# Hypergraph Ramsey Numbers: Triangles versus Cliques

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**Abstract**

A celebrated result in Ramsey Theory states that the order of magnitude of the triangle-complete graph Ramsey numbers $R(3,t)$ is $t^2/\log t$. In this paper, we consider an analogue of this problem for uniform hypergraphs. A *triangle* is a hypergraph consisting of edges $e, f, g$ such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. For all $r \geq 2$, let $R(C_3, K_t^r)$ be the smallest positive integer $n$ such that in every red-blue coloring of the edges of the complete $r$-uniform hypergraph $K_n^r$, there exists a red triangle or a blue $K_t^r$. We show that there exist constants $a, b_r > 0$ such that for all $t \geq 3$,

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leq R(C_3, K_t^3) \leq b_3 t^{\frac{3}{2}}$$

and for $r \geq 4$

$$\frac{t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}+o(1)}} \leq R(C_3, K_t^r) \leq b_r t^{\frac{3}{2}}.$$

This determines up to a logarithmic factor the order of magnitude of $R(C_3, K_t^r)$. We conjecture that $R(C_3, K_t^r) = o(t^{3/2})$ for all $r \geq 3$. We also study a generalization to hypergraphs of cycle-complete graph Ramsey numbers $R(C_k, K_t)$ and a connection to $r_3(N)$, the maximum size of a set of integers in $\{1, 2, \ldots, N\}$ not containing a three-term arithmetic progression.

## 1 Introduction

A *hypergraph* is a pair $(V, E)$ where $V$ is a set whose elements are called *vertices* and $E$ is a family of subsets of $V$ called *edges*. If all edges have size $r$, then the hypergraph is referred to as an *$r$-graph*. Throughout this paper, $C_k$ denotes a *loose $k$-cycle*, namely the hypergraph with edges $e_1, \ldots, e_k$ such that $|e_i \cap e_{i+1}| = 1$ for $i = 1, \ldots, k-1$, $|e_1 \cap e_k| = 1$, and $e_i \cap e_j = \emptyset$ otherwise. In particular, a *loose triangle* is a hypergraph consisting of three edges $e, f, g$ such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. Since we consider only loose cycles and triangles, we will omit the word "loose". A hypergraph is *linear* if any pair of distinct edges of the hypergraph intersect in at most one vertex.

An *independent set* in a hypergraph is a set of vertices containing no edges of the hypergraph. Let $K_t^r$ denote the $t$-vertex *complete $r$-graph*, i.e., the $t$-vertex $r$-graph whose edges are all $r$-element subsets of the vertex set. In this paper we consider the cycle versus complete hypergraph Ramsey

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numbers $R(C_k, K_t^r)$ – this is the minimum $n$ such that every $n$-vertex $r$-graph contains either a cycle $C_k$ or an independent set of $t$ vertices. Our main effort will be on the triangle-complete hypergraph Ramsey number $R(C_3, K_t^r)$. A celebrated result of Kim [15] together with earlier bounds by Ajtai, Komlós and Szemerédi [2] shows that

$$R(C_3, K_t) = \Theta\Big(\frac{t^2}{\log t}\Big) \quad \text{as } t \to \infty.$$

This establishes the order of magnitude of these Ramsey numbers for graphs.

### 1.1 Triangle-free hypergraphs

The study of the independence number in triangle-free hypergraphs was initiated by Ajtai, Komlós, Pintz, Spencer and Szemerédi [1] and used to give a counterexample to a conjecture of Erdős on the Heilbronn problem [21] on the largest area of a triangle with vertices from $n$ points in the unit square. Motivated also by the triangle-complete graph Ramsey numbers, in this paper we determine for $r \geq 3$ the order of magnitude of the triangle-complete Ramsey numbers for $r$-graphs up to logarithmic factors:

**Theorem 1.1.** *There exist constants $a, b_3 > 0$ such that for all $t \geq 1$,*

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \leq R(C_3, K_t^3) \leq b_3 t^{\frac{3}{2}}.$$

*For each $r > 3$, there exist constants $a_r, b_r > 0$ such that for all $t \geq 1$,*

$$\frac{t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4} + \frac{a_r}{\sqrt{\log\log t}}}} \leq R(C_3, K_t^r) \leq b_r t^{\frac{3}{2}}.$$

We shall see that $b_r \leq (2r)^{9/2}$ for all $r \geq 3$. The upper bound in Theorem 1.1 is proved in Section 3. The lower bound in Theorem 1.1 comes from a construction that combines randomness and linear algebra and a construction of triangle-free hypergraphs coming from sets with no three-term arithmetic progressions, presented in Section 5. The preliminaries required to analyze this construction are presented in Section 4. Some of the ideas of the construction were recently used in [16] to study a related problem. In light of Theorem 1.1, we make the following conjecture:

**Conjecture 1.1.** *For all fixed $r \geq 3$,*

$$R(C_3, K_t^r) = o(t^{3/2}) \quad \text{as } t \to \infty.$$

We shall see in Section 2 that if $H$ is a triangle-free hypergraph (the edges may have arbitrary size) on $n$ vertices, then $H$ contains an independent set of size at least $\lfloor \sqrt{n} \rfloor$. By Theorem 1.1, this is not tight for $r$-uniform hypergraphs for each fixed $r \geq 3$. It would be interesting to see if it is tight when edges whose size depends on $n$ are allowed.

### 1.2 Linear triangle-free hypergraphs

We indicate a connection between independent sets in *linear* triangle-free hypergraphs and Roth's Theorem [21] on arithmetic progressions. Let $r_3(N)$ denote the largest size of a set of integers in $\{1, 2, \ldots, N\}$ containing no three-term arithmetic progressions. This problem has attracted much attention, starting with the original theorem of Roth [21] showing that $r_3(N) = o(N)$. The best current known bounds are as follows: for some constant $c > 0$,

$$\frac{N}{e^{c\sqrt{\log N}}} \leq r_3(N) \leq \frac{N}{(\log N)^{1-o(1)}}.$$

The lower bound, which comes from a construction of Behrend [5], is essentially unchanged for more than sixty years. The upper bound, due to Sanders [24] improves many earlier results which gave smaller powers of $\log N$ in the denominator. Let $RL(C_3, K_t^3)$ denote the minimum $n$ such that every linear triangle-free 3-graph on at least $n$ vertices contains an independent set of size $t$. We prove the following theorem:

**Theorem 1.2.** *There are constants $\tilde{a}, \tilde{b} > 0$ such that for all $t \geq 1$*

$$\frac{t^{\frac{3}{2}}}{e^{\tilde{a}\sqrt{\log t}}} \leq RL(C_3, K_t^3) \leq \frac{\tilde{b}t^{\frac{3}{2}}}{\sqrt{\log t}}.$$

*Furthermore, if for some $c > 0$, $RL(C_3, K_t^3) = O(t^{3/2}(\log t)^{-3/4-c})$, then*

$$r_3(N) = O\Big(\frac{N}{(\log N)^{\frac{4c}{3}}}\Big).$$

It would be interesting if one could prove that $r_3(N) = o(N)$ using Theorem 1.2 above. The bound $RL(C_3, K_t^3) = O(t^{3/2}/\sqrt{\log t})$ may also be evidence for Conjecture 1.1, that $R(C_3, K_t^3) = o(t^{3/2})$.

### 1.3 $k$-Cycle-free hypergraphs

The construction used in Theorem 1.1 extends more generally to give lower bounds on all *cycle-complete hypergraph Ramsey numbers*. The cycle $C_3$ is precisely a hypergraph triangle. We give for all $k, r \geq 3$ a construction of $C_k$-free $r$-graphs with low independence number, based on known results on the $C_k$-free bipartite Ramanujan graphs of Lubotzky, Phillips and Sarnak [18]. Specifically, we prove the following theorem by a suitable and fairly straightforward modification of the construction. We write $f = O^*(g)$ to denote that for some constant $c > 0$, $f(t) = O((\log t)^c g(t))$, and $f = \Omega^*(g)$ is equivalent to $g = O^*(f)$.

**Theorem 1.3.** *For fixed $r, k \geq 3$,*

$$R(C_k, K_t^r) = \Omega^*\left(t^{1+\frac{1}{3k-1}}\right) \quad \text{as } t \to \infty.$$

The key point of this theorem is that the exponent $1 + 1/(3k-1)$ of $t$ is bounded away from 1 by a constant independent of $r$, and strictly improves for all $r, k \geq 5$ the lower bounds given by considering appropriate random hypergraphs, namely

$$R(C_k, K_t^r) = \Omega^*\left(t^{1+\frac{1}{kr-r-k}}\right) \quad \text{as } t \to \infty.$$

In the case $r = 2$, namely for graphs, the best available constructions for lower bounds on $r(C_k, K_t^r)$ indeed come from appropriate random graphs; in particular the $C_k$-free random graph process studied by Bohman and Keevash [8].

By using the known constructions of extremal bipartite graphs of girth 12, arising from generalized hexagons, we obtain the following improvement of the lower bound in Theorem 1.3 for $C_5$ i.e. for *loose pentagons*:

**Theorem 1.4.** *For fixed $r \geq 3$, there exists a constant $c_r > 0$ such that*

$$R(C_5, K_t^r) \geq c_r \Big(\frac{t}{\log t}\Big)^{\frac{5}{4}} \qquad \text{as } t \to \infty.$$

The main part of this theorem is the exponent 5/4; we suspect that this exponent may be tight as $t \to \infty$, and perhaps even more generally, that $r(C_k, K_t^r) = \Theta^*(t^{k/(k-1)})$ for all $r, k \geq 3$. Our second conjecture is as follows:

**Conjecture 1.5.** *For all $r \geq 3$,*

$$R(C_5, K_t^r) = O(t^{5/4}) \qquad \text{as } t \to \infty.$$

For graphs, the best current bounds are $a_2 t^{\frac{4}{3}} / \log t \leq R(C_5, K_t) \leq b_2 t^{3/2}/\sqrt{\log t}$. for some constants $a_2 > 0$ and $b_2 > 0$, where the upper bound is due to Caro, Li, Rousseau and Zhang [10] and the lower bound is from Bohman and Keevash [8].

## 2 Non-uniform hypergraphs

The goal of this section is to give a simple proof that any triangle-free hypergraph on $n$ vertices has an independent set of size at least $\lfloor \sqrt{n} \rfloor$. Recall that the *chromatic number* $\chi(H)$ of a hypergraph $H$ is the minimum $k$ such that there is an assignment of $k$ colors to the vertices such that no subset of vertices of the same color forms an edge of $H$.

**Theorem 2.1.** *Let $H$ be any hypergraph on $n$ vertices not containing a triangle and in which $|e| \geq 2$ for all $e \in H$. Then*

$$\alpha(H) \geq \lfloor \sqrt{n} \rfloor.$$

*Proof.* Suppose for a contradiction that $\alpha(H) < \lfloor \sqrt{n} \rfloor$. Then $\chi(H) > k := \lfloor \sqrt{n} \rfloor$. So, $H$ contains a $(k+1)$-vertex-critical subgraph $H'$, which means that $\chi(H') = k + 1$ but $\chi(H' - v) \leq k$ for every $v \in V(H')$. By Corollary 3 on Page 431 of [7] (see also [27] and [17]), the *strong degree* of each vertex in $H'$ is at least $k$, i.e. for each $v \in V(H')$ there are $k$ edges $e_1, e_2, \ldots, e_k$ such that $e_i \cap e_j = \{v\}$ for all $1 \leq i < j \leq k$. In words, the $e_i$s share $v$ and nothing else. Choose a vertex $v_i$ in each $e_i \backslash \{v\}$. Since $H'$ has no triangles, the set $\{v_1, \ldots, v_k\}$ is an independent set of $H$ of size $k \geq \lfloor \sqrt{n} \rfloor$, which is a contradiction. □

This result is almost tight since $R(C_3, t) = \Theta(t^2/(\log t))$, so there are $n$-vertex triangle-free graphs with independence number of order $\sqrt{n \log n}$. It would be interesting to see if for hypergraphs (not necessarily uniform) where every edge has size at least three, the above lower bound on the independence number is tight.

## 3 Proof of Theorem 1.1 : Upper Bound

The aim of this section is to prove the upper bound of Theorem 1.1. For $r \geq 3$, let $\triangle_r$ denote the family of all triangle-free hypergraphs each of whose edges has size at least three and size at most $r$. The upper bound on $R(C_3, K_t^r)$ in Theorem 1.1 will be derived as a direct consequence of the following more general statement about hypergraphs in $\triangle_r$:

**Theorem 3.1.** *For every $r \geq 3$ and $G \in \triangle_r$, $\alpha(G) \geq |V(G)|^{2/3}/(8r^3)$.*

This section is devoted to the proof of Theorem 3.1, which gives the constant $b_r = (8r^3)^{3/2} = (2r)^{9/2}$ in the upper bound in Theorem 1.1.

### 3.1 Expandable sets

In this section we state and prove a sequence of preliminary results needed for the proof of Theorem 3.1.

A set $S$ of vertices of $G \in \triangle_r$ is called *expandable* if, for every $T \subseteq V(G) - S$ with $|T| \leq 2r$, there is an edge of $G$ containing $S$ and disjoint from $T$, otherwise $S$ is *non-expandable*. For example, if $S$ is an edge of $G$, then it is expandable, and every set $S \subset V(G)$ of size more than $r$ is non-expandable.

Let $G$ be an $n$-vertex graph in $\triangle_r$ with the smallest $\sum_{e\in E(G)} |e|$ for which Theorem 3.1 fails. Certainly $G$ has at least one edge.

**Lemma 3.2.** *No three expandable sets in $G$ form a triangle.*

*Proof.* If sets $S_1, S_2, S_3$ form a triangle, then by the definition of expandable sets, there is an edge $e_1 \supseteq S_1$ disjoint from $(S_2 \cup S_3) \setminus S_1$, there is an edge $e_2 \supseteq S_2$ disjoint from $(e_1 \cup S_3) \setminus S_2$, and there is an edge $e_3 \supseteq S_3$ disjoint from $(e_1 \cup e_2) \setminus S_3$. Now $e_1, e_2, e_3$ form a triangle in $G$, contradicting $G \in \triangle_r$. □

**Lemma 3.3.** *Let $S \subset V(G)$ be an expandable set and $|S| \geq 3$. Then no edge of $G$ of size more than $|S|$ contains $S$.*

*Proof.* Suppose an expandable set $S$ with $|S| \geq 3$ is contained in $e \in E(G)$ with $|e| \geq |S| + 1$. Let $V(G') = V(G)$ and $E(G') = E(G) - e + S$. By Lemma 3.2, $G' \in \triangle_r$. Since $\sum_{e\in E(G')} |e| < \sum_{e\in E(G)} |e|$, by the minimality of $G$, $\alpha(G') \geq |V(G')|^{2/3}/8r^3 = n^{2/3}/8r^3$. But every independent set in $G'$ is also independent in $G$, and so $\alpha(G) \geq \alpha(G') \geq n^{2/3}/8r^3$, a contradiction to the choice of $G$. □

**Lemma 3.4.** *For every $3 \leq i < j \leq r$ no $i$-element subset of $V(G)$ is contained in more than $(2r)^{j-i}$ edges of size $j$.*

*Proof.* We use induction on $j - i$. If a $(j-1)$-element $S \subset V(G)$ is contained in $2r + 1$ edges of size $j$ in $G$, then $S$ is expandable, a contradiction to Lemma 3.3. Suppose now that $3 \leq i \leq j - 2$ and an $i$-element $S \subset V(G)$ is contained in $m \geq (2r)^{j-i} + 1$ edges $e_1, e_2, \dots, e_m \in E(G)$ of size $j$. By Lemma 3.3, $S$ is not expandable. This means that for some set $T$ of $2r$ vertices of $V(G)\setminus S$, we have $(e_i \setminus S) \cap T \neq \emptyset$ for every $1 \leq i \leq m$. In other words, $T$ intersects the part outside $S$ of every $e_i$.

By the pigeonhole principle, there is an $x \in T$ such that the set $S \cup \{x\}$ is contained in at least $(2r)^{j-i-1} + 1$ edges among $e_1, e_2, \dots, e_m$, a contradiction. □

**Corollary 3.5.** *For every $3 \le j \le k$ each 2-element non-expandable subset of $V(G)$ is contained in at most $(2k)^{j-2}$ edges of size $j$.*

*Proof.* Suppose that $S = \{x, y\}$ is a non-expandable pair of vertices in $G$ is contained in $m \ge (2k)^{j-2} + 1$ edges $e_1, \dots, e_m$ of size $j$. Then some $2k$ vertices $x_1, \dots, x_{2k}$ outside $S$ intersect all edges of $G$ containing $S$, and in particular, all edges $e_1, \dots, e_m$. Then by the pigeonhole principle, for some $1 \le t \le 2k$, the 3-element set $S + x_t$ is contained in at least $(2k)^{j-3} + 1$ edges among $e_1, \dots, e_m$, a contradiction to Lemma 3.4. $\square$

## 3.2 Proof of Theorem 3.1

In this section we complete the proof of Theorem 3.1. For $3 \le i \le r$, let $G_i$ be the subgraph of $G$ consisting of all edges of size $i$, that is, $E(G_i) = \{e \in E(G) \ : \ |e| = i\}$. For convenience, denote $n = |V(G)|$.

**Lemma 3.6.** *For every $3 < j \le r$, $|E(G_j)| \le (2r)^{j-2}\binom{n}{2}$.*

*Proof.* Let $e \in E(G_j)$ and $x, y, z \in e$. By Lemma 3.2, at least one of the pairs $\{x, y\}, \{x, z\}$ and $\{y, z\}$ is non-expandable and thus, by Corollary 3.5, is contained in at most $(2r)^{j-2}$ edges of $G_j$. Since every $e \in E(G_j)$ contains such a pair, the lemma follows. $\square$

**Lemma 3.7.** $|E(G_3)| \ge n^{5/3}/4r^2$.

*Proof.* Suppose that $|E(G_3)| < n^{5/3}/4r^2$. Let $p = n^{-1/3}/4r^2$ and let $W$ be a random subset of $V(G)$ where each $v \in V(G)$ is in $W$ with probability $p$ independently of all other vertices. By Lemma 3.6, for $j \ge 4$, the expected number of edges of size $j$ in $G[W]$ is at most

$$|E(G_j)|p^j \le (2r)^{j-2}\binom{n}{2}(4r^2)^{-j}n^{-j/3} \le (2r)^{-j}n^{2/3}.$$

By assumption, the expected number of edges of size 3 in $G[W]$ is at most

$$n^{5/3}p^3/4r^2 = (2r)^{-8}n^{2/3} \le (2r)^{-5}n^{2/3}.$$

So, the expectation of $|W| - |E(G[W])|$ is at least

$$pn - \sum_{j=4}^{r}(2r)^{-j}n^{2/3} - (2r)^{-5}n^{2/3} \ge pn - 2(2r)^{-4}n^{2/3} = \left(1 - \frac{1}{2r^2}\right)pn.$$

Thus there is a particular subset $U$ of $V(G)$ with $|U| - |E(G[U])| \ge 0.9pn$. Then deleting a vertex from each edge in $G[U]$ we obtain an independent subset $U'$ of $U$ with $|U'| \ge 0.9pn$, so $\alpha(G) \ge n^{2/3}/5r^2 > n^{2/3}/8r^3$, a contradiction to the choice of $G$. $\square$

The key part of the proof will be to produce an independent set in $H = G_3$ of size at least $n^{2/3}/8r^3$ that is *also an independent set in* $G$, using the preceding lemmas. By Lemma 3.7, $|E(H)| \ge (2r)^{-2}n^{5/3}$. Let $d = 3|E(H)|/n$ be the average degree of $H$, so $d \ge 3n^{2/3}/4r^2$. An edge $e \in H$ is called *$k$-light* if exactly $k$ pairs of vertices of $e$ have codegree in $H$ at most $r$. An edge is *heavy* if it is 0-light. We see quickly that $H$ has no heavy edges: for a heavy edge $\{x, y, z\} \in H$, since $r + 1 \ge 4$, we can greedily choose distinct vertices $a, b, c \notin \{x, y, z\}$ such that edges $\{a, x, y\}, \{b, y, z\}, \{c, x, z\}$ form a triangle, since each of the pairs $(x, y), (y, z), (z, x)$ has codegree at least $r + 1 \geq 4$. We now consider two cases.

**Case 1.** *The number of edges in $H$ that are 2-light or 3-light is at least $2|E(H)|/3$.*

For each vertex $v$, let $d'(v)$ be the number of edges $e$ of $H$ containing $v$ such that $e$ is either 2-light or 3-light and $v$ is incident to two light pairs of $e$. Then $\sum_v d'(v)$ counts each such $e$ one or three times so $\sum_v d'(v) \geq 2|E(H)|/3$. Therefore some vertex $v$ of $H$ is in at least $2|E(H)|/3n = 2d/9$ edges, where two pairs of codegree (in $H$) at most $r$ in each edge contain $v$. Let $e_1, e_2, \dots, e_m$ be such a set of edges on $v$ with $m \geq 2d/9$. Then the link graph $L(v)$ consisting of pairs $e_i \backslash \{v\}$ has maximum degree at most $r$. It follows by Vizing's Theorem that $L(v)$ has a matching of size $\ell \geq m/(r+1)$. This means that we have found edges, say $e_1, e_2, \dots, e_\ell$ sharing no vertices other than $v$, and such that in each $e_i$ the two pairs containing $v$ have codegree at most $r$. Now pick $x_1, x_2, \dots, x_\ell$ where $x_i \in e_i \backslash \{v\}$ for $1 \leq i \leq \ell$. We claim that this is an independent set in the entire hypergraph $G$. If not, then say $e = \{x_1, \dots, x_j\} \in E(G)$. Then $\{e, e_1, e_2\}$ is a triangle in $G$, since $e_1$ and $e_2$ share only $v$, $e$ and $e_1$ share only $x_1$, and $e$ and $e_2$ share only $x_2$ – see Figure 1. This independent set has size $\ell \geq m/(r+1) \geq 2d/9(r+1) \geq n^{2/3}/8r^3$. This completes the proof in Case 1.

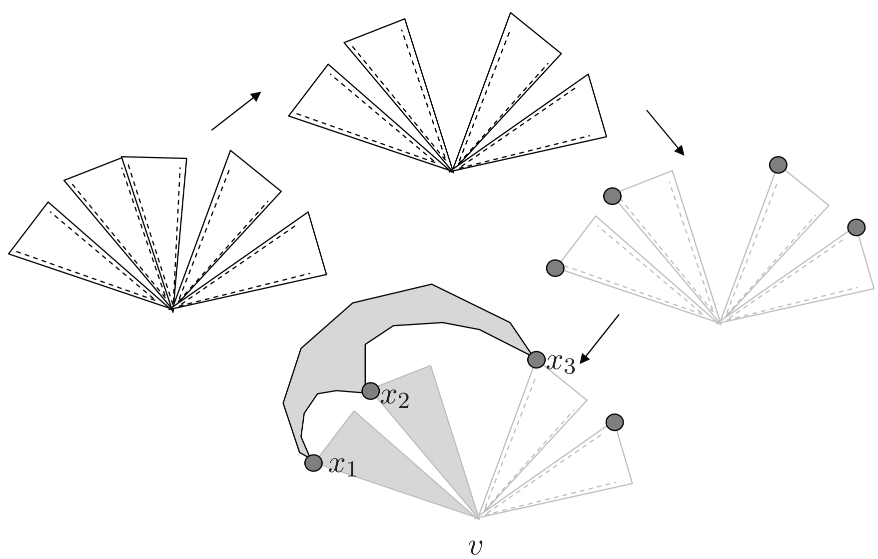


Figure 1 : Finding an independent set in Case 1.

**Case 2.** *The number of 1-light edges in $H$ is at least $|E(H)|/3$.*

For each vertex $v$, let $d''(v)$ be the number of edges $e$ of $H$ containing $v$ such that $e$ is 1-light and $v$ is incident to the light pair of $e$. Then $\sum_v d''(v)$ counts each such $e$ exactly twice so $\sum_v d''(v) \geq 2|E(H)|/3$. By averaging, some $v$ in $H$ lies in at least $2|E(H)|/3n = 2d/9$ 1-light edges such that the pair of codegree in $H$ at most $r$ in each edge contains $v$. Then there are at least $2d/9r$ distinct vertices

$x_1, x_2, \ldots, x_m$ such that the codegree of $(v, x_i)$ is at most $r$, and there is a 1-light edge $e_i \supset \{v, x_i\}$ for all $i \in \{1, 2, \ldots, m\}$. Since each $e_i$ is 1-light, exactly two pairs of vertices in $e_i$ have codegree at least $1 + r$, and in particular, all $e_i$s are distinct. We claim that $\{x_1, x_2, \ldots, x_m\}$ is again an independent set in $G$. Suppose not, and that $\{x_1, \ldots, x_j\}$ is an edge. Let $e_i \backslash \{x_i, v\} = \{y_i\}$. Note that every $y_i$ is disjoint from $\{x_1, \ldots, x_j\}$, otherwise if say $y_i = x_j$, then $\{v, x_i\}$ and $\{v, x_j\}$ both have codegree less than $1 + r$, but they lie in the edge $e_i$, which has only one pair of codegree less than $1 + r$ – a contradiction. So every $y_i$ is disjoint from $\{x_1, \ldots, x_j\}$. Now we claim that $y_1 = y_2 = \ldots = y_j$. If say $y_1 \neq y_2$ (left drawing in Figure 2 below), consider the triples $\{v, x_1, y_1\}, \{v, x_2, y_2\}$ and the edge $\{x_1, x_2, \ldots, x_j\}$. Since $y_1, y_2, x_1, \ldots, x_j$ are all distinct, this is a triangle. So $y_1 = y_2 = \ldots = y_j = y$. Now consider the pairs $\{y, x_1\}, \{y, x_2\}$ (shown in black bold lines in the right drawing in Figure 2 below). Since $\{y, x_1\}$ and $\{y, x_2\}$ are pairs in $e_1$ and $e_2$, respectively, and they do not contain $v$, by the choice of $e_1$ and $e_2$, those pairs have codegree at least $1 + r$. So we can pick $z_1 \neq z_2$ with $z_1, z_2 \notin \{x_1, \ldots, x_j, y, v\}$ such that $\{x_1, y, z_1\}, \{x_2, y, z_2\}, \{x_1, x_2, \ldots, x_j\}$ is a triangle – namely $z_1, z_2, x_1, \ldots, x_j$ are all distinct. This shows that $\{x_1, x_2, \ldots, x_m\}$ is independent, and it has size at least $2d/9r \geq n^{2/3}/6r^3$.

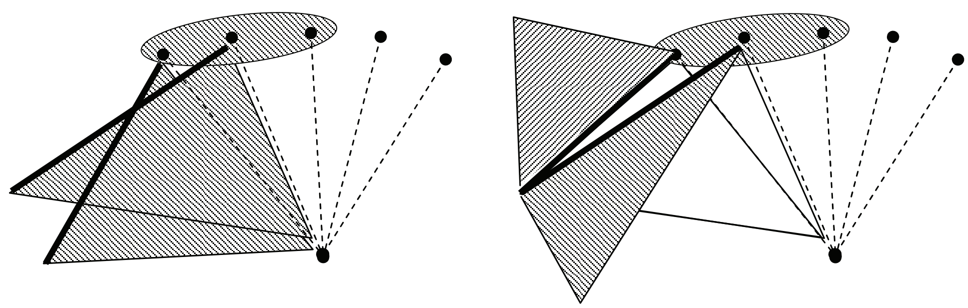

Figure 2 : Finding an independent set in Case 2.

## 4 Generalized quadrangles and a spectral lemma

Generalized quadrangles were first constructed by Tits [26] and described as graphs by Benson [6]. Let $G_q$ denote a *generalized quadrangle of order* $q$, which is a $(q+1)$-regular $(q+1)$-uniform $C_2, C_3$-

free hypergraph on $q^3 + q^2 + q + 1$ vertices. Generalized quadrangles of order $q$ exist whenever $q$ is a prime power.

### 4.1 A general spectral lemma

In this section, we employ a lemma which relates the distribution of edges in a bipartite graph to spectral properties of its adjacency matrix. This lemma is an analog of a well-known spectral lemma in graph theory (see for example [3]) which is frequently referred to as the expander mixing lemma, and is used especially in the context of $(n, d, \lambda)$-graphs and pseudorandom graphs. The lemma we give, which may be referred to as the expander mixing lemma for bipartite graphs, appears in a different form in [13] and in [14]. For completeness, we give the proof here and it is very similar to the proof for non-bipartite graphs in [3].

**Lemma 4.1.** *Let $G(U, V)$ be a $d$-regular bipartite graph with adjacency matrix $A$ and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of $A$. Let $\lambda = \max\{|\lambda_i| : i \notin \{1, N\}\}$. Then for any sets $X \subseteq U$ and $Y \subseteq V$, the number $e(X, Y)$ of edges from $X$ to $Y$ satisfies*

$$\left| e(X,Y) - \frac{d}{|V|}|X||Y| \right| \leq \lambda\sqrt{|X||Y|}.$$

*Proof.* Let $\chi_X$ and $\chi_Y$ denote the characteristic vectors of $X$ and $Y$. Let $x_1, x_2, \ldots, x_N$ be an orthonormal basis of eigenvectors of $A$, where $x_i$ is the eigenvector corresponding to $\lambda_i$, and write

$$\chi_X = \sum_{i=1}^{N} s_i x_i \quad \chi_Y = \sum_{i=1}^{N} t_i x_i.$$

Then

$$e(X,Y) = \langle A\chi_X, \chi_Y \rangle = \lambda_1 s_1 t_1 + \lambda_N s_N t_N + \sum_{i=2}^{N-1} \lambda_i s_i t_i.$$

The values of $s_1, t_1, s_N$ and $t_N$ are recovered quickly from the knowledge of the first and last eigenvectors, $x_1$ and $x_N$, recalling $x_1$ is the constant unit vector and $x_N$ is the unit vector which is constant on $V(G_q)$ and minus that constant on $E(G_q)$. Noting that $\|\chi_X\|^2 = |X|$ and $\|\chi_Y\|^2 = |Y|$, and using $\lambda_1 = d = -\lambda_N$, it is straightforward to see

$$e(X,Y) = \frac{d}{|V|}|X||Y| + \sum_{i=2}^{N-1} \lambda_i s_i t_i.$$

Finally, by Cauchy-Schwarz,

$$\sum_{i=2}^{N-1} \lambda_i s_i t_i \leq \lambda(A) \left( \sum_{i=1}^{N} s_i^2 \right)^{1/2} \left( \sum_{i=1}^{N} t_i^2 \right)^{1/2}$$

and the sums are $\|\chi_X\| = \sqrt{|X|}$ and $\|\chi_Y\| = \sqrt{|Y|}$ respectively. □

This lemma will be used in the context of hypergraphs (in particular for the hypergraph $H = G_q$) in the following way: if $H$ is a hypergraph, then the *bipartite incidence graph* of $H$ is the bipartite graph $B(H)$ whose parts are $V(H)$ and $E(H)$, and $\{v, e\} \in E(B(H))$ if and only if $v \in e$. We denote by $A(H)$ the adjacency matrix of the bipartite incidence graph $B(H)$, and when $|V(H)| = |E(H)|$ we denote by $\lambda(H)$ the largest absolute value of the eigenvalues of $A(H)$ other than $\lambda_1$ and $\lambda_N$. Lemma 4.1 is applied to $B(H)$ to give the following hypergraph formulation:

**Lemma 4.2.** *Let $H$ be a $d$-uniform $d$-regular hypergraph and let $X \subseteq V(H)$ and $Y \subseteq E(H)$. Then*

$$\Big|\sum_{e\in Y} |X \cap e| - \frac{d}{|V|}|X||Y|\Big| \ \leq \ \lambda(H)\sqrt{|X||Y|}.$$

*In particular, if $\lambda(H) \leq \delta\sqrt{d}$ and $|X| \geq 2\tau n/d$, then the number of edges $e \in E(H)$ such that $|X \cap e| \geq \tau$ is at least $n - 2\delta^2 n/\tau$.*

*Proof.* For the first inequality, if $H$ is a $d$-uniform $d$-regular hypergraph, then $B(H)$ is $d$-regular. Applying Lemma 4.1 gives

$$\Big|e(X,Y) - \frac{d}{|V|}|X||Y|\Big| \ \leq \ \lambda(H)\sqrt{|X||Y|}.$$

We note that

$$e(X,Y) = \sum_{e\in Y} |X \cap e|.$$

This gives the first inequality of Lemma 4.2. Applying this inequality with $\lambda(H) \leq \delta\sqrt{d}$, we obtain for any $Z \subseteq E(H)$,

$$\Big|\sum_{e\in Z} |X \cap e| - \frac{d}{n}|X||Z|\Big| \leq \delta\sqrt{d|X||Z|}.$$

Now let $Y = \{e \in E(H) : |X \cap e| \geq \tau\}$ and $Z = E(H)\backslash Y$. Suppose for a contradiction that $|Z| > 2\delta^2 n/\tau$. By definition of $Z$,

$$\sum_{e\in Z} |X \cap e| < \tau|Z|.$$

By the preceding inequality,

$$\tau|Z| > \sum_{e\in Z} |X \cap e| > \frac{d}{n}|X||Z| - \delta\sqrt{d|X||Z|}.$$

Since $|X| \geq 2\tau n/d$, we get

$$\tau|Z| < \delta\sqrt{2\tau n|Z|}.$$

This contradicts $|Z| > 2\delta^2 n/\tau$. □

We remark that for fixed $|X|$, $d|X|/|V|$ is exactly the expected value of $|X \cap e|$ when $X$ is a random set whose elements are chosen from $V(H)$ independently with probability $|X|/|V|$.

### 4.2 Spectral properties of $A(G_q)$

In order to apply Lemma 4.2 to $G_q$, we determine $\lambda(G_q)$. Since $G_q$ is $(q+1)$-uniform and $(q+1)$-regular, the bipartite incidence graph $B(G_q)$ is $(q+1)$-regular. Since $B(G_q)$ is connected, this implies $q+1$ and $-(q+1)$ are eigenvalues of $A = A(G_q)$ with multiplicity 1. By the definition of a generalized quadrangle, for every vertices $x$ and $y$ in distinct partite sets of $B(G_q)$, there exists exactly one $x,y$-path of length 3. Since each entry $a^3_{i,j}$ of $A^3$ is the number of $i,j$-walks of length 3, we have

$$A^3 = J + qA$$

where $J$ is the block matrix

$$J = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}.$$

and $K$ is the square all 1 matrix with appropriate dimensions. If $\lambda \not\in \{-(q+1), q+1\}$ is an eigenvalue of $A$, then an eigenvector $x$ for $\lambda$ is orthogonal to the constant unit vector and so $Kx = 0$. It follows that $\lambda^3 = q\lambda$ and therefore $\lambda \in \{-\sqrt{q}, 0, \sqrt{q}\}$. Since the eigenvalues of $A(G_q)$ other than $-(q+1)$ and $(q+1)$ are not all zero, $\lambda(G_q) = \sqrt{q}$. A more complete analysis of these eigenvalues and their multiplicities was achieved by Haemers [13]. Since we have $\lambda(G_q) = \sqrt{q}$, Lemma 4.2 gives the following:

**Corollary 4.3.** *Let $X \subseteq V(G_q)$ where $|X| \geq 2\tau n/(q+1)$, and let $Y$ be the set of $e \in E(G_q)$ such that $|X \cap e| \geq \tau$. Then*

$$|Y| \geq n - \frac{2n}{\tau}.$$

*Proof.* Since $\lambda(G_q) = \sqrt{q}$, applying Lemma 4.2 with $\delta = 1$ and $d = q+1$ gives the result. □

## 5 Proof of Theorem 1.1: Lower Bound

Based on the generalized quadrangle $G_q$, we now specify the construction of a triangle-free $n$-vertex hypergraph $H_q$ with independence number $O(n^{2/3}\sqrt{\log n})$, which gives the lower bound in Theorem 1.1 for $r = 3$. Let $\tau = \lfloor 4 \log q \rfloor$. The idea is to place randomly a carefully chosen triangle-free 3-graph $F_q$ on $q+1$ vertices into each of the edges of $G_q$, independently for each edge of $G_q$, to form a new hypergraph $H_q$ with $n = q^3 + q^2 + q + 1$ vertices. We then use the spectral result in the form of Corollary 4.3 to deduce that a set of $2\tau n/q$ vertices of $G_q$ must intersect almost all edges of $G_q$ in roughly $\tau$ vertices. Together with some basic probability, we use this to deduce that the expected number of independent sets of size $2\tau n/q$ in $H_q$ is $o(1)$, and therefore some $H_q$ has independence number $2\tau n/q$, as required. A similar idea will be used in the lower bound in Theorems 1.2 and 1.4, and the appropriate modifications to $F_q$ will be made in Section 5.3 to obtain the lower bound in Theorem 1.1 for $r > 3$.

### 5.1 The hypergraph $F_q$

Throughout this section, $\tau = \lfloor 8\sqrt{\log q} \rfloor$. To describe $H_q$, we use the auxiliary hypergraph $F_q$ with vertex set $[q+1]$, defined as follows. Let $V = \{v_{ij} : 1 \leq i, j \leq \tau\}$ be a $\tau^2$-element subset of $[q+1]$

and let $S_1, \dots, S_\tau, T_1, \dots, T_\tau$ be a partition of $[q+1] - V$ into sets whose sizes differ by at most one. Let $S = \bigcup_{i=1}^{\tau} S_i$ and $T = \bigcup_{j=1}^{\tau} T_j$. The edge set of $F_q$ is the set of all triples $\{v_{ij}, a, b\}$ such that $a \in S_i$ and $b \in T_j$. Note that $F_q$ is actually 3-partite, with parts $V, S$ and $T$. Then $H_q$ is constructed by taking independently for each $e \in G_q$ a random bijection $\pi_e$ from $V(F_q)$ to $e$ and letting a triple in $e$ be an edge if its pre-image is an edge in $F_q$.

**Lemma 5.1.** *$H_q$ is triangle-free.*

*Proof.* Since $G_q$ is linear and triangle-free, it is sufficient to verify that $F_q$ is triangle-free. Suppose $F_q$ has a triangle. Since $F_q$ is 3-partite, some vertex in $V$ belongs to two of the edges of the triangle. Let this vertex be $v_{ij} \in V$, and these two edges be $\{v_{ij}, s_i, t_j\}$ and $\{v_{ij}, s'_i, t'_j\}$. Now the third edge must be either $\{v, s_i, t'_j\}$ or $\{v, s'_i, t_j\}$ for some $v \in V$. By definition of $F_q$, this implies that $v = v_{ij}$, a contradiction. □

Next we bound from above the probability that a set of $\tau$ vertices of $e \in E(G_q)$ is an independent set in $H_q$.

**Lemma 5.2.** *Let $I$ be a $\tau$-element subset of $e \in E(G_q)$. Then as $q \to \infty$, the probability that $I$ is independent in $H_q$ is at most $1 - \frac{\tau^3 - o(\tau^3)}{4eq}$.*

*Proof.* Let $N$ be the number of $\tau$-sets $X$ of $V(F_q) = [q+1]$ that are not independent in $F_q$. A lower bound for $N$ is obtained by picking an element $v_{ij} \in V$, an element $s \in S_i$, an element $t \in T_j$ and $\tau - 3$ elements in $[q+1] - (V \cup S_i \cup T_j)$. As $q \to \infty$, this gives

$$
\begin{aligned}
N &\geq \sum_{v_{ij} \in V} |S_i||T_j| \binom{q+1-|S_i|-|T_j|-|V|}{\tau-3} \\
&\geq \tau^2 \cdot \left\lfloor \frac{q}{2\tau} \right\rfloor^2 \binom{q+1-2\lceil \frac{q+1-\tau^2}{2\tau} \rceil - \tau^2}{\tau-3} \\
&\geq \tau^2 \cdot \left\lfloor \frac{q}{2\tau} \right\rfloor^2 \binom{(1-1/\tau)(q+1)-\tau^2}{\tau-3} \\
&= (1-o(1))\tau^2 \left(\frac{q}{2\tau}\right)^2 \binom{(1-1/\tau)q}{\tau-3} \\
&= (1-o(1))\frac{q^2}{4}(1-1/\tau)^{\tau-3}\frac{q^{\tau-3}}{(\tau-3)!} \\
&= (1-o(1))\frac{\tau^3}{4eq}\frac{(q+1)^\tau}{\tau!} \\
&= (1-o(1))\frac{\tau^3}{4eq}\binom{q+1}{\tau}.
\end{aligned}
$$

Now the probability that $I \subset e$ is not independent in $H_q$ is

$$
\frac{|\{\pi_e : \ I \text{ is not independent under } \pi_e\}|}{(q+1)!} = \frac{N\tau!(q+1-\tau)!}{(q+1)!} = \frac{N}{\binom{q+1}{\tau}}.
$$

The lower bound on $N$ now gives the desired result. □

## 5.2 Independence number of $H_q$

If $n = q^3 + q^2 + q + 1$ for some prime power $q$, then we show that with positive probability, $H_q$ has no independent set of size more than $2\tau n/q$ if $n$ is large enough and $\tau = \lfloor 8\sqrt{\log q}\rfloor$. Note that $2\tau n/q < 16n^{2/3}\sqrt{\log n}$ if $n$ is large enough. If $n$ is not of this form, pick the smallest prime power $q$ such that $n \le q^3 + q^2 + q + 1$, and remove $q^3 + q^2 + q + 1 - n$ vertices from $H_q$. The new hypergraph $H'_q$ has $\alpha(H'_q) \le \alpha(H_q)$. Since it is well-known that there exists a prime $q : n^{1/3} \le q \le 2n^{1/3}$, $H'_q$ has no independent set of size more than $2\tau n/q = O(n^{2/3}\sqrt{\log n})$, as required to finish the proof of the lower bound in Theorem 1.1.

Suppose that $X \subset V(H_q) = V(G_q)$ is an independent set of size $\lceil 2\tau n/q\rceil$ in $H_q$. By Corollary 4.3, at least $n - 2n/\tau$ of the edges of $G_q$ contain at least $\tau$ vertices of $I$. Let $Y = Y(X)$ be this set of edges. For each $e \in Y$, $X \cap e$ is an independent set in the random hypergraph $F_q$ on $e$. Let $B_e$ be the event that $X \cap e$ is independent in $F_q$. By Lemma 5.2,

$$P(B_e) \le 1 - \frac{\tau^3}{11q}$$

provided $q$ is large enough. The events $B_e$ are independent over $e \in Y$, and therefore the expected number of independent sets of size $2\tau n/q$ in $H_q$ is at most

$$\begin{aligned}\sum_{X:|X|=\lceil 2\tau n/q\rceil}\prod_{e\in Y} P(B_e) &\le \Big(1 - \frac{\tau^3 - o(\tau^3)}{4eq}\Big)^{n-2n/\tau}\binom{n}{\lceil 2\tau n/q\rceil}\\ &\le \exp\Big(-\frac{n(\tau^3 - o(\tau^3))}{4eq} + \frac{2\tau n}{q}\log\frac{n}{q}\Big). \qquad (1)\end{aligned}$$

Since $\tau = \lfloor 8\sqrt{\log q}\rfloor$ and $n = q^3 + q^2 + q + 1$ , as $q \to \infty$, we have

$$-\frac{n(\tau^3 - o(\tau^3))}{4eq} + \frac{2\tau n}{q}\log\frac{n}{q} \le \frac{n\tau}{q}\left[-\frac{\tau^2 - o(\tau^2)}{4e} + 2\log q^2\right] \le \frac{n\tau}{q}\left[-\frac{(1-o(1))64\log q}{4e} + 4\log q\right].$$

Thus the quantity in (1) decays to zero. Therefore with high probability, $H_q$ has no independent set of size more than $2\tau n/q < 16n^{2/3}\sqrt{\log n}$ if $n$ is large enough. This proves the lower bound in Theorem 1.1 for $r = 3$. We next turn to the case $r > 3$.

## 5.3 The hypergraph $H_{q,r}$

In this section we prove the lower bound in Theorem 1.1 for $r > 3$. Take $H_{q,r}$ to consist of randomly placed copies of a carefully chosen hypergraph $F_{q,r}$ on $q + 1$ vertices in the edges of $G_q$. The hypergraph $F_{q,r}$ takes the role of the hypergraph $F_q$ in the preceding section. To describe $F_{q,r}$, we first review a known construction of linear $r$-graphs based on a construction of dense sets without three-term arithmetic progressions.

## 5.4 Description of $F_{q,r}$

Erdős, Frankl and Rödl [12] showed that for every $r \ge 3$ there is a constant $c_r > 0$ such that for each $N \in \mathbb{N}$ there exists a linear triangle-free $r$-partite $r$-graph $J(N, r)$ with $N$ vertices in each part and at least $N^2/\exp(c_r\sqrt{\log N})$ edges. Their construction is based on and generalizes the construction of Ruzsa and Szemerédi [23] for $r = 3$ of a dense linear triangle-free 3-graph. The Ruzsa-Szemerédi construction is in turn derived from the Behrend's construction [5] of relatively dense sets of integers with no three-term arithmetic progressions. Using the Erdős-Frankl-Rödl construction, we describe a triangle-free (but not linear) $r$-graph $F_{q,r}$ on $q+1$ vertices for each $r > 3$. This is key in the description of $H_{q,r}$ for the proof of Theorem 1.1.

Fix $r > 3$, and let $C_r > 0$ be a constant depending on $r$, to be chosen later. Let $J$ be the Erdős–Frankl–Rödl hypergraph $J(\tau, r-1)$, where

$$\tau = \lceil (\log q)^{1/2} \exp(-C_r\sqrt{\log\log q})\rceil = (\log q)^{1/2-o(1)}.$$

For convenience let $m = |E(J)|$ and let $V_1, \dots, V_{r-1}$ be the parts of $J$. To define $V(F_{q,r})$, associate pairwise disjoint sets $S_v$ to the vertices $v \in V(J)$, and let $W$ be a set of $m$ vertices disjoint from all the sets $S_v$ and indexed by the edges of $J$, namely $W = \{v_e : e \in E(J)\}$. Then let

$$V(F_{q,r}) = W \cup \bigcup_{v\in V(J)} S_v$$

where the $S_v$ are as equal in size as possible subject to

$$q + 1 = m + \bigcup_{v\in V(J)} S_v.$$

This ensures that $F_{q,r}$ has exactly $q+1$ vertices. The edges of $F_{q,r}$ are defined as follows. For every $e = \{v_1, \dots, v_{r-1}\} \in J(\tau, r-1)$ with $v_i \in V_i$, recall that $v_e \in W$, and let

$$F_e = \{v_e \cup \{x_1, \dots, x_{r-1}\} : x_i \in S_{v_i}, i = 1, \dots, r-1\}.$$

Then

$$E(F_{q,r}) = \bigcup_{e\in J(\tau,r-1)} F_e.$$

Loosely speaking, the edges $e \in E(J)$ are being replaced with complete $(r-1)$-partite $(r-1)$-graphs $K(e)$ with parts of size roughly $q/(r-1)\tau$, and then we form the edges of $F_{q,r}$ by enlarging each of the edges of $K(e)$ with the new vertex $v_e$. It is straightforward to check (by both the linearity of $J$ and the fact that $J$ is triangle-free) that $F_{q,r}$ is triangle-free (although it is not linear). The key lemma about $F_{q,r}$ is now as follows:

**Lemma 5.3.** *Let $r \geq 3$ and $I$ be a $\tau$-element subset of $e \in E(G_q)$. Then for some $d_r > 0$, the probability that $I$ is independent in $H_{q,r}$ is at most*

$$1 - \frac{\tau^{3-d_r/\sqrt{\log\tau}}}{q}.$$

*Proof.* Let $N$ be the number of $\tau$-element subsets of $V(F_{q,r}) = [q+1]$ that are not independent in $F_{q,r}$. Since every $\tau$-element set obtained by picking an element $w_e \in W$, an element from each set $S_v$ such that $v \in e$, and then $\tau - r$ elements in $[q+1] \setminus (W \cup \bigcup_{v\in e} S_v)$ is not independent, we have

$$N \geq \sum_{w_e\in W} \left(\prod_{v\in e} |S_v|\right) \binom{q+1-\sum_{v\in e}|S_v| - |W|}{\tau - r}.$$

Since all $S_v$ have almost the same cardinality, as $q \to \infty$ the right-hand side is at least

$$(m + o(m)) \cdot \Big(\frac{q}{\tau}\Big)^{r-1} \cdot \Big(\frac{q+1-q/\tau}{\tau - r}\Big)^{\tau - r} \geq (m + o(m)) \cdot \frac{\tau}{qer^r}\binom{q+1}{\tau}.$$

So we can choose $d_r > 0$ depending only on $r$ such that the last expression is at least

$$\frac{\tau^{3-d_r/\sqrt{\log \tau}}}{q}\binom{q+1}{\tau}.$$

This bound proves the lemma. □

The rest of the proof for $H_{q,r}$ carries through as for $H_q$, except at the end, the expected number of independent sets of size $2\tau n/q$ in $H_{q,r}$ is now by Lemma 5.3 at most

$$\Big(1 - \frac{\tau^{3-d_r/\sqrt{\log \tau}}}{q}\Big)^{n-2n/\tau}\binom{n}{2\tau n/q} < \exp\Big(-\frac{\tau^{3-d_r/\sqrt{\log \tau}}n}{2q} + \frac{2\tau n \log n}{q}\Big).$$

We have chosen $\tau$ to ensure

$$\tau^{3-d_r/\sqrt{\log \tau}} > 6\tau \log n.$$

This ensures that the expected number of independent sets of size $2\tau n/q$ in $H_{q,r}$, for large enough $n$, is less than

$$\exp\Big(-\frac{\tau n \log n}{q}\Big) < \exp\Big(-n^{2/3} \log n\Big) < 1.$$

We conclude that with positive probability, for large enough $n$ and a large enough constant $C_r$,

$$\alpha(H_{q,r}) \leq 2\tau n/q \leq 2n^{2/3}(\log n)^{1/2 + C_r/\sqrt{\log\log n}}.$$

This gives the lower bound on Ramsey numbers in Theorem 1.1. □

# 6 Proof of Theorem 1.2

To prove the upper bound in Theorem 1.2, it is sufficient to show that every $n$-vertex linear triangle-free 3-graph has an independent set of size $\Omega(n^{2/3}(\log n)^{1/3})$. Let $H$ be such a 3-graph. By the main theorem in [1],

$$\alpha(H) = \Omega\Big(\frac{n\sqrt{\log d}}{\sqrt{d}}\Big)$$

where $d$ is the average degree of $H$. The union of all pairs $e \backslash \{v\}$ for edges $e$ containing a vertex $v$ of degree at least $d$ in $H$ is an independent set of $2d$ vertices in $H$, since $H$ is linear and triangle-free. Therefore

$$\alpha(H) = \Omega\Big(\min_d \max\Big\{d, \frac{n\sqrt{\log d}}{\sqrt{d}}\Big\}\Big) = \Omega(n^{2/3}(\log n)^{1/3}).$$

This completes the proof of the upper bound in Theorem 1.2.

### 6.1 Proof of Theorem 1.2: Lower Bound

Based on the hypergraph $G_q$, for $n = q^3 + q^2 + q + 1$ and $q$ a prime power, we construct an $n$-vertex linear triangle-free 3-graph $H_q^*$ with $\alpha(H_q^*) \leq n^{2/3}\exp(A\sqrt{\log n})$ for some $A > 0$. If $n$ is not of that form, then as in the proof of Theorem 1.1 we use the distribution of primes and a large subhypergraph of $H_q^*$ to obtain the same result with perhaps a slightly larger implicit constant. Let $N = \lfloor (q+1)/3 \rfloor$ and let $F_q^* = J(N, 3)$, where $J(N, 3)$ is defined in Section 5.4. Then $|E(F_q^*)| = |E(J)| = \Omega(qr_3(q))$. The main lemma we require counts independent sets of size $\tau$ in $F_q^*$.

**Lemma 6.1.** *As $q \to \infty$ the number of independent sets of size $\tau$ in $F_q^*$ is at most*

$$\Big(1 - \Omega\Big(\frac{\tau^3 r_3(q)}{q^2}\Big)\Big)\binom{q+1}{\tau}.$$

*Proof.* Let $N$ be the number of non-independent sets of size $\tau$ in $F_q^*$. It is sufficient to show

$$N = \Omega\Big(\frac{\tau^3 r_3(q)}{q^2}\Big)\binom{q+1}{\tau}.$$

Since $M := |E(F_q^*)| = \Omega(qr_3(q))$, by inclusion-exclusion,

$$\begin{aligned} N &\geq M \cdot \binom{q-2}{\tau-3} - \binom{M}{2}\binom{q-4}{\tau-5} \\ &= M \cdot \binom{q+1}{\tau}\frac{\tau(\tau-1)(\tau-2)}{(q+1)q(q-1)}\Big(1 - \frac{(M-1)(\tau-3)(\tau-4)}{2(q-2)(q-3)}\Big) \\ &= \Omega\Big(\frac{\tau^3 r_3(q)}{q^2}\Big)\binom{q+1}{\tau}. \end{aligned}$$

This is the required bound on $N$. □

As before, we construct $H_q^*$ by placing a randomly permuted copy of $F_q^*$ in each edge of $G_q$. The expected number of independent sets of size $\lceil 2\tau n/q \rceil$ in $H_q^*$ is then at most

$$\Big(1 - O\Big(\frac{\tau^3 r_3(q)}{q^2}\Big)\Big)^{n-2n/\tau}\binom{n}{\lceil 2\tau n/q \rceil}$$

using Lemma 6.1 and Corollary 4.3 as in the proof of Theorem 1.1. Choose $\tau$ to satisfy

$$\frac{4\tau n \log n}{q} < \frac{n\tau^3 r_3(q)}{q^2}$$

which ensures that the the expected number of independent sets is $o(1)$. It is sufficient to take

$$\tau^2 = (1 + o(1))\frac{4q \log n}{r_3(q)}.$$

Then with high probability

$$\alpha(H_q^*) < \frac{2\tau n}{q} < \frac{8n\sqrt{q \log n}}{q\sqrt{r_3(q)}}.$$

To obtain from this the lower bound on $RL(C_3, K_t^3)$, let $n = RL(C_3, K_t^3)$ so that

$$\frac{8n\sqrt{q \log n}}{q\sqrt{r_3(q)}} > t.$$

Since $r_3(q) > q/\exp(c\sqrt{\log q})$ for some $c > 0$, this gives the lower bound on $RL(C_3, K_t^r)$ in Theorem 1.2.

Finally, we connect a bound on Ramsey numbers to $r_3(N)$. According to the above proof, if $n = RL(C_3, K_t^3) = O(t^{3/2}(\log t)^{-3/4-c})$, then

$$\frac{n\sqrt{q \log n}}{q\sqrt{r_3(q)}} = \Omega(t).$$

Put $N = q$. Recalling $n = N^3 + o(N^3)$,

$$r_3(N) = O\Big(\frac{N^5 \log N}{t^2}\Big).$$

The definition of $n$ in terms of $t$ gives

$$t = \Omega(n^{2/3}(\log n)^{1/2+2c/3}) = \Omega(N^2(\log N)^{1/2+2c/3}).$$

Therefore

$$r_3(N) = O\Big(\frac{N}{(\log N)^{4c/3}}\Big).$$

This completes the proof of Theorem 1.2. □

# 7 Proof of Theorem 1.3

For Theorem 1.3, which states that

$$R(C_k, K_t^r) = \Omega^*\Big(t^{1+\frac{1}{3k-1}}\Big),$$

we let $G_{k,q}$ be an $n$-vertex $(q+1)$-uniform $(q+1)$-regular hypergraph with no cycles of length at most $k$, such that $q$ is a maximum relative to $n$ and such that $\lambda(G_{k,q}) \leq 2\sqrt{q}$.

A construction of hypergraphs $G_{k,q}$ for primes $q \equiv 1 \bmod 4$ can be obtained from the construction of Ramanujan graphs of Lubotzsky, Phillips and Sarnak [18]. These $G_{k,q}$ are constructed from the following bipartite graphs of [18]: Let $p, q$ be primes congruent to 1 modulo 4 with $p > 16$. If $(\frac{p}{q}) = -1$, then $B_{p,q}$ is a bipartite $(q+1)$-regular graph with $p(p^2-1)$ vertices in each part and no cycle of length less than $4\log_q(p/4)$. If $(\frac{p}{q}) = 1$, then $B_{p,q}$ is a bipartite $(q+1)$-regular graph with $p(p^2-1)/2$ vertices in each part and no cycle of length less than $2\log_q p$. In both cases $B_{p,q}$ has no cycle of length less than $2\log_q p$ since $p > 16$, and the second largest eigenvalue in absolute value except the first and last is at most $2\sqrt{q}$.

So, given $k \geq 4$, we first choose a prime $q \equiv 1 \bmod 4$, then choose a smallest prime $p \equiv 1 \bmod 4$ with $p > q^k$. By the previous paragraph, for $n \in \{\frac{1}{2}p(p^2-1), p(p^2-1)\}$, there exists a $2n$-vertex bipartite $(q+1)$-regular graph $B_{p,q}$ of girth greater than $2k$. This $B_{p,q}$ is the bipartite incidence graph of a $C_k$-free $(q+1)$-graph $G_{k,q}$ on $n$ vertices. And if we choose the smallest possible $p$, then $n < (1+o(1)q^{3k}$. Furthermore, it follows that $\lambda(G_{k,q}) \leq 2\sqrt{q}$.

Let $F_{k,q,r}$ denote the $r$-graph consisting of a vertex-disjoint union of $\tau = \lfloor 4\log q \rfloor$ stars of size $\lfloor q/\tau \rfloor$ on $q$ vertices. In each edge of $G_{k,q}$, put a randomly permuted copy of $F_{k,q,r}$ to get the $r$-graph $H_{k,q,r}$. Corollary 4.3 shows that if $X$ is a set of at least $2\tau n/q$ vertices of $H_{k,q,r}$, then at least $n - 8n/\tau$ edges of $G_{k,q}$ contain at least $\tau$ vertices of $X$. The expected number of independent sets in $H_{k,q,r}$ of size $2\tau n/q$ is at most

$$\Big(1 - \frac{\tau^2}{10q}\Big)^{n-8n/\tau}\binom{n}{2\tau n/q} < \exp\Big(-\frac{\tau^2 n}{20q} + \frac{2\tau n\log n}{q}\Big)$$

provided $q$ is large enough. The choice of $\tau$ ensures this decays to zero. Therefore with positive probability,

$$\alpha(H_{k,q,r}) = O\Big(\frac{\tau n}{q}\Big) = O\Big(n^{1-1/3k}\log n\Big),$$

as long as $q > c_k n^{1/3k}$ for some constant $c_k$ depending only on $k$.

Now suppose we are given $k \geq 4$ and an integer $n$ not of the form required to construct $B_{p,q}$ and hence $G_{k,q}$ and $H_{k,q,r}$. For such an $n$, we will choose $p, q$ so that the construction above is possible on $n'$ vertices with $n < n' < 8n$, and then restrict the resulting $H_{k,q,r}$ (which has $n'$ vertices) to a subhypergraph with only $n$ vertices. The resulting $n$-vertex $r$-graph would again have independence number $O\Big(n^{1-1/3k}\log n\Big)$.

Given $k \geq 4$ and a sufficiently large $n$, choose a prime $q \equiv 1 \bmod 4$ such that

$$\frac{1}{2}(2n)^{1/3k} < q < (2n)^{1/3k}.$$

Such a $q$ exists by the prime number theorem in arithmetic progressions. Next choose a prime $p \equiv 1 \bmod 4$ such that

$$(3n)^{1/3} < p < 2n^{1/3}.$$

Again, by the prime number theorem in arithmetic progressions, we can find such a $p$ because $n$ is sufficiently large. Now set $n' = p(p^2-1)/2$ or $p(p^2-1)$ depending on whether $(\frac{p}{q})$ is 1 or $-1$, and construct $H_{k,q,r}$ as described above. The resulting $(q+1)$-graph $H_{k,q,r}$ contains no $C_k$ as $q < (2n)^{1/3k} < (3n)^{1/3k} < p^{1/k}$. Finally, observe that

$$n' > p^3/2 - p/2 > 3n/2 - n^{1/3} > n$$

and $n' < p^3 < 8n$. Moreover, $q > c_k n^{1/3k}$ so the above bound on the independence number holds as $n \to \infty$.

This shows that for any $r \geq 3$ and $k \geq 4$,

$$R(C_k, K_t^r) = \Omega^*\Big(t^{1+\frac{1}{3k-1}}\Big).$$

### 7.1 Proof of Theorem 1.5

The specialization of the above arguments to $k = 5$ comes from the existence of generalized hexagons (see [9] or [25]). The generalized hexagons $G_q$ exist for prime powers $q$ and can be viewed as $(q+1)$-uniform $(q+1)$-regular hypergraphs $G_q$ on $q^5 + q^4 + q^3 + q^2 + q + 1$ vertices containing no cycles of length at most five, and moreover the associated matrix $A(G_q)$ has $\lambda(G_q) = \sqrt{q}$ once more. Using the hypergraph $F_{k,q,r}$ in each edge of the hypergraph $G_q$ as before gives the result: we obtain a hypergraph $H_{5,q,r}$ with

$$\alpha(H_{5,q,r}) = O(n^{4/5} \log n)$$

from which the lower bound on Ramsey numbers $R(C_5, K_t^r) = \Omega(t^{5/4}(\log t)^{-5/4})$ for all $r \geq 3$ follows.

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