula in Theorem 4; it can be motivated by the notion that there must be an integer in the gap in order to move to the next interval.

Theorem 4 *If S is a finite set of integers greater than 1 with interval description $(g_1, l_1, \ldots, g_m, l_m)$, then S is the feasible set of a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ such that*

$$|X| = g_1 + l_1 \left(l_1 + \cdots \left(g_{m-1} + l_{m-1} \left(l_{m-1} + (g_m + l_m)\right)\right)\right).$$

Proof. Let $n(S)$ be the claimed formula for the number of vertices in the mixed hypergraph realizing S. The construction is by induction on the number of intervals.

Let $Q_{g,l}$ be the join of K_g with the mixed hypergraph on l vertices having a single universal $\mathcal{D}$-edge. The feasible set of $Q_{g,l}$ has interval description (g, l), and $Q_{g,l}$ has $g + l$ vertices. This completes the case $m = 1$. Note that in this case $\max(S) = g + l$; the number of vertices is always at least the number of colors, so when $m = 1$ this construction is optimal.

For $m > 1$, let S' be the finite set with interval description $(g_2, l_2, \ldots, g_m, l_m)$. By the induction hypothesis, S' is the feasible set of a mixed hypergraph $\mathcal{H}'$ with $n(S')$ vertices. We now shift S' to make room for the first interval in S by taking the join of $\mathcal{H}'$ with K_{l_1}, invoking Lemma 2. We now have $l_1 + n(S')$ vertices.

By Lemma 5, applying Construction 3 to $\mathcal{H}'$ and Q_{0,l_1} adds $[2, \ldots, l_1]$ to the current feasible set and multiplies the number of vertices by l_1. To obtain a mixed hypergraph with feasible set S, we now take the join with K_{g_1}, which adds g_1 vertices. The resulting hypergraph realizes S and has $g_1 + l_1[l_1 + n(S')] = n(S)$ vertices. □

When S consists of m intervals of lengths at most l and the gaps have lengths bounded by a constant times l, the mixed hypergraph resulting from Theorem 4 has $O(l^m)$ vertices.

4 Special Families of Mixed Hypergraphs

In this section we consider two special types of mixed hypergraphs. The first family may have gaps in its feasible sets, despite its specialized structure, but the second family cannot have gaps.

A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a *bihypergraph* if $\mathcal{C} = \mathcal{D}$. It is r*-uniform* if each $\mathcal{C}$-edge and each $\mathcal{D}$-edge consists of r vertices.

Theorem 5 *For each integer $r \geq 3$, there exists an r-uniform bihypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ whose feasible set contains a gap.*

Proof. Let $A = \{a_1, \ldots, a_{r-1}\}$ and $B = \{b_1, \ldots, b_n\}$, where $n \geq (r-1)(r-2) + 2$. Let G be the complete bipartite graph with bipartition A, B. Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph in which $X = E(G)$ and both $\mathcal{C}$ and $\mathcal{D}$ consist of the edge sets of all copies of F in G, where F is the double-star with r edges obtained by adding a pendant edge to a leaf of $K_{1,r-1}$. Thus $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is an r-uniform bihypergraph. Strict colorings of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ correspond to edge-colorings of G such that each copy of F has two edges with the same color and two edges with distinct colors.

We show first that $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ has a strict $(r-1)$-coloring and a strict n-coloring. Assign color i to all edges incident to a_i in G. In this $(r-1)$-coloring, any two edges incident to the same vertex in A have the same color, while any two edges incident to the same vertex in B have distinct colors. A copy of F in G contains a vertex u from A and a vertex v from B as nonleaf vertices. Two edges incident to u have the same color, while two edges incident to v have distinct colors. A strict n-coloring can be defined similarly by assigning color j to all edges incident to b_j.

Next, let c be a strict m-coloring of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where $m \geq (r-1)(r-2) + 1$. We argue that $m = n$, thereby establishing a gap at each number from $(r-1)(r-2) + 1$ to $n - 1$. Since $m \geq (r-1)(r-2) + 1$, the pigeonhole principle yields a vertex a_i in A that is incident to edges of at least $r - 1$ colors. If some color α appears on no edge incident to a_i, then an edge $a_j b_k$ with color α expands to a polychromatic copy of F by including $a_i b_k$ and $r - 2$ additional edges at a_i with distinct colors.

Thus a_i is incident to edges of all m colors in c. Since $m \geq r$, this means that every edge of the form $b_j a_k$ must have the same color as $a_i b_j$; otherwise we could again add $r-2$ distinct colors at a_i to obtain a polychromatic F. Thus all edges incident to a vertex in B have the same color, which yields $m \leq n$. If $m < n$, then there exist distinct b_j and b_k in B such that all edges incident to either of b_j, b_k have the same color. In this case, we have a monochromatic copy of F consisting of $r-1$ edges incident to b_k and one edge incident to b_j. Hence $m = n$. □

Finally we consider a family of mixed hypergraphs in which the chromatic spectrum cannot have gaps. A mixed hypergraph is a *mixed interval hypergraph* if each element of $\mathcal{C} \cup \mathcal{D}$ is an interval in an underlying linear order on X. In a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ the subfamily $\mathcal{C}' \subseteq \mathcal{C}$ is a *sieve* if for all distinct $C, C' \in \mathcal{C}'$, every two elements of $C \cap C'$ form a $\mathcal{D}$-edge of size 2. The *sieve number* $s(\mathcal{H})$ of a mixed hypergraph $\mathcal{H}$ is the maximum cardinality of a sieve in $\mathcal{H}$.

Mixed interval hypergraphs and sieves were introduced and studied in [2], where it was shown in particular that when a mixed interval hypergraph has a proper coloring, its lower chromatic number is at most 2 (with equality only when $\mathcal{D} \neq \emptyset$), and its upper chromatic number is $n - s(\mathcal{H})$.

Theorem 6 *The chromatic spectrum of a mixed interval hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is gap-free.*

Proof. We use induction on $n = |X|$. For $n = 2, 3$ the statement is evident. Consider $n > 3$. Let $x_1, \ldots, x_n$ be an ordering of X such that each element of $\mathcal{C} \cup \mathcal{D}$ is an interval in this ordering.

We do not change a spectrum by restricting $\mathcal{C}$ and $\mathcal{D}$ to their elements which are minimal with respect to inclusion. Therefore we may assume that x_n appears in at most one $\mathcal{C}$-edge and in at most one $\mathcal{D}$-edge. Deleting the vertex x_n and the $\mathcal{C}$-edge and/or $\mathcal{D}$-edge containing it yields a mixed interval hypergraph $\mathcal{H}'$. By the induction hypothesis, $\mathcal{H}'$ has a gap-free chromatic spectrum.

Passing from $\mathcal{H}'$ to $\mathcal{H}$ increases the lower chromatic number from 1 to 2 if $\mathcal{H}'$ had no $\mathcal{D}$-edges but $\mathcal{H}$ has. It increases the upper chromatic number (by exactly one) if the $\mathcal{C}$-edge containing x_n does not increase the sieve number of $\mathcal{H}'$.

Therefore it is enough to show for $i \geq 2$ that if $\mathcal{H}'$ has a strict i-coloring, then also $r_i(\mathcal{H}) > 0$. Consider a strict i-coloring c of $\mathcal{H}'$; we obtain a strict i-coloring of $\mathcal{H}$. Let $k-1$ be the highest index such that $c(x_{k-1}) \neq c(x_{n-1})$ (this exists since $i \geq 2$). If x_n belongs to a $\mathcal{C}$-edge of size at least three or to no $\mathcal{C}$-edge, let $c(x_n) = c(x_{k-1})$; this extends c to a proper coloring of $\mathcal{H}$.

If x_n belongs to a $\mathcal{C}$-edge of size 2, then setting $c(x_n) = c(x_{n-1})$ yields a proper coloring unless x_n belongs to a $\mathcal{D}$-edge D contained in $\{x_k, \ldots, x_n\}$. Let j be the lowest index such that each consecutive pair in $\{x_j, \ldots, x_n\}$ forms a $\mathcal{C}$-edge. These pairs cannot exhaust D, since otherwise $\mathcal{H}$ is uncolorable. We obtain the proper coloring of $\mathcal{H}$ by giving color $c(x_{k-1})$ to all of $x_j, \ldots, x_n$. □

Finally, we present a combining operation that preserves the absence of gaps in the chromatic spectrum. A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is *connected* if it is possible to move from each vertex to every other via steps taken within single $\mathcal{C}$-edges or $\mathcal{D}$-edges (in other words, the underlying hypergraph $(X, \mathcal{C} \cup \mathcal{D})$ is connected).

A mixed hypergraph is *uniquely colorable* if it has only one feasible partition. Thus its feasible set has size 1 and the nonzero element of the spectrum equals 1. This does not reduce to the usual definition of *uniquely colorable* in graphs.

In a connected mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the nonempty set $X_0 \subset X$ is a *separator* [11, 15] if $X - X_0$ has a partition into sets X_1, X_2 such that no element of $\mathcal{C} \cup \mathcal{D}$ contains elements of both X_1 and X_2. The mixed subhypergraph $\mathcal{H}[X_0]$ *induced* by $X_0 \subseteq X$ is the mixed subhypergraph with vertex set X_0 whose $\mathcal{C}$-edges and $\mathcal{D}$-edges are those of $\mathcal{H}$ contained in X_0. Given a separator X_0 and resulting partition X_1, X_2 of $X - X_0$, the induced mixed subhypergraphs $\mathcal{H}_1 = \mathcal{H}[X_1 \cup X_0]$ and $\mathcal{H}_2 = \mathcal{H}[X_2 \cup X_0]$ are the mixed hypergraphs *derived* from the separator.

We need the following separator theorem proven in [11].

Theorem 7 *Let X_0 be a separator inducing a uniquely colorable mixed hypergraph $\mathcal{H}_0 = \mathcal{H}[X_0]$ in $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. For the derived mixed subhypergraphs $\mathcal{H}_1 = \mathcal{H}[X_1 \cup X_0]$ and $H_2 = \mathcal{H}[X_2 \cup X_0]$, the following equalities hold:*

1) $\chi(\mathcal{H}) = \max\{\chi(\mathcal{H}_1), \chi(\mathcal{H}_2)\}$;

2) $\bar{\chi}(\mathcal{H}) = \bar{\chi}(\mathcal{H}_1) + \bar{\chi}(\mathcal{H}_2) - \bar{\chi}(\mathcal{H}_0)$.

Next we show that identification at a uniquely colorable mixed subhypergraph preserves the absence of gaps in the chromatic spectrum.

Theorem 8 *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ be a mixed hypergraph with a uniquely colorable separator $\mathcal{H}_0$, resulting in the derived subhypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$. If the feasible sets of $\mathcal{H}_1$ and $\mathcal{H}_2$ have no gaps, then the feasible set of $\mathcal{H}$ also has no gaps.*

Proof. Choose i such that $\chi(\mathcal{H}) \leq i \leq \bar{\chi}(\mathcal{H})$. By Theorem 7, there exist i_0, i_1, i_2 such that $\chi(\mathcal{H}_1) \leq i_1 \leq \bar{\chi}(\mathcal{H}_1)$, $\chi(\mathcal{H}_2) \leq i_2 \leq \bar{\chi}(\mathcal{H}_2)$, $\chi(\mathcal{H}_0) \leq i_0$, and $i = i_1 + i_2 - i_0$. Since $S(\mathcal{H}_1)$ and $S(\mathcal{H}_2)$ have no gaps, we can take strict colorings of $\mathcal{H}_1$ with colors $1, \ldots, i_1$ and $\mathcal{H}_2$ with colors $1, 2, \ldots, i_0, i_1 + 1, i_1 + 2, \ldots, i_1 + i_2 - i_0$.

Since both $\mathcal{H}_1$ and $\mathcal{H}_2$ contain $\mathcal{H}_0$, and $\mathcal{H}_0$ is uniquely colorable, we can permute the names in the coloring of $\mathcal{H}_2$ so that this coloring agrees with the coloring of $\mathcal{H}_1$ on X_0. Permuting the colors in this way yields a proper coloring of $\mathcal{H}$ with i colors. □

The *neighborhood* of a vertex v in a mixed hypergraph is the set of vertices other than v belonging to edges containing v. A *simplicial vertex* in $\mathcal{H}$ is a vertex whose neighborhood induces a uniquely colorable mixed hypergraph. A *simplicial elimination ordering* of $\mathcal{H}$ is an ordering $x_1, \ldots, x_n$ of X such that each x_i is a simplicial vertex in the subhypergraph induced by $x_i, \ldots, x_n$. A mixed hypergraph is *pseudo-chordal* (see [15]) if it has a simplicial elimination ordering.

Corollary 2 *Every pseudo-chordal mixed hypergraph has a gap-free chromatic spectrum.*

Proof. Either the pseudo-chordal mixed hypergraph is itself uniquely colorable, or the neighborhood of a simplicial vertex is a separator inducing a uniquely colorable mixed hypergraph, and induction applies. □

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