List coloring triangle-free hypergraphs

Jeff Cooper [*] Dhruv Mubayi [†]

October 30, 2013

Abstract

A triangle in a hypergraph is a collection of distinct vertices u, v, w and distinct edges e, f, g with $u, v \in e$, $v, w \in f$, $w, u \in g$ and $\{u, v, w\} \cap e \cap f \cap g = \emptyset$. Johansson [10] proved that every triangle-free graph with maximum degree Δ has list chromatic number $O(\Delta / \log \Delta)$. Frieze and the second author [7] proved that every *linear* (meaning that every two edges share at most one vertex) triangle-free triple system with maximum degree Δ has chromatic number $O(\sqrt{\Delta / \log \Delta})$. The restriction to linear triple systems was crucial to their proof.

We provide a common generalization of both these results for rank 3 hypergraphs (edges have size 2 or 3). Our result removes the *linear* restriction from [7], while reducing to the (best possible) result [10] for graphs.

As an application, we prove that if $\mathcal{C}_3$ is the collection of 3-uniform triangles, then the Ramsey number $R(\mathcal{C}_3, K_t^3)$ satisfies

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(\mathcal{C}_3, K_t^3) \leq \frac{bt^{3/2}}{(\log t)^{1/2}}$$

for some positive constants a and b. The upper bound makes progress towards the recent conjecture of Kostochka, the second author, and Verstraëte [13] that $R(C_3, K_t^3) = o(t^{3/2})$ where C_3 is the linear triangle.

1 Introduction

A hypergraph $H = (V, E)$ is a tuple consisting of a set of vertices V and a set of edges E, which are subsets of V. The hypergraph has rank k if every edge contains at most

[*]Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA; email: jcoope8@uic.edu

[†]Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago IL 60607, USA; research supported in part by NSF grant 0969092; email: mubayi@uic.edu

k vertices and is called k-uniform if every edge contains exactly k vertices. A *proper coloring* of H is an assignment of colors to the vertices so that no edge is monochromatic. The *chromatic number* of H, $\chi(H)$, is the minimum number of colors needed in a proper coloring of H.

The chromatic number of graphs (2-uniform hypergraphs) has been studied extensively. A greedy coloring algorithm can be used to show that for any graph G with maximum degree Δ, $\chi(G) \leq \Delta+1$; this bound is tight for complete graphs and odd cycles. Brooks [4] extended this by showing that if G is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta$.

A natural question to ask is what other structural properties can be put on a graph to decrease its chromatic number. One approach is to fix a graph K and consider the family of graphs which contain no copy of K. For example, if K is a tree on e edges and G contains no copy of K, then $\chi(G) \leq e$; this follows from the fact that if G contains no copy of K, then G contains a vertex of degree at most $e-1$ (see [19], pg. 70).

When K is a cycle, the problem becomes more difficult. Kim [11] showed that if G contains no 4-cycles or 3-cycles, then $\chi(G) \leq (1+o(1))\Delta/\log\Delta$ as $\Delta \to \infty$, which is within a factor of 2 of the best possible bound. Shortly after, Johansson [10] showed that if G contains no 3-cycles, then $\chi(G) = O(\Delta/\log\Delta)$. Using Johansson's result, Alon, Krivelevich, and Sudakov [2] showed that if K is any graph containing a vertex x such that $K-x$ is bipartite, then $\chi(G) = O(\Delta/\log\Delta)$.

Some analogous results for hypergraphs are known. Using the local lemma, one can show that $\chi(H) = O(\Delta^{1/(k-1)})$ for any k-uniform hypergraph H. Bohman, Frieze, and the second author [3] showed that if K is a fixed k-uniform hypertree on e edges and H is a k-uniform hypergraph containing no copy of K, then $\chi(H) \leq 2(k-1)(e-1)+1$; Loh [14] improved this to $\chi(H) \leq e$, matching the result for graphs.

A hypergraph is *linear* (or contains no 2-cycles) if any two of its edges intersect in at most one vertex. A *triangle* in a linear hypergraph is a set of three pairwise intersecting edges with no common point. In [7], Frieze and the second author showed that if H is a 3-uniform, linear, triangle-free hypergraph, then $\chi(H) = O(\sqrt{\Delta}/\sqrt{\log\Delta})$. They subsequently removed the triangle-free condition and generalized their result from 3 to k, showing that $\chi(H) = O((\Delta/\log\Delta)^{1/(k-1)})$ for any k-uniform, linear hypergraph H [?]. As shown in [3], these results are tight apart from the implied constants.

1.1 Our Result

Our contribution is to remove the linear condition from [7]. However, in doing so, we also widen the definition of a triangle.

Definition 1. A *triangle* in a hypergraph H is a set of three distinct edges $e, f, g \in H$ and three distinct vertices $u, v, w \in V(H)$ such that $u, v \in e$, $v, w \in f$, $w, u \in g$ and $\{u, v, w\} \cap e \cap f \cap g = \emptyset$.

For example, the three triangles in a 3-uniform hypergraph are the loose triangle $C_3 = \{abc, cde, efa\}$, $F_5 = \{abc, bcd, aed\}$, and $K_4^- = \{abc, bcd, abd\}$.

Given a set $L(v)$ of colors for every vertex $v \in V(H)$, a *proper list coloring* of H is a proper coloring where every vertex v receives a color from $L(v)$. The list chromatic number of H, $\chi_l(H)$, is the minimum l so that if $|L(v)| \geq l$ for all v, then H has a proper list coloring. It is not hard to see that $\chi(H) \leq \chi_l(H)$. As in [11] and [10], our main theorem can be stated in terms of list chromatic number. If H is a rank k hypergraph and $i \leq k$, the i-degree of a vertex v is the number of size i edges containing v.

Theorem 2. *Suppose H is a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Then*

$$\chi_l(H) \leq c_1 \max\{(\frac{\Delta}{\log \Delta})^{\frac{1}{2}}, \frac{\Delta_2}{\log \Delta_2}\},$$

for some constant c_1.

Theorem 2 generalizes the results of [10] and [7]. Additionally, it strengthens [7] by removing the linear hypothesis, which was a crucial ingredient in the proof. We prove Theorem 2 by using a semi-random algorithm to properly color the hypergraph. Our algorithm is similar to the algorithm in [7], however, several new ideas are developed to deal with the non-linear case.

As mentioned above, for n-vertex 3-uniform hypergraphs H with maximum degree Δ, one can easily show that the independence number of H is $\Omega(n/\sqrt{\Delta})$ and $\chi(H) = O(\sqrt{\Delta})$; however, adding a local restriction to the hypergraph in order to significantly improve either of these bounds appears to be a hard problem. There are two conjectures in this regard. De Caen [5] conjectured that if we add the hypothesis that every vertex subset S spans at most $c|S|^2$ edges (for some fixed constant c), and $\Delta = \Theta(n)$, then the lower bound on the independence number can be improved by a factor that tends to infinity with Δ. More recently, [7] conjectured that if there is a fixed hypergraph F with $F \not\subset H$, then $\chi(H) < c_F\sqrt{\Delta/\log \Delta}$. Guruswami and Sinop [8] showed that this conjecture implies certain hardness results in computer science.

1.2 Application to Hypergraph Ramsey Numbers

Let $\mathcal{C}_3^r$ be the collection of r-uniform hypergraph triangles. Notice that for graphs, C_3^2 consists of only the 3-vertex cycle, and for triple systems, $\mathcal{C}_3^3 = \{C_3, F_5, K_4^-\}$. The hypergraph Ramsey number $R(\mathcal{C}_3^r, K_t^r)$ is the smallest n so that in every red-blue coloring of the edges of the complete r-uniform hypergraph K_n^r, there exists a red triangle or a blue K_t^r. Ajtai-Komlós-Szemerédi [1] and Kim [12] proved that $R(\mathcal{C}_3^2, K_t^2) = \Theta(t^2/\log t)$.

In [13], Kostochka, the second author, and Verstraëte proved a version of this result for $r = 3$. In this setting, $R(C_3, K_t^3)$ is the smallest n so that in every red-blue coloring of the edges of the complete 3-uniform hypergraph K_n^3, there exists a red C_3 or a blue K_t^3. [13] showed that there exist constants a, b such that

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(C_3, K_t^3) \leq bt^{3/2},$$

and they conjectured that the upper bound could be reduced to $o(t^{3/2})$. In proving the upper bound, they showed that if H is a 3-uniform, C_3-free hypergraph with n vertices and average degree $d \geq n^{2/3}/12$, then $\alpha(H) \geq d/18$. Our result implies that any 3-uniform, $\mathcal{C}_3$-free hypergraph H with n vertices and average degree d satisfies $\alpha(H) = \Omega(\frac{n}{d^{1/2}} \log^{1/2} d)$. Setting $d = n^{2/3} \log^{1/3} n$, we obtain $R(\mathcal{C}_3^3, K_t^3) = O(t^{3/2}/\sqrt{\log t})$. Since the C_3-free construction given in [13] is also F_5 and K_4^- free, this implies that for some constants a and b,

$$\frac{at^{3/2}}{(\log t)^{3/4}} \leq R(\mathcal{C}_3^3, K_t^3) \leq b\frac{t^{3/2}}{(\log t)^{1/2}}.$$

1.3 Organization

In Section 2, we present the probabilistic tools we will need to analyze our algorithm. In Section 3, we describe our algorithm. The presentation is similar to Vu's description in [18] of Johansson's algorithm. Section 4 contains an analysis of our algorithm. This analysis does not use triangle-free anywhere, but is instead based on parameters which can be given to the algorithm. In Section 5, we show how triangle-free can be used to set these parameters in a way that implies Theorem 2.

2 Tools

2.1 Local Lemma

Asymmetric Local Lemma ([17]). *Consider a set $\mathcal{E} = \{A_1, \ldots, A_n\}$ of (typically bad) events that such each A_i is mutually independent of $\mathcal{E} - (\mathcal{D}_i \cup A_i)$, for some $\mathcal{D}_i \subset \mathcal{E}$. If for each $1 \le i \le n$*

- $\Pr[A_i] \le 1/4$, *and*
- $\sum_{A_j \in \mathcal{D}_i} \Pr[A_j] \le 1/4$,

then with positive probability, none of the events in $\mathcal{E}$ occur.

2.2 Concentration Theorems

The first result is due to Hoeffding [9].

Theorem 3. *Suppose that $X = X_1 + \cdots + X_m$, where the X_i are independent random variables satisfying $|X_i| \le a_i$ for all i. Then for any $t > 0$,*

$$\Pr[X \ge \mathbf{E}[X] + t] \le e^{-\frac{2t^2}{\sum_{i=1}^m a_i^2}},$$

and

$$\Pr[X \le \mathbf{E}[X] - t] \le e^{-\frac{2t^2}{\sum_{i=1}^m a_i^2}}.$$

We will also use the following theorem, which is Theorem 2.7 from [16].

Theorem 4. *Suppose that $X = X_1 + \cdots + X_m$, where the X_i are independent random variables satisfying $X_i \le \mathbf{E}[X_i] + b$ for all i. Then for any $t > 0$,*

$$\Pr[X \ge \mathbf{E}[X] + t] \le e^{-\frac{t^2}{2\,\mathbf{Var}[X] + bt}}.$$

McDiarmid [15] proved the following generalization of Theorem 3.

Theorem 5. *Let $Z_1, \ldots, Z_n$ be independent random variables, with Z_i taking values in a set $\mathcal{A}_i$ for each i. Suppose that the (measurable) function $g : \prod \mathcal{A}_k \to \mathbb{R}$ satisfies $|g(x) - g(x')| \le d_i$ whenever the vectors x and x' differ only in the i^{th} coordinate. Let W be the random variable $g(Z_1, \ldots, Z_n)$. Then for any $t > 0$,*

$$\Pr[W > \mathbf{E}[W] + t] \le e^{-2t^2/\sum_{i=1}^n d_i^2}.$$

Note that in the above theorem, we may view $\prod \mathcal{A}_k$ as a probability space induced by the random variables $Z_1, \ldots, Z_n$. We will use the following corollary, which resembles Theorem 7.2 from [6].

Corollary 6. *Let $X_1, \ldots, X_n$ be independent random variables, with X_i taking values in a set $\mathcal{B}_i$ for each i. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be events, where each $\mathcal{A}_i \subset \mathcal{B}_i$. Set $\mathcal{A} = \prod_{i=1}^n \mathcal{A}_i$. Suppose that the (measurable) function $f : \prod \mathcal{B}_k \to \mathbb{R}$ is non-negative and satisfies $|f(x) - f(x')| \leq d_i$ for any two vectors $x, x' \in \mathcal{A}$ differing only in the i^{th} coordinate. Let Y be the random variable $f(X_1, \ldots, X_n)$. Then*

$$\Pr\left[Y > \mathbf{E}[Y]/\Pr[\mathcal{A}] + t\right] \leq e^{-2t^2/\sum_{i=1}^n d_i^2} + \Pr[\bar{\mathcal{A}}].$$

Proof. Define $g : \mathcal{A} \to \mathbb{R}$ by $g(x) := f(x)$ (in other words, $g = f|\mathcal{A}$). For each i, let $Z_i : X_i^{-1}(\mathcal{A}_i) \to \mathcal{A}_i$ be the random variable with $Z_i(s) = X_i(s)$ for all $s \in X_i^{-1}(\mathcal{A}_i)$. Let W be the random variable $g(Z_1, \ldots, Z_n)$. Since the X_i are independent, the Z_i are also independent, so we will be able to apply Theorem 5 to bound $\Pr[W > \mathbf{E}[W] + t]$.

By total probability and the non-negativity of f,

$$\mathbf{E}[Y] = \mathbf{E}[Y|\mathcal{A}]\Pr[\mathcal{A}] + \mathbf{E}[Y|\bar{\mathcal{A}}]\Pr[\bar{\mathcal{A}}] \geq \mathbf{E}[Y|\mathcal{A}]\Pr[\mathcal{A}]$$

so

$$\mathbf{E}[W] = \mathbf{E}[Y|\mathcal{A}] \leq \mathbf{E}[Y]/\Pr[\mathcal{A}].$$

Combining this with Theorem 5 implies

$$\begin{aligned}
\Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t\right] &= \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t \middle| \mathcal{A}\right]\Pr[\mathcal{A}] + \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t \middle| \bar{\mathcal{A}}\right]\Pr[\bar{\mathcal{A}}] \\
&\leq \Pr\left[Y > \frac{\mathbf{E}[Y]}{\Pr[\mathcal{A}]} + t \middle| \mathcal{A}\right] + \Pr[\bar{\mathcal{A}}] \\
&\leq \Pr\left[Y > \mathbf{E}[Y|\mathcal{A}] + t | \mathcal{A}\right] + \Pr[\bar{\mathcal{A}}] \\
&= \Pr\left[W > \mathbf{E}[W] + t\right] + \Pr[\bar{\mathcal{A}}] \\
&\leq e^{-2t^2/\sum_{i=1}^n d_i^2} + \Pr[\bar{\mathcal{A}}].
\end{aligned}$$

□

3 Coloring Algorithm

The input to our algorithm is a rank 3 hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. Let H denote the input hypergraph restricted to its size 3 edges, and let G denote the input hypergraph restricted to its size 2 edges. At the beginning, each vertex u has a list $C(u)$ of acceptable colors. We assume $|C(u)| = C$ for all vertices u. For each vertex u and color c, we set

$$p_u^0(c) = \begin{cases} 1/C, & \text{if } c \in C(u) \\ 0, & \text{if } c \notin C(u). \end{cases}$$

We define a parameter $\hat{p}$, which will serve as an upper bound on the weights $p_u^i(c)$. Set $W^0(u) = \{p_u^0(c) : c \in \cup_v C(v)\}$. We start with the hypergraph $H^0 = H$ and the collection $\{W^0(u)\}_u$. For each color c, we also construct a graph G_c^0, which is initially a copy of the 2-graph G. Finally, we assign to each vertex an empty set $B^0(u)$.

At the $(i+1)^{th}$ step, $i = 0, 1, \ldots, T-1$, our input to the algorithm is a quadruple, $(H^i, \{G_c^i\}_c, \{W_u^i\}_u, \{B^i(u)\}_u)$. We generate a small random set of colors at each vertex u as follows: For each color c, we choose c with probability $\theta p_u^i(c)$. Let

$$\gamma_u^i(c) = \begin{cases} 1, & \text{if } c \text{ is chosen at } u, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the $\gamma_u^i(c)$ are independent random variables.

Consider a vertex u. We define the set of colors lost at u as

$$L^i(u) = \{c : \exists e \in E(H^i) \cup E(G_c^i) \text{ such that } u \in e \text{ and } \gamma_v^i(c) = 1 \ \forall v \in e - u\}.$$

We say a color c *survives* at u if $c \notin B^i(u)$ and $c \notin L^i(u)$. For $c \notin B^i(u)$, we define

$$q_u^i(c) := \Pr[c \text{ survives at } u] = \Pr[\bigcap_{\substack{\{v,w\}: \\ uvw \in H^i}} (\gamma_v^i(c) = 0 \cup \gamma_w^i(c) = 0) \bigcap_{v: uv \in G_c^i} \gamma_v^i(c) = 0]. \quad (3.1)$$

In other words, if $c \notin B^i(u)$, then $q_u^i(c) = \Pr[c \notin L^i(u)]$. Note that at the $(i+1)^{th}$ step, $q_u^i(c)$ is a fixed number, which can be computed given H^i, G_c^i, and all of the $p_v^i(c)$; it does not depend on the random variables $\gamma_u^i(c)$. In the analysis below, we will use the bound

$$\begin{aligned} q_u^i(c) &= 1 - \Pr[\bigcup_{uvw \in H^i} (\gamma_v^i(c) = 1 \cap \gamma_w^i(c) = 1) \bigcup_{uv \in G_c^i} \gamma_v(c) = 1] \\ &\geq 1 - \sum_{uvw \in H^i} \theta^2 p_v^i(c) p_w^i(c) - \sum_{uv \in G_c^i} \theta p_v^i(c). \end{aligned} \quad (3.2)$$

Let $\mathbf{I}[X]$ denote the $0,1$ indicator variable for the event X. Define $p_u^{i+1}(c)$ as:

- If $p_u^i(c)/q_u^i(c) < \hat{p}$ and $c \notin B^i(u)$, then

$$p_u^{i+1}(c) = p_u^i(c) \frac{\mathbf{I}[c \text{ survives at } u]}{\Pr[c \text{ survives at } u]} = \begin{cases} p_u^i(c)/q_u^i(c), & \text{if } c \text{ survives at } u, \\ 0, & \text{else.} \end{cases} \quad (3.3)$$

- If $p_u^i(c)/q_u^i(c) \geq \hat{p}$ or $c \in B^i(u)$, then we toss a biased coin with $\Pr[Head] = p_u^i(c)/\hat{p}$. We then set
$$\eta_u^i(c) = \mathbf{I}[Head],$$
and
$$p_u^{i+1}(c) = p_u^i(c)\frac{\mathbf{I}[Head]}{\Pr[Head]} = \begin{cases} \hat{p}, & \text{if } \eta_u^i(c) = 1 \\ 0, & \text{else.} \end{cases} \tag{3.4}$$

Crucially, (3.3) and (3.4) imply

$$\mathbf{E}[p_u^{i+1}(c)] = p_u^i(c). \tag{3.5}$$

Color u with c if c survives at u and $\gamma_u^i(c) = 1$ (if there are multiple such c, pick one arbitrarily). Let U^{i+1} denote the set of uncolored vertices in H after the iteration i. Let H^{i+1} be the hypergraph induced from H by U^{i+1}, let $B^{i+1}(u) = \{c : p_u^{i+1}(c) = \hat{p}\}$, and let $W_u^{i+1} = \{p_u^{i+1}(c)\}$. To form G_c^{i+1}, start with G_c^i, and for each triple $u, v, w \in U^i$ with $u, v \in U^{i+1}$, $uv \notin G_c^i$, and w colored c, add an edge uv to G_c^{i+1}. Then delete any vertex from G_c^{i+1} that is not in U^{i+1}.

Observe that if uvw is an edge in H^i and u and v are both colored c in the current round, then $p_w^{i+1}(c) \in \{0, \hat{p}\}$; in particular, c is never considered for w in a future round. Similarly, if $vw \in G_c^i$ and v is colored with c in the current round, then c is never considered for w in the future. Thus the algorithm always maintains a proper partial coloring of H.

After T iterations, some vertices will remain uncolored. We color these in one final step, which is described in Section 4.5.

3.1 Parameters and Notation

We summarize all of the variables used in the algorithm and its analysis in the two tables below. The first table contains descriptions of the independent variables in our algorithm. We set them for one family of hypergraphs in Section 5, when we prove that our algorithm works for triangle-free hypergraphs. The values of the remaining parameters are defined in the second table.

Our algorithm requires that the parameter ω_0 satisfy the following properties:

- For any edge $uvw \in H^i$ and any color c,

$$\Pr[c \notin L^i(u) \cup L^i(v) \cup L^i(w)] \leq q_u^i(c)q_v^i(c)q_w^i(c)(1 + 1/\omega_0). \tag{3.6}$$

- For any color c and any pair u, v with an edge $uvw \in H^i$ for some w,

$$\Pr[c \notin L^i(u) \cup L^i(v)] \leq q_u^i(c) q_v^i(c)(1 + 1/\omega_0). \tag{3.7}$$

- For any color c and any edge $uv \in G_c^i$,

$$\Pr[c \notin L^i(u) \cup L^i(v)] \leq q_u^i(c) q_v^i(c)(1 + 1/\omega_0). \tag{3.8}$$

	Description
Δ	Maximum degree of 3-graph
Δ_2	Maximum degree of 2-graph
δ	Maximum codegree
ω	Color bound, tending to ∞ with Δ
ϵ	Small constant
ω_0	Error term depending on H
$\hat{p}$	Threshold probability

	Value	Description
C	$\sqrt{\Delta}/\sqrt{\omega}$	Number of colors
T	$(5\omega/\epsilon)\log\omega$	Number of iterations
θ	ϵ/ω	Activation probability
m	21	Used to control codegrees

We will use the following notation:

$$\begin{aligned}
N_H^i(u) &= \{v \in V(H^i) - u : \exists e \in H^i \text{ with } u, v \in e\} \\
N_H^i(u, v) &= \{w \in V(H^i) - \{u, v\} : \{u, v, w\} \in H^i\} \\
N_c^i(u) &= \{v \in V(G_c^i) - u : \exists e \in G_c^i \text{ with } u, v \in e\} \\
N^i(u) &= N_H^i(u) \cup \cup_c N_c^i(u) \\
N_G^0(u) &= \{v \in V(G) : uv \in E(G)\} \\
d_H^i(u) &= |\{e \in H^i : u \in e\}| \\
d_H^i(u, v) &= |\{e \in H^i : u, v \in e\}| \\
d_{G_c}^i(u) &= |\{v \in G_c^i : uv \in G_c\}|.
\end{aligned}$$

At the beginning of iteration i of the algorithm, we also define the following parameters:

$$\begin{aligned}
w(p_u^i) &= \sum_{c \in C(u)} p_u^i(c) \\
f_u^i(c) &= \sum_{v : uv \in G_c^i} p_v^i(c)
\end{aligned}$$

$$
\begin{aligned}
f_u^i &= \sum_{c\in C(u)} \sum_{v:uv\in G_c^i} p_u^i(c)p_v^i(c)\\
e_{uvw}^i &= \sum_{c\in C(u)} p_u^i(c)p_v^i(c)p_w^i(c)\\
e_u^i &= \sum_{\{v,w\}:uvw\in H^i} e_{uvw}^i\\
e_u^i(c) &= \sum_{\{v,w\}:uvw\in H^i} p_v^i(c)p_w^i(c)\\
h_u^i &= -\sum_{c\in C(u)} p_u^i(c)\log p_u^i(c), \text{ where } x\log x := 0 \text{ if } x=0\ .
\end{aligned}
$$

Our analysis assumes that the parameters of the algorithm satisfy the following relations. All asymptotic notation assumes $\Delta \to \infty$.

(R1) $\theta \log(\hat{p}C) \geq 149$

(R2) $\omega = \Delta^{o(1)}$.

(R3) $\omega_0 > \omega^4$

(R4) $\epsilon \leq 1/72$

(R5) $\delta \leq \Delta^{6/10}$

(R6) $\Delta_2 \leq \sqrt{\Delta}\sqrt{\omega}$

(R7) $\Delta^{-1/2} \leq \hat{p} \leq \Delta^{-11/24}$.

The analysis in Section 4 only requires that (3.6), (3.7), (3.8), and (R1)-(R7) hold; the parameters ω, ϵ, $\hat{p}$, and ω_0 depend on the structure of the hypergraph. For instance, we will use the following bounds when applying the analysis to triangle-free hypergraphs:

$$\epsilon = 1/72 \qquad \omega = (1/24)(\epsilon/150)\log\Delta \qquad \hat{p} = \Delta^{-11/24} \qquad \omega_0 = 1/19\theta\hat{p}.$$

4 Analysis of Algorithm

Theorem 7. *If* (3.6), (3.7), (3.8), *and (R1)-(R7) hold and* $|C(u)| \leq C$ *for all vertices* u*, then the algorithm produces a proper list coloring of* $H \cup G$*.*

Proof. By Lemma 8, our algorithm proceeds for T iterations, coloring most of the vertices. Since Lemmas 8, 9 and 11 hold after iteration T, we may color the remaining vertices as described in Section 4.5. $\square$

Lemma 8 (Main Lemma)**.** *If* (3.6), (3.7), (3.8), *and (R1)-(R7) hold, then for each* $i = 0, 1, \ldots T$*, the following properties hold:*

(P1) $|1 - w(p_u^i)| \leq i/(T \log C)$.

(P2) $e_u^i \leq (1 - \theta/3)^i \omega + i/\omega^2$

(P3) $f_u^i \leq 16(1 - \theta/4)^i \omega$

(P4) $h_u^i \geq h_u^0 - 37\epsilon \sum_{j=0}^{i-1} (1 - \theta/4)^j$

(P5) $d_H^i(u) \leq (1 - \theta/3)^i \Delta$

(P6) $d_{G_c}^i(u) \leq 3i\theta\Delta^{5/4}\hat{p}$.

The proof of the Main Lemma relies on the next three lemmas.

Lemma 9. *For any* $i = 0, 1, \ldots T - 1$*, if* (3.6), (3.7), (3.8), *and (R1)-(R7) hold and* $|B^i(u)| \leq \epsilon/\hat{p}$ *for all* $u \in U^i$*, then there is an assignment of colors to the vertices in* U^i *so that the following properties hold:*

(Q1) $|w(p_u^{i+1}) - w(p_u^i)| \leq 1/(T \log C)$

(Q2) $e_{uvw}^{i+1} \leq e_{uvw}^i + 1/(\Delta\omega^2)$

(Q3) $f_u^{i+1} \leq f_u^i(1 - \theta/2) + 3\theta e_u^i + 1/\omega^2$

(Q4) $h_u^i - h_u^{i+1} \leq 2\theta(f_u^i + e_u^i) + 1/\omega^2$

(Q5) $d_H^{i+1}(u) \leq (1 - \theta/2) d_H^i(u) + \Delta^{19/20}$

(Q6) $d_{G_c}^{i+1}(u) \leq d_{G_c}^i(u) + 2\theta\Delta^{5/4}\hat{p}$.

Lemma 10. *If (Q1)-(Q6) hold for* i *and (P1)-(P6) hold for* i*, then (P1)-(P6) hold for* $i + 1$.

Lemma 11. *If (P1)-(P6) hold for* $i + 1$ *and (R1) holds, then* $|B^{i+1}(u)| \leq \epsilon/\hat{p}$.

4.1 Proof of Main Lemma

The proof relies on Lemmas 9, 10 and 11. Assuming these lemmas, we proceed inductively as follows: properties (P1)-(P6) hold for $i = 0$ ((P3) holds by (R6)). Assume (P1)-(P6) hold for i. By Lemma 11, $|B^i(u)| \leq \epsilon/\hat{p}$, so by Lemma 9, (Q1)-(Q6) hold for i. Thus Lemma 10 implies (P1)-(P6) hold for $i + 1$.

4.2 Proof of Lemma 10

Proof of (P1). By (P1) (for i) and (Q1),

$$\begin{aligned}|1-w(p_u^{i+1})| &= |1-w(p_u^i)+w(p_u^i)-w(p_u^{i+1})| \\ &\le |1-w(p_u^i)|+|w(p_u^{i+1})-w(p_u^i)| \\ &\le (i+1)/(T\log C).\end{aligned}$$

Proof of (P5). Recall that (see [19], pg. 434)

$$(1-p)^n \to e^{-pn} \text{ if } p^2n\to 0 \text{ as } n\to\infty.$$

Since $\theta^2 T = o(1)$,

$$(\theta\Delta/6)(1-\theta/3)^T \to (\theta\Delta/6)e^{-T\theta/3} = (\theta\Delta/6)e^{-\frac{5}{3}\log\omega} > \Delta^{19/20}. \tag{4.1}$$

Using (P5) (for i),

$$\begin{aligned} d_H^{i+1}(u) \overset{(Q5)}{\le} (1-\theta/2)d_H^i(u)+\Delta^{19/20} &\overset{(P5)}{\le} (1-\theta/2)(1-\theta/3)^i\Delta+\Delta^{19/20} \\ &= (1-\theta/3)^{i+1}\Delta-\frac{\theta}{6}(1-\theta/3)^i\Delta+\Delta^{19/20} \\ &\le (1-\theta/3)^{i+1}\Delta-\frac{\theta}{6}(1-\theta/3)^T\Delta+\Delta^{19/20} \\ &\overset{(4.1)}{<} (1-\theta/3)^{i+1}\Delta.\end{aligned}$$

Proof of (P2). By (Q2),

$$e_{uvw}^{i+1} \le e_{uvw}^0+(i+1)/\Delta\omega^2 \le C(1/C^3)+(i+1)/\Delta\omega^2 = \omega/\Delta+(i+1)/\Delta\omega^2.$$

So by (P5) (for $i+1$),

$$e_u^{i+1} = \sum_{uvw} e_{uvw}^{i+1} \le (1-\theta/3)^{i+1}\Delta(\omega/\Delta+(i+1)/\Delta\omega^2) \le (1-\theta/3)^{i+1}\omega+(i+1)/\omega^2.$$

Proof of (P3). Since $\theta^2T=o(1)$,

$$\theta\omega(1-\theta/4)^T \to \theta\omega e^{-\theta T/4} = \epsilon e^{-\frac{5}{4}\log\omega} = \epsilon\omega^{-5/4} > 3(\theta T+1/3)\omega^2. \tag{4.2}$$

Using this with (P3) and (P2) (for i),

$$\begin{aligned} f_u^{i+1} &\overset{(Q3)}{\le} f_u^i(1-\theta/2)+3\theta e_u^i+1/\omega^2 \\ &\overset{(P3)}{\le} 16(1-\theta/4)^i\omega(1-\theta/2)+3\theta e_u^i+1/\omega^2 \\ &\overset{(P2)}{\le} 16(1-\theta/4)^i\omega(1-\theta/2)+3\theta\omega(1-\theta/3)^i+3\theta T/\omega^2+1/\omega^2\end{aligned}$$

$$
\begin{aligned}
&= 16(1-\theta/4)^i\omega(1-\theta/4-\theta/4) + \theta\omega(1-\theta/3)^i + 3(\theta T + 1/3)/\omega^2 \\
&= 16(1-\theta/4)^{i+1}\omega - 4\theta\omega(1-\theta/4)^i + \theta\omega(1-\theta/3)^i + 3(\theta T + 1/3)/\omega^2 \\
&< 16(1-\theta/4)^{i+1}\omega - \theta\omega(1-\theta/4)^i + 3(\theta T + 1/3)/\omega^2 \\
&\overset{(4.2)}{<} 16(1-\theta/4)^{i+1}\omega.
\end{aligned}
$$

Proof of (P4). Again using $\theta^2 T = o(1)$,

$$
\omega(1-\theta/4)^i \geq \omega(1-\theta/4)^T \to \omega e^{-\theta T/4} = \omega^{-1/4} > T/\omega^2. \tag{4.3}
$$

Since $T = (5\omega/\epsilon)\log\omega$, this implies

$$
\epsilon(1-\theta/4)^i > (5\log\omega)/\omega^2 > 1/\omega^2. \tag{4.4}
$$

Therefore, using $\epsilon = \omega\theta$ and (P4) (for i),

$$
\begin{aligned}
h_u^{i+1} &\overset{(Q4)}{\geq} h_u^i - 2\theta(f_u^i + e_u^i) - 1/\omega^2 \\
&\overset{(P3)}{\geq} h_u^i - 2\theta(16(1-\theta/4)^i\omega + e_u^i) - 1/\omega^2 \\
&\overset{(P2)}{\geq} h_u^i - 2\theta(16(1-\theta/4)^i\omega + (1-\theta/3)^i\omega + T/\omega^2) - 1/\omega^2 \\
&\geq h_u^i - 2\theta(17(1-\theta/4)^i\omega + T/\omega^2) - 1/\omega^2 \\
&\overset{(4.3)}{>} h_u^i - 2\theta(18(1-\theta/4)^i\omega) - 1/\omega^2 \\
&= h_u^i - 36\epsilon(1-\theta/4)^i - 1/\omega^2 \\
&\overset{(4.4)}{\geq} h_u^i - 37\epsilon(1-\theta/4)^i \\
&\overset{(P4)}{\geq} h_u^0 - 37\epsilon\sum_{j=0}^{i-1}(1-\theta/4)^j - 37\epsilon(1-\theta/4)^i \\
&= h_u^0 - 37\epsilon\sum_{j=0}^{i}(1-\theta/4)^j.
\end{aligned}
$$

Proof of (P6). By (R6) and (R7), $\Delta_2 < \theta\Delta^{5/4}\hat{p}$. Using this with (Q6),

$$
d_{G_c}^{i+1}(u) \overset{(Q6)}{\leq} \Delta_2 + 2(i+1)\theta\Delta^{5/4}\hat{p} \leq 3(i+1)\theta\Delta^{5/4}\hat{p}.
$$

4.3 Proof of Lemma 11

First,

$$
\begin{aligned}
|B^{i+1}(u)|\hat{p}\log(\hat{p}C) = \sum_{c\in B^{i+1}(u)} \hat{p}\log(\hat{p}C) &= \sum_{c\in B^{i+1}(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C) \\
&\leq \sum_{c\in C(u)} p_u^{i+1}(c)\log(p_u^{i+1}(c)C)
\end{aligned}
$$

$$= \sum_{c \in C(u)} p_u^{i+1}(c) \log p_u^{i+1}(c) + \sum_{c \in C(u)} p_u^{i+1}(c) \log C$$
$$= -h_u^{i+1} + \log C \sum_{c \in C(u)} p_u^{i+1}(c). \tag{4.5}$$

Using $p_u^0(c) = 1/C$ for all $c \in C(u)$,

$$\begin{aligned} h_u^0 &= -\sum_{c \in C(u)} p_u^0(c) \log p_u^0(c) \\ &= \log C \sum_{c \in C(u)} p_u^0(c) \\ &= \log C \sum_{c \in C(u)} (p_u^0(c) - p_u^{i+1}(c)) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &= \log C(1 - w(p_u^{i+1})) + \log C \sum_{c \in C(u)} p_u^{i+1}(c) \\ &\overset{\text{(P1)}}{\geq} -1 + \log C \sum_{c \in C(u)} p_u^{i+1}(c). \end{aligned}$$

Using $\sum_{j=0}^{i} (1 - \theta/4)^j \leq 4/\theta$, the above inequality, and inequality (4.5),

$$\begin{aligned} h_u^{i+1} \overset{\text{(P4)}}{\geq} h_u^0 - 37\epsilon \sum_{j=0}^{i} (1-\theta/4)^j \geq h_u^0 - 148\epsilon/\theta \geq \log C \sum_{c \in C(u)} p_u^{i+1}(c) - 149\epsilon/\theta \\ \overset{(4.5)}{\geq} h_u^{i+1} + |B^{i+1}(u)| \hat{p} \log(\hat{p}C) - 149\epsilon/\theta. \end{aligned}$$

So

$$|B^{i+1}(u)| \leq \frac{149\epsilon}{\theta \hat{p} \log(\hat{p}C)} \overset{\text{(R1)}}{\leq} \epsilon/\hat{p}.$$

4.4 Proof of Lemma 9

Throughout this section, we drop the notation $i+1$ and i, and use, for instance, $p'_u(c)$ and $p_u(c)$ to denote values in iterations $i+1$ and i, respectively.

We are going to apply the Local Lemma. Our probability space is determined by coin flips at each vertex which determine the random variables $\gamma_u(c)$ and $\eta_u(c)$. Recall that

$$N(u) = N^i(u) = N_H^i(u) \cup \cup_c N_c^i(u),$$

where

$$N_c^i(u) = \{v \in V(G_c^i) - u : \exists e \in G_c^i \text{ with } u, v \in e\}.$$

The random variable $p'_u(c)$ is determined by the coin flips in $N(u) + u$. Thus an event "(Qk) fails to hold for u" (or "(Q6) fails to hold for uvw") does not depend on the coin flips outside of $N(N(u))$ (or $N(N(u)) \cup N(N(v)) \cup N(N(w))$). Consequently, if $v \notin N(N(N(N(N(N(u))))))$ (sixth neighborhood), then the events "(Qk) fails to hold for u (or ubc)" and "(Ql) fails to hold for v (or vwx)" are mutually independent. Since $|N(u)| \leq 2\Delta + \Delta_2$, (R6) implies

$$|N(N(N(N(N(N((u)))))))| \leq (2\Delta + \Delta_2))^6 < (3\Delta)^6.$$

To apply the Local Lemma, it therefore suffices to show that the probability that each (Qk) fails is less than $(3\Delta)^{-6}/4$. We prove this for (Q1), (Q2), (Q4), and (Q6) first, and then move on to (Q3) and (Q5).

Proof of (Q1). By (3.5), $\mathbf{E}[p'_u(c)] = p_u(c)$ for each color c. By linearity of expectation,

$$\mathbf{E}[w(p'_u)] = w(p_u).$$

By (R7), $C\hat{p}^2 \leq \Delta^{-10/24}$. Since $w(p'_u)$ is the sum of C independent non-negative random variables, each bounded by $\hat{p}$, Theorem 3 implies

$$\Pr[|w(p'_u) - w(p_u)| \geq 1/(T \log C)] \leq 2e^{-2/(C\hat{p}^2(T\log C)^2)} < 2e^{-7\log\Delta}.$$

Proof of (Q2). Suppose $uvw \in H$. We first prove

$$\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] \leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0). \tag{4.6}$$

Assume that $p'_u(c)$, $p'_v(c)$, and $p'_w(c)$ are determined by (3.3). If $c \in L(u) \cup L(v) \cup L(w)$, then $p'_u(c)p'_v(c)p'_w(c) = 0$, so by (3.6),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\frac{p_w(c)}{q_w(c)}\Pr[c \notin L(u) \cup L(v) \cup L(w)] \\ &\leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0).\end{aligned}$$

Suppose $p'_u(c)$ and $p'_v(c)$ are determined by (3.3), and $p'_w(c)$ is determined by (3.4). Then $p'_w(c)$ is independent of $p'_u(c)$ and $p'_v(c)$, so by (3.7),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &= \mathbf{E}[p'_u(c)p'_v(c)]\,\mathbf{E}[p'_w(c)] \\ &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u) \cup L(v)]p_w(c) \\ &\leq p_u(c)p_v(c)p_w(c)(1 + 1/\omega_0).\end{aligned}$$

If at least two of $p'_u(c)$, $p'_v(c)$, and $p'_w(c)$ are determined by (3.4), then all three are independent of each other, and

$$\mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] = p_u(c)p_v(c)p_w(c),$$

finishing the proof of (4.6).

By definition, $e^0_{uvw} \le C/C^3 = \omega/\Delta$. By (R3), $\omega^3 + T < \omega_0/2$. So by (Q2) (for i),

$$e_{uvw}/\omega_0 \overset{(Q2)}{\le} (e^0_{uvw} + \frac{i}{\Delta\omega^2})\frac{1}{\omega_0} \le (\frac{\omega}{\Delta} + \frac{T}{\Delta\omega^2})\frac{1}{\omega_0} = \frac{\omega^3+T}{\omega_0}\frac{1}{\Delta\omega^2} < 1/(2\Delta\omega^2).$$

So by (4.6),

$$\begin{aligned}
\mathbf{E}[e'_{uvw}] = \sum_{c\in C(u)} \mathbf{E}[p'_u(c)p'_v(c)p'_w(c)] &\le \sum_{c\in C(u)} p_u(c)p_v(c)p_w(c)(1+1/\omega_0)\\
&= e_{uvw}(1+1/\omega_0)\\
&< e_{uvw} + 1/(2\Delta\omega^2).
\end{aligned}$$

Now e'_{uvw} is the sum of C independent random variables, each bounded by $\hat{p}^3$. By (R7), $\Delta^2 C\hat{p}^6 \le \Delta^{-6/24}$. Thus Theorem 3 yields

$$\begin{aligned}
\Pr[e'_{uvw} \ge e_{uww} + 1/(\Delta\omega^2)] &\le \Pr[e'_{uvw} \ge e_{uvw} + 1/(2\Delta\omega^2) + 1/(2\Delta\omega^2)]\\
&\le \Pr[e'_{uvw} \ge \mathbf{E}[e'_{uvw}] + 1/(2\Delta\omega^2)]\\
&< e^{-2/(4\Delta^2\omega^4 C\hat{p}^6)}\\
&< e^{-7\log\Delta}.
\end{aligned}$$

Proof of (Q4). By (3.3) and (3.4), $p'_u(c) = p_u(c)\,\mathbf{I}[A]/\Pr[A]$ for some event A. Thus, using $x\log x = 0$ for $x \in \{0,1\}$,

$$\begin{aligned}
\mathbf{E}[p'_u(c)\log p'_u(c)] &= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log(p_u(c)\,\mathbf{I}[A]/\Pr[A])]\\
&= \mathbf{E}[p_u(c)\,\mathbf{I}[A]/\Pr[A]\log p_u(c) + p_u(c)\,\mathbf{I}[A]/\Pr[A]\log(\mathbf{I}[A]/\Pr[A])]\\
&= \frac{p_u(c)\log p_u(c)}{\Pr[A]}\mathbf{E}[\mathbf{I}[A]] + \frac{p_u(c)}{\Pr[A]}\mathbf{E}[\mathbf{I}[A]\log(\mathbf{I}[A]/\Pr[A])]\\
&= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\mathbf{E}[\mathbf{I}[A]\log\mathbf{I}[A]] - \frac{p_u(c)}{\Pr[A]}\mathbf{E}[\mathbf{I}[A]\log\Pr[A]]\\
&= p_u(c)\log p_u(c) + \frac{p_u(c)}{\Pr[A]}\mathbf{E}[0] - p_u(c)\log\Pr[A]\\
&= p_u(c)\log p_u(c) - p_u(c)\log\Pr[A].
\end{aligned}$$

Recall that

$$q_u(c) = \Pr[\bigcap_{\substack{\{v,w\}:\\ uvw\in H}} (\gamma_v(c)=0 \cup \gamma_w(c)=0) \bigcap_{v:uv\in G_c} \gamma_v(c)=0]$$

Also, $1-rx \geq (1-x)^r$ for $r, x \in (0,1)$. Finally, $\mathbf{I}[\gamma_v(c)=0 \cup \gamma_w(c)=0]$ and $\mathbf{I}[\gamma_v(c)=0]$ are increasing functions of the indicators $\mathbf{I}[\gamma_v(c)=0]$, so by the FKG inequality,

$$\begin{aligned}
q_u(c) &= \mathbf{E}\left[\prod_{uvw\in H} \mathbf{I}[\gamma_v(c)=0\cup\gamma_w(c)=0] \prod_{uv\in G_c} \mathbf{I}[\gamma_v(c)=0]\right] \\
&\overset{\text{FKG}}{\geq} \prod_{uvw\in H} \mathbf{E}[\mathbf{I}[\gamma_v(c)=0\cup\gamma_w(c)=0]] \prod_{uv\in G_c} \mathbf{E}[\mathbf{I}[\gamma_v(c)=0]] \\
&= \prod_{uvw\in H} \Pr[\gamma_v(c)=0\cup\gamma_w(c)=0] \prod_{uv\in G_c} \Pr[\gamma_v(c)=0] \\
&= \prod_{uvw\in H} (1-\theta^2 p_v(c)p_w(c)) \prod_{uv\in G_c} (1-\theta p_v(c)) \\
&\geq \prod_{uvw\in H} (1-\theta)^{\theta p_v(c)p_w(c)} \prod_{uv\in G_c} (1-\theta)^{p_v(c)}.
\end{aligned}$$

By the algorithm, $\Pr[A] \geq q_u(c)$. Also, $\log(1-x) \geq -x-x^2$ for $x\in[0,1/3]$. Combining these inequalities with the previous inequality, we obtain

$$\begin{aligned}
\log\Pr[A] \geq \log q_u(c) &\geq \log\Big(\prod_{uvw\in H}(1-\theta)^{\theta p_v(c)p_w(c)} \prod_{uv\in G_c}(1-\theta)^{p_v(c)}\Big) \\
&= \sum_{uvw\in H} \theta p_v(c)p_w(c)\log(1-\theta) + \sum_{uv\in G_c} p_v(c)\log(1-\theta) \\
&\geq \sum_{uvw\in H} \theta p_v(c)p_w(c)(-\theta-\theta^2) + \sum_{uv\in G_c} p_v(c)(-\theta-\theta^2) \\
&= (-\theta^2-\theta^3)\sum_{uvw\in H} p_v(c)p_w(c) + (-\theta-\theta^2)\sum_{uv\in G_c} p_v(c) \\
&= -(\theta^2+\theta^3)e_u(c) - (\theta+\theta^2)f_u(c).
\end{aligned}$$

Therefore, using the definition of h_u and $\theta < 1/2$,

$$\begin{aligned}
\mathbf{E}[h_u - h_u'] &= h_u + \sum_{c\in C(u)} \mathbf{E}[p_u'(c)\log p_u'(c)] \\
&= h_u + \sum_{c\in C(u)} p_u(c)\log p_u(c) - \sum_{c\in C(u)} p_u(c)\log\Pr[A]) \\
&= -\sum_{c\in C(u)} p_u(c)\log\Pr[A] \\
&\leq \sum_{c\in C(u)} p_u(c)((\theta+\theta^2)f_u(c) + (\theta^2+\theta^3)e_u(c)) \\
&= (\theta+\theta^2)f_u + (\theta^2+\theta^3)e_u \\
&< 2\theta(f_u+e_u).
\end{aligned}$$

The terms in $\sum_c -p_u'(c)\log p_u'(c)$ are independent and, since $-x\log x$ is increasing for $0<x\leq\hat{p}$, bounded by $-\hat{p}\log\hat{p}$. Also $x^2\log^2 x$ is increasing, so by (R7), $C(-\hat{p}\log\hat{p})^2 \leq$

$\Delta^{-10/24-o(1)}$. Thus, by Theorem 3,

$$\Pr[h_u - h'_u \geq 2\theta(f_u + e_u) + 1/\omega^2] < e^{-2/(\omega^4 C(-\hat{p}\log\hat{p})^2)} < e^{-7\log\Delta}.$$

Proof of (Q6). Fix $c \in C(u)$. For each $v \in N_H(u)$, set

$$X_v = d_H(u,v)\gamma_v(c),$$

and set

$$X = \sum_{v \in N_H(u)} X_v.$$

Then

$$\mathbf{E}[X] = \sum_{v \in N_H(u)} d_H(u,v)p_v(c)\theta \leq \hat{p}\theta \sum_{v \in N_H(u)} d_H(u,v) \leq 2\Delta\hat{p}\theta.$$

Since the X_v are independent from each other (because the $\gamma_v(c)$ are independent), and $x(1-x)$ is increasing for $x < 1/2$,

$$\begin{aligned}
\mathbf{Var}[X] = \sum_{v \in N_H(u)} \mathbf{Var}[X_v] &= \sum_{v \in N_H(u)} (\mathbf{E}[X_v^2] - \mathbf{E}[X_v]^2) \\
&= \sum_{v \in N_H(u)} (d_H(u,v)^2 p_v(c)\theta - d_H(u,v)^2 p_v(c)^2\theta^2) \\
&\leq \sum_{v \in N_H(u)} d_H(u,v)^2 \hat{p}\theta(1-\hat{p}\theta) \\
&= \hat{p}\theta(1-\hat{p}\theta) \sum_{v \in N_H(u)} d_H(u,v)^2 \\
&\leq \hat{p}\theta(1-\hat{p}\theta)\delta \sum_{v \in N_H(u)} d_H(u,v) \\
&= \hat{p}\theta(1-\hat{p}\theta)2\Delta\delta \\
&< \hat{p}\theta 2\Delta\delta.
\end{aligned}$$

If $uv \notin G_c$ and $uv \in G'_c$, then there exists an edge $uvw \in H$ such that $\gamma_w(c) = 1$. Hence

$$d'_{G_c}(u) - d_{G_c}(u) \leq \sum_{uvw \in H} (\gamma_v(c) + \gamma_w(c)) = \sum_{v \in N_H(u)} d_H(u,v)\gamma_v(c) = X.$$

Since $\hat{p} \geq \Delta^{-1/2}$ and $\delta \leq \Delta^{6/10}$ ((R7) and (R5)), $\Delta^{5/4}\hat{p}/\delta \geq \Delta^{3/20}$. Applying Theorem 4 (with $b = \delta$),

$$\begin{aligned}
\Pr[d'_{G_c}(u) - d_{G_c}(u) \geq 2\Delta^{5/4}\hat{p}\theta] &\leq \Pr[X \geq \Delta^{5/4}\hat{p}\theta + \Delta^{5/4}\hat{p}\theta] \\
&\leq \Pr[X \geq \mathbf{E}[X] + \Delta^{5/4}\hat{p}\theta] \\
&\leq e^{-\Delta^{10/4}\hat{p}^2\theta^2/(4\hat{p}\theta\Delta\delta + \delta\Delta^{5/4}\hat{p}\theta)}
\end{aligned}$$

$$
\begin{aligned}
&< e^{-\Delta^{10/4}\hat{p}^2\theta^2/5\delta\Delta^{5/4}\hat{p}\theta} \\
&= e^{-\Delta^{5/4}\hat{p}\theta/5\delta} \\
&< e^{-7\log\Delta}.
\end{aligned}
$$

We now prove (Q3) and (Q5). The following two claims will be used in both proofs.

Claim 12. *For any $v \in U$ and $c \in C(v)$,*

$$\Pr[v \notin U' | c \notin L(v)] \geq \Pr[v \notin U'] \geq 3\theta/4,$$

and if $uv \in G_c$, then

$$
\begin{aligned}
\Pr[v \notin U' | c \notin L(u)] &\geq \Pr[v \notin U'] - \theta\hat{p} \geq 5\theta/8, \\
\Pr[v \notin U' | c \notin L(u) \cup L(v)] &\geq \Pr[v \notin U'] - \theta\hat{p} \geq 5\theta/8.
\end{aligned}
$$

Proof of claim. The vertex v is colored (i.e., $v \notin U'$) if and only if for some color $d \notin B(v)$, $\gamma_v(d) = 1$ and $d \notin L(v)$. Let R_d denote the event that $\gamma_v(d) = 1$ and $d \notin L(v)$. If $c \in B(v)$, then v cannot be colored c, so the event $v \notin U'$ is independent of the events $c \notin L(v)$ and $c \notin L(u)$; hence

$$\Pr[v \notin U'] = \Pr[v \notin U' | c \notin L(v)] = \Pr[v \notin U' | c \notin L(u)] = \Pr[v \notin U' | c \notin L(u) \cup L(v)].$$

Otherwise,

$$
\begin{aligned}
\Pr[v \notin U' | c \notin L(v)] &= \frac{\Pr[v \notin U', c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[\cup_{d \in C(v)-B(v)} R_d, c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup R_c), c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup \gamma_v(c) = 1), c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \frac{\Pr[(\cup_{d \in C(v)-B(v)-c} R_d \cup \gamma_v(c) = 1)] \Pr[c \notin L(v)]}{\Pr[c \notin L(v)]} \\
&= \Pr[(\cup_{d \in C(v)-B(v)-c} R_d) \cup (\gamma_v(c) = 1)] \\
&\geq \Pr[(\cup_{d \in C(v)-B(v)-c} R_d) \cup R_c] \\
&= \Pr[v \notin U'].
\end{aligned}
$$

Suppose $uv \in G_c$. If $c \notin L(u)$, then $\gamma_w(c) = 0$ for all $w \in N_{G_c}(u)$, so in particular, $\gamma_v(c) = 0$. Consequently,

$$\Pr[R_c | c \notin L(u) \cup L(v)] = \Pr[\gamma_v(c) = 1 \cap c \notin L(v) | c \notin L(u) \cup L(v)] = 0.$$

So by the independence of colors and the inequality

$$\Pr[\cup_{d\in C(v)-B(v)} R_d] \le \Pr[\cup_{d\in C(v)-B(v)-c} R_d] + \Pr[R_c],$$

we obtain

$$\begin{aligned}
\Pr[v \notin U' | c \notin L(u) \cup L(v)] &= \Pr[\cup_{d\in C(v)-B(v)} R_d | c \notin L(u) \cup L(v)] \\
&= \Pr[\cup_{d\in C(v)-B(v)-c} R_d] \\
&\ge \Pr[\cup_{d\in C(v)-B(v)} R_d] - \Pr[R_c] \\
&\ge \Pr[v \notin U'] - \theta\hat{p}.
\end{aligned}$$

Since we only used the condition $c \notin L(u)$, this also implies

$$\Pr[v \notin U' | c \notin L(u)] \ge \Pr[v \notin U'] - \theta\hat{p}.$$

To finish the proof of the claim, we now show $\Pr[v \notin U'] \ge 3\theta/4$. First,

$$\begin{aligned}
\Pr[v \notin U'] &= \Pr[\cup_{d\in C(v)-B(v)} R_d] \\
&\ge \sum_{d\in C(v)-B(v)} \Pr[R_d] - \sum_{d,d'\in C(v)-B(v)} \Pr[R_d]\Pr[R_{d'}] \\
&= \sum_{d\in C(v)-B(v)} \theta p_v(d) q_v(d) - \sum_{d,d'\in C(v)-B(v)} \theta^2 p_v(d) p_v(d') q_v(d) q_v(d') \\
&\ge \theta \sum_{d\in C(v)} p_v(d) q_v(d) - \theta \sum_{d\in B(v)} p_v(d) q_v(d) - \theta^2 \sum_{d,d'\in C(v)-B(v)} p_v(d) p_v(d') \\
&\ge \theta \sum_{d\in C(v)} p_v(d) q_v(d) - \theta |B(v)| \hat{p} - \theta^2 \sum_{d,d'\in C(v)-B(v)} p_v(d) p_v(d').
\end{aligned}$$

By (3.2),

$$\begin{aligned}
q_v(d) &\ge 1 - \sum_{uvw\in H} \theta^2 p_u(d) p_w(d) - \sum_{uv\in G_d} \theta p_u(d) \\
&= 1 - \theta^2 \sum_{uvw\in H} p_u(d) p_w(d) - \theta \sum_{uv\in G_d} p_u(d) \\
&= 1 - \theta^2 e_v(d) - \theta f_v(d).
\end{aligned}$$

Since $\sum_{d\in C(v)} p_v(d) \le \sqrt{2}$ (by (P1) and $1/\log C = o(1)$),

$$\theta^2 \sum_{d,d'\in C(v)-B(v)} p_v(d) p_v(d') \le \frac{1}{2}\theta^2 \sum_{d\in C(v)} \sum_{d'\in C(v)-d} p_v(d) p_v(d') \le \frac{1}{2}\theta^2 (\sum_{d\in C} p_v(d))^2 \le \theta^2.$$

By our lemma's assumption, $|B(v)| \le \epsilon/\hat{p}$. By (P3), $f_v < 16\omega$, so $\theta f_v < 16\epsilon$. By (P2), $e_v \le \omega + T/\omega^2$, so by definition of T and θ, $\theta^2 e_v \le \epsilon/3$. Using these three inequalities, $\sum_{d\in C(v)} p_v(d) \ge (1-\epsilon/3)$, and (R4), we finally obtain

$$\Pr[v \notin U'] \ge \theta \sum_{d\in C(v)} p_v(d)(1 - \theta^2 e_v(d) - \theta f_v(d)) - \theta |B(v)| \hat{p} - \theta^2$$

$$
\begin{aligned}
&= \theta \sum_{d\in C(v)} p_v(d) - \theta^3 \sum_{d\in C(v)} p_v(d)e_v(d) - \theta^2 \sum_{d\in C(v)} p_v(d)f_v(d) - \theta|B(v)|\hat{p} - \theta^2 \\
&\geq \theta \sum_{d\in C(v)} p_v(d) - \theta^3 \sum_{d\in C(v)} p_v(d)e_v(d) - \theta^2 \sum_{d\in C(v)} p_v(d)f_v(d) - \theta\epsilon - \theta^2 \\
&= \theta \sum_{d\in C(v)} p_v(d) - \theta^3 e_v - \theta^2 f_v - \theta\epsilon - \theta^2 \\
&\geq \theta(1-\epsilon/3) - \theta\epsilon/3 - 16\theta\epsilon - \theta\epsilon - \theta\epsilon/3 \\
&= \theta(1-18\epsilon) \\
&\geq 3\theta/4.
\end{aligned}
$$

□

Recall that m is a fixed constant.

Claim 13. *For each $l = 0, \ldots, m-2$, let*

$$N^0(u,l) = \{v \in N^0_H(u) - N^0_G(u) : \Delta^{l/2m} < d^0_H(u,v) \leq \Delta^{(l+1)/2m}\},$$

and for $l = m-1$, let

$$N^0(u,l) = \{v \in N^0_H(u) : d^0_H(u,v) > \Delta^{l/2m}\} \cup N^0_G(u).$$

For each l and color c, let $\mathcal{A}_{c,l}$ be the event that $\gamma_v(c) = 1$ for at most $\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Let $\mathcal{A}$ denote the event that $\mathcal{A}_{c,l}$ holds for all l and c. Then

$$\Pr[\bar{\mathcal{A}}] \leq e^{-10\log\Delta}.$$

Proof of claim. Suppose $l < m-1$. Since each $v \in N^0(u,l)$ contributes at least $\Delta^{l/2m}$ edges to $d^0_H(u)$, and each edge is counted at most twice,

$$|N^0(u,l)| \leq 2\Delta/\Delta^{l/2m} = 2\Delta^{1-l/2m}.$$

If $l = m-1$,

$$|N^0(u,l)| \leq 2\Delta/\Delta^{l/2m} + \Delta_2 = 2\Delta^{1-l/2m} + \Delta_2 \overset{\text{(R6)}}{<} 3\Delta^{1-l/2m}.$$

Thus $|N^0(u,l)| < 3\Delta^{1-l/2m}$ for each l.

Since $\Pr[\gamma_v(c) = 1] \leq \hat{p}\theta$ and $3e\theta < 1/e$,

$$
\begin{aligned}
\Pr[\bar{\mathcal{A}}_{c,l}] \leq \binom{|N^0(u,l)|}{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} &\leq \binom{3\Delta^{1-l/2m}}{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}} \\
&\leq (\frac{3e}{\hat{p}})^{\Delta^{1-l/2m}\hat{p}} (\hat{p}\theta)^{\Delta^{1-l/2m}\hat{p}}
\end{aligned}
$$

$$
\begin{aligned}
&= (3e\theta)^{\Delta^{1-l/2m}\hat{p}} \\
&< e^{-\Delta^{1-l/2m}\hat{p}} \\
&\overset{(\mathrm{R7})}{\leq} e^{-\Delta^{(m+1)/2m}\Delta^{-1/2}} \\
&= e^{-\Delta^{1/2m}}.
\end{aligned}
$$

So by the union bound,

$$
\Pr[\bar{\mathcal{A}}] \leq Cme^{-\Delta^{1/2m}} \leq e^{-10\log\Delta}.
$$

$\square$

Proof of (Q3). Observe that

$$
\begin{aligned}
f'_u &= \sum_{c\in C(u)} \sum_{v:uv\in G'_c} p'_u(c)p'_v(c) \\
&= \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[uv \in G'_c] + \sum_{c\in C(u)} \sum_{\substack{v:uv\notin G_c,\\ uv\in G'_c}} p'_u(c)p'_v(c) \\
&\leq \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[v \in U'] \\
&+ \sum_{c\in C(u)} \sum_{\substack{\{v,w\}:\\ uvw\in H}} (p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c) = 1] + p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c) = 1]) \\
&= D_1 + D_2,
\end{aligned}
$$

where

$$
D_1 = \sum_{c\in C(u)} \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[v \in U'],
$$

and

$$
D_2 = \sum_{c\in C(u)} \sum_{\substack{\{v,w\}:\\ uvw\in H}} (p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c) = 1] + p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c) = 1]).
$$

4.4.1 Bound on D_1

To bound D_1, we first prove that for $uv \in G_c$,

$$
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] \leq p_u(c)p_v(c)(1 - 9\theta/16). \tag{4.7}
$$

Note that by (R3),

$$
1/\omega_0 < 1/\omega^4 = o(\theta). \tag{4.8}
$$

First assume that $p'_u(c)$ and $p'_v(c)$ are determined by (3.3). If $c \in L(u) \cup L(v)$, then $p'_u(c)p'_v(c) = 0$, so using (3.8), Claim 12, and then (4.8),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)|v \in U']\Pr[v \in U'] \\
&\le \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u) \cup L(v)|v \in U']\Pr[v \in U'] \\
&= \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[v \in U'|c \notin L(u) \cup L(v)]\Pr[c \notin L(u) \cup L(v)] \\
&\overset{(3.8)}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)\Pr[v \in U'|c \notin L(u) \cup L(v)] \\
&\overset{\text{C.12}}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(4.8)}{\le} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

Suppose $p'_u(c)$ is determined by (3.3) and $p'_v(c)$ is determined by (3.4). Then $p'_u(c)$ and $p'_v(c)$ are independent of each other, and $p'_v(c)$ is independent of the event $v \in U'$, so

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)|v \in U']\Pr[v \in U'] \\
&= \mathbf{E}[p'_u(c)|v \in U']\,\mathbf{E}[p'_v(c)]\Pr[v \in U'] \\
&\overset{(3.5)}{\le} \mathbf{E}[p'_u(c)|v \in U']p_v(c)\Pr[v \in U'] \\
&\le \frac{p_u(c)}{q_u(c)}\Pr[c \notin L(u)|v \in U']\Pr[v \in U']p_v(c) \\
&= \frac{p_u(c)}{q_u(c)}\Pr[v \in U'|c \notin L(u)]\Pr[c \notin L(u)]p_v(c) \\
&= p_u(c)p_v(c)\Pr[v \in U'|c \notin L(u)] \\
&\overset{\text{C.12}}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(4.8)}{\le} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

Similarly, if $p'_u(c)$ is determined by (3.4) and $p'_v(c)$ is determined by (3.3),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &\le p_u(c)p_v(c)\Pr[v \in U'|c \notin L(v)] \\
&\overset{\text{C.12}}{\le} p_u(c)p_v(c)(1 + 1/\omega_0)(1 - 5\theta/8) \\
&\overset{(4.8)}{\le} p_u(c)p_v(c)(1 - 9\theta/16).
\end{aligned}$$

If $p'_u(c)$ and $p'_v(c)$ are both determined by (3.4),

$$\begin{aligned}
\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v \in U']] &= \mathbf{E}[p'_u(c)p'_v(c)]\Pr[v \in U'] \\
&= \mathbf{E}[p'_u(c)]\,\mathbf{E}[p'_v(c)]\Pr[v \in U'] \\
&\overset{(\text{C.12})}{\le} p_u(c)p_v(c)(1 - 3\theta/4) \\
&< p_u(c)p_v(c)(1 - 9\theta/16),
\end{aligned}$$

concluding the proof of (4.7).

By (4.7),

$$\begin{aligned}\mathbf{E}[D_1] &= \sum_{c\in C(u)}\sum_{v:uv\in G_c}\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[v\in U']]\\ &\le \sum_{c\in C(u)}\sum_{v:uv\in G_c} p_u(c)p_v(c)(1-9\theta/16)\\ &= f_u(1-9\theta/16).\end{aligned}$$

For $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

Then each T_c is a (vector valued) random variable, and the set of random variables $\{T_c : c \in C(u)\}$ are mutually independent and determine the variable D_1. We will now apply Corollary 6 with parameters:

- $\mathcal{B}_c = \{0,1\}^{2|N(N(u))|}$, for each $c \in C(u)$
- Independent random variables $T_c : \{c\} \to \{0,1\}^{2|N(N(u))|}$, for each $c \in C(u)$
- Events $\mathcal{A}_c = \cap_{l=1}^m \mathcal{A}_{c,l}$, for each $c \in C(u)$ (where $\mathcal{A}_{c,l}$ is from Claim 13)
- $\mathcal{A} = \prod_{c\in C(u)} \mathcal{A}_c$, for each $c \in C(u)$ (this is the same $\mathcal{A}$ as in Claim 13)
- D_1 (which is non-negative) in the role of Y, so that $D_1 : \prod_{c\in C(u)} \mathcal{B}_c \to \mathbb{R}$.
- $d_c = d_{G_c(u)}\hat{p}^2 + 2m\hat{p}^3\Delta^{1+1/2m}$, for each $c \in C(u)$.

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ such that x and x' differ only in coordinate c. Our goal is to show that $|D_1(x) - D_1(x')| \le d_c$. Note first that

$$\begin{aligned}D_1 &= \sum_{v:uv\in G_c} p'_u(c)p'_v(c)\,\mathbf{I}[v\in U'] + \sum_{l=0}^{m-1}\sum_{v\in N^0(u,l)}\mathbf{I}[v\in U']\sum_{\substack{d\in C(u)-c:\\ uv\in G_d}} p'_u(d)p'_v(d)\\ &:= D_{1,1} + \sum_{l=0}^{m-1}\sum_{v\in N^0(u,l)} D_{1,2}^v.\end{aligned}$$

Since $0 \le D_{1,1} \le d_{G_c}(u)\hat{p}^2$,

$$|D_{1,1}(x) - D_{1,1}(x')| \le d_{G_c}(u)\hat{p}^2. \tag{4.9}$$

Fix l. Let $v \in N^0(u,l)$. If $l \le m-2$, then $uv \in G_d$ only if there exists a vertex w such that $uvw \in H^0$ and w received color d in a previous round. Thus uv is in at most $d_H^0(u,v) \le \Delta^{(l+1)/2m}$ graphs G_d, which implies

$$|D_{1,2}^v(x) - D_{1,2}^v(x')| \le \Delta^{(l+1)/2m}\hat{p}^2, \quad l \le m-2. \tag{4.10}$$

If $l = m-1$, then uv is in at most C graphs G_d, so

$$|D^v_{1,2}(x) - D^v_{1,2}(x')| \leq C\hat{p}^2 \leq \Delta^{1/2}\hat{p}^2, \quad l = m-1. \tag{4.11}$$

Note that x and x' are vectors of length $|C(u)|$. The c^{th} coordinate of each vector is a vector of length $2|N(N(u))|$, and the entries in these vectors correspond to the values assigned to the random variables $\gamma_v(c)$ and $\eta_v(c)$ for all $v \in N(N(u))$. Let $x_c(\gamma, v)$ and $x'_c(\gamma, v)$ denote the values of $\gamma_v(c)$ corresponding to x and x', respectively.

Suppose $x_c(\gamma, v) = x'_c(\gamma, v) = 0$. Then v cannot be colored c during the current iteration. Thus changing the value of $\gamma_w(c)$ for any $w \in N(N(u)) - v$ does not affect whether or not v is colored. Therefore $\mathbf{I}[v \in U'](x) = \mathbf{I}[v \in U'](x')$. In addition, $p'_u(d)p'_v(d)(x) = p'_u(d)p'_v(d)(x')$ for any $d \in C(u) - c$. Thus $D^v_{1,2}(x) = D^v_{1,2}(x')$ if $x_c(\gamma, v) = x'_c(\gamma, v) = 0$.

By definition of $\mathcal{A}_{c,l}$, $x_c(\gamma, v) = 1$ for at most $\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u, l)$. Therefore $D^v_{1,2}(x) \neq D^v_{1,2}(x')$ for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u, l)$. So by (4.11) and (4.10),

$$\begin{aligned}\sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} |D^v_{1,2}(x) - D^v_{1,2}(x')| &\leq 2\Delta^{1+1/2m}\hat{p}^3 + \sum_{l=0}^{m-2}(2\Delta^{1-l/2m}\hat{p})(\Delta^{(l+1)/2m}\hat{p}^2) \\ &= 2m\Delta^{1+1/2m}\hat{p}^3.\end{aligned}$$

Combining this with (4.9),

$$|D_1(x) - D_1(x')| \leq d_{G_c}(u)\hat{p}^2 + 2m\Delta^{1+1/2m}\hat{p}^3 = d_c.$$

Since $\sum_{c \in C(u)} d_{G_c}(u) \leq \Delta + \Delta_2 < 2\Delta$ and, by (P6), $d_{G_c}(u) \leq 3T\theta\Delta^{5/4}\hat{p}$,

$$\begin{aligned}&\sum_{c \in C(u)} (d_{G_c}(u)\hat{p}^2 + 2m\hat{p}^3\Delta^{1+1/2m})^2 \\ &\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + \sum_{c \in C(u)} (\hat{p}^4 d_{G_c}(u)^2 + d_{G_c}(u)4m\hat{p}^5\Delta^{1+1/2m}) \\ &\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + \hat{p}^4 \sum_{c \in C(u)} d_{G_c}(u)^2 \\ &\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + 3\hat{p}^5T\theta\Delta^{5/4} \sum_{c \in C(u)} d_{G_c}(u) \\ &\leq 4Cm^2\hat{p}^6\Delta^{2+1/m} + 8m\hat{p}^5\Delta^{2+1/2m} + 6T\theta\hat{p}^5\Delta^{9/4}.\end{aligned}$$

By (R7), $\hat{p}^5\Delta^{9/4} \leq \Delta^{-1/24}$, $\hat{p}^5\Delta^{2+1/2m} \leq \Delta^{-15/56}$, and $C\hat{p}^6\Delta^{2+1/m} \leq \Delta^{-17/84}$. Together with Claim 13, Corollary 6 now implies

$$\begin{aligned}\Pr[D_1 > f_u(1-\theta/2) + 1/2\omega^2] &\leq \Pr[D_1 > f_u(1-9\theta/16)/\Pr[\mathcal{A}] + 1/2\omega^2] \\ &\leq \Pr[D_1 > \mathbf{E}[D_1]/\Pr[\mathcal{A}] + 1/2\omega^2]\end{aligned}$$

$$\overset{\text{C.6}}{\leq} e^{-1/4\omega^4(6T\theta\hat{p}^5\Delta^{9/4}+8m\hat{p}^5\Delta^{2+1/2m}+4Cm^2\hat{p}^6\Delta^{2+1/m})} + \Pr[\bar{\mathcal{A}}]$$
$$\leq e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}]$$
$$\overset{\text{C.13}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta}$$
$$< 2e^{-7\log\Delta}.$$

4.4.2 Bound on D_2

We now bound D_2. We first prove that for any edge uvw,

$$\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1] \leq p_u(c)p_v(c)(1+1/\omega_0). \tag{4.12}$$

Assume that both $p'_u(c)$ and $p'_v(c)$ are determined by (3.3). Since the function $\mathbf{I}[c \notin L(u)\cup L(v)]$ is decreasing and $\mathbf{I}[\gamma_w(c)=1]$ is increasing, the FKG inequality implies

$$\begin{aligned}\Pr[c \notin L(u)\cup L(v)|\gamma_w(c)=1] &= \frac{\Pr[c \notin L(u)\cup L(v), \gamma_w(c)=1]}{\Pr[\gamma_w(c)=1]}\\ &\leq \frac{\Pr[c \notin L(u)\cup L(v)]\Pr[\gamma_w(c)=1]}{\Pr[\gamma_w(c)=1]}\\ &= \Pr[c \notin L(u)\cup L(v)]. \end{aligned} \tag{4.13}$$

Similarly,

$$\Pr[c \notin L(u)|\gamma_w(c)=1] \leq \Pr[c \notin L(u)]. \tag{4.14}$$

If $c \in L(u)$ or $c \in L(v)$, then $p'_u(c)p'_v(c)=0$, so by (3.8),

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1] &\leq \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u)\cup L(v)|\gamma_w(c)=1]\\ &\overset{(4.13)}{\leq} \frac{p_u(c)}{q_u(c)}\frac{p_v(c)}{q_v(c)}\Pr[c \notin L(u)\cup L(v)]\\ &\overset{(3.8)}{\leq} p_u(c)p_v(c)(1+1/\omega_0).\end{aligned}$$

Suppose $p'_u(c)$ is determined by (3.3) and $p'_v(c)$ is determined by (3.4). Then $p'_u(c)$ and $p'_v(c)$ are independent of each other, and $p'_v(c)$ is independent of the event $\gamma_w(c)=1$, so

$$\begin{aligned}\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1] &= \mathbf{E}[p'_u(c)|\gamma_w(c)=1]\,\mathbf{E}[p'_v(c)]\\ &\overset{(3.5)}{=} \mathbf{E}[p'_u(c)|\gamma_w(c)=1]p_v(c)\\ &\leq \frac{p_u(c)}{q_u(c)}\Pr[c \notin L(u)|\gamma_w(c)=1]p_v(c)\\ &\overset{(4.14)}{\leq} \frac{p_u(c)}{q_u(c)}\Pr[c \notin L(u)]p_v(c)\\ &= p_u(c)p_v(c)\end{aligned}$$

$$< p_u(c)p_v(c)(1+1/\omega_0).$$

If $p'_u(c)$ and $p'_v(c)$ are both determined by (3.4), then

$$\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1] = \mathbf{E}[p'_u(c)p'_v(c)] = \mathbf{E}[p'_u(c)]\,\mathbf{E}[p'_v(c)] \overset{(3.5)}{=} p_u(c)p_v(c),$$

which establishes (4.12).

Now, by (4.12),

$$\begin{aligned}
\mathbf{E}[D_2] &= \sum_{c\in C(u)}\sum_{uvw}(\mathbf{E}[p'_u(c)p'_v(c)\,\mathbf{I}[\gamma_w(c)=1]] + \mathbf{E}[p'_u(c)p'_w(c)\,\mathbf{I}[\gamma_v(c)=1]])\\
&= \sum_{c\in C(u)}\sum_{uvw}\mathbf{E}[p'_u(c)p'_v(c)|\gamma_w(c)=1]\Pr[\gamma_w(c)=1]\\
&+ \sum_{c\in C(u)}\sum_{uvw}\mathbf{E}[p'_u(c)p'_w(c)|\gamma_v(c)=1]\Pr[\gamma_v(c)=1]\\
&\le (1+1/\omega_0)\sum_{c\in C(u)}\sum_{uvw}(p_u(c)p_v(c)\Pr[\gamma_w(c)=1] + p_u(c)p_w(c)\Pr[\gamma_v(c)=1])\\
&= (1+1/\omega_0)\sum_{c\in C(u)}\sum_{uvw}(p_u(c)p_v(c)\theta p_w(c) + p_u(c)p_w(c)\theta p_v(c))\\
&= (1+1/\omega_0)2\theta e_u.
\end{aligned}$$

We prove concentration of D_2 using the same setup that we used for D_1. Again, for $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

Then D_2 is determined by the set of random variables $\{T_c : c \in C(u)\}$. Observe that

$$\begin{aligned}
D_2 &= \sum_{c\in C(u)}\sum_{l=0}^{m-1}\sum_{v\in N_H(u)\cap N^0(u,l)}\mathbf{I}[\gamma_v(c)=1]\sum_{w\in N_H(u,v)}p'_u(c)p'_w(c)\\
&:= \sum_{c\in C(u)}\sum_{l=0}^{m-1}\sum_{v\in N_H(u)\cap N^0(u,l)} D_2^{v,c}.
\end{aligned}$$

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ (from Claim 13) such that x and x' differ only in coordinate c. Fix l and $v \in N_H(u) \cap N^0(u,l)$. Note that for $d \in C(u) - c$,

$$D_2^{v,d}(x) = D_2^{v,d}(x').$$

By definition of $\mathcal{A}_{c,l}$ (from Claim 13), $\mathbf{I}[\gamma_v(c)=1](x) = 1$ or $\mathbf{I}[\gamma_v(c)=1](x') = 1$ for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Furthermore,

$$\sum_{w\in N_H(u,v)} p'_u(d)p'_w(d) \le d_H(u,v)\hat{p}^2.$$

Thus

$$\begin{aligned}
|D_2(x) - D_2(x')| &\le \sum_{l=0}^{m-1} \sum_{v \in N_H(u) \cap N^0(u,l)} |D_2^{v,c}(x) - D_2^{v,c}(x')| \\
&\le \sum_{l=0}^{m-1} \sum_{\substack{v \in N_H(u) \cap N^0(u,l): \\ \mathbf{I}[\gamma_v(c)=1](x)=1 \text{ or} \\ \mathbf{I}[\gamma_v(c)=1](x')=1}} d_H(u,v)\hat{p}^2 \\
&\le \sum_{l=0}^{m-2} (2\Delta^{1-l/2m}\hat{p})\Delta^{(l+1)/2m}\hat{p}^2 + (2\Delta^{1-(m-1)/2m}\hat{p})\delta\hat{p}^2 \\
&< 2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3.
\end{aligned}$$

Recall that $\hat{p} \le \Delta^{-11/24}$ and $\delta \le \Delta^{6/10}$ ((R7) and (R5)). Thus

$$C(2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3)^2 \le \Delta^{1/2}(3\Delta^{-211/840})^2 \le 9\Delta^{-1/420}.$$

By Corollary 6 and Claim 13,

$$\begin{aligned}
\Pr[D_2 > 3\theta e_u + 1/2\omega^2] &\le \Pr[D_2 > (1+1/\omega_0)2\theta e_u / \Pr[\mathcal{A}] + 1/2\omega^2] \\
&\overset{\text{C.6}}{\le} e^{-2/(4\omega^4 C(2m\Delta^{1+1/2m}\hat{p}^3 + 2\delta\Delta^{1/2+1/2m}\hat{p}^3)^2)} + \Pr[\bar{\mathcal{A}}] \\
&\le e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}] \\
&\overset{\text{C.13}}{\le} e^{-7\log\Delta} + e^{-10\log\Delta} \\
&\le 2e^{-7\log\Delta}.
\end{aligned}$$

Therefore, with probability at least $1 - 2\Delta^{-5}$,

$$\begin{aligned}
f'_u &\le f_u(1-\theta/2) + 1/2\omega^2 + 3\theta e_u + 1/2\omega^2 \\
&\le f_u(1-\theta/2) + 3\theta e_u + 1/\omega^2.
\end{aligned}$$

Proof of (Q5). Since

$$d'_H(u) = \frac{1}{2} \sum_{v \in N_H(u)} \sum_{w \in N_H(u,v)} \mathbf{I}[v, w \in U'] \le \frac{1}{2} \sum_{v \in N_H(u)} d_H(u,v)\,\mathbf{I}[v \in U'],$$

Claim 12 implies

$$\mathbf{E}[d'_H(u)] \le \frac{1 - 3\theta/4}{2} \sum_{v \in N_H(u)} d_H(u,v) = (1 - 3\theta/4)d_H(u).$$

We prove concentration in the same way as in the proof of (Q3). For $c \in C(u)$, let

$$T_c = \{\gamma_v(c) : v \in N(N(u))\} \cup \{\eta_v(c) : v \in N(N(u))\}.$$

The random variable $d'_H(u)$ is determined by the set of random variables $\{T_c : c \in C(u)\}$. Observe that

$$d'_H(u) \leq \frac{1}{2}\sum_{l=0}^{m-1} \sum_{v \in N^0(u,l)} d^0_H(u,v)\, \mathbf{I}[v \in U'].$$

Fix $c \in C(u)$. Let $x, x' \in \mathcal{A}$ (from Claim 13) such that x and x' differ only in coordinate c. Fix l. As in the proof of (Q3) (see the two paragraphs after (4.11)),

$$\mathbf{I}[v \in U'](x) \neq \mathbf{I}[v \in U'](x')$$

for at most $2\Delta^{1-l/2m}\hat{p}$ vertices $v \in N^0(u,l)$. Further, if $l \leq m-2$, then $d^0_H(u,v) \leq \Delta^{(l+1)/2m}$, and if $l = m-1$, then $d^0_H(u,v) \leq \delta$. Therefore

$$\begin{aligned} |d'_H(u)(x) - d'_H(u)(x')| &\leq \sum_{l=0}^{m-2} \Delta^{1-l/2m}\hat{p}\Delta^{(l+1)/2m} + \Delta^{1-(m-1)/2m}\hat{p}\delta \\ &< m\Delta^{1+1/2m}\hat{p} + \Delta^{1/2+1/2m}\hat{p}\delta. \end{aligned}$$

By (R7) and (R5),

$$C(m\Delta^{1+1/2m}\hat{p} + \Delta^{1/2+1/2m}\hat{p}\delta)^2 \leq \Delta^{1/2}(2\Delta^{559/840})^2 = \Delta^{769/420}.$$

By Corollary 6 and Claim 13,

$$\begin{aligned} \Pr[d'_H(u) > (1-\theta/2)d_H(u) + \Delta^{19/20}] &\leq \Pr[d'_H(u) > (1-3\theta/4)d_H(u)/\Pr[\mathcal{A}] + \Delta^{19/20}] \\ &\overset{\text{C.6}}{\leq} e^{-2\Delta^{38/20}/C(m\Delta^{1+1/2m}\hat{p}+\Delta^{1/2+1/2m}\hat{p}\delta)^2} + \Pr[\bar{\mathcal{A}}] \\ &\leq e^{-2\Delta^{38/20-769/420}} + \Pr[\bar{\mathcal{A}}] \\ &\leq e^{-7\log\Delta} + \Pr[\bar{\mathcal{A}}] \\ &\overset{\text{C.13}}{\leq} e^{-7\log\Delta} + e^{-10\log\Delta} \\ &\leq 2e^{-7\log\Delta}. \end{aligned}$$

4.5 Final Step

After the iterative portion of the algorithm, some vertices will still be uncolored. Assuming (R1)-(R7) and Lemmas 8, 9, and 11 hold, we color them using the Asymmetric Local Lemma as follows. Suppose u has not been colored. By (P1), Lemma 11, and (R4),

$$\begin{aligned} \sum_{c \in C(u)-B^T(u)} p^T_u(c) = \sum_{c \in C(u)} p^T_u(c) - \sum_{c \in B^T(u)} p^T_u(c) &\overset{\text{(P1)}}{\geq} 1 - 1/\log C - |B^T(u)|\hat{p} \\ &\overset{\text{L.11}}{\geq} 1 - o(1) - \epsilon \end{aligned}$$

$$\overset{(R4)}{\geq} 1/2.$$

For each $c \notin B^T(u)$, define

$$p_u^*(c) := \frac{p_u^T(c)}{\sum_{c \in C(u)-B^T(u)} p_u^T(c)} \leq 2p_u^T(c).$$

For each uncolored vertex u, randomly assign u one color from the distribution given by p_u^*. For an edge $e = uvw \in H^T$, let A_{uvw} denote the event that u, v, and w receive the same color. By definition of T, $T/\omega^2 = o(\omega)$. So by (Q2),

$$e_{uvw}^T \leq e_{uvw}^0 + T/\Delta\omega^2 = 1/C^2 + o(\omega/\Delta) = \omega/\Delta + o(\omega/\Delta).$$

Therefore

$$\Pr[A_{uvw}] = \sum_{c \in C(u)} p_u^*(c)p_v^*(c)p_w^*(c) \leq 8 \sum_{c \in C(u)} p_u^T(c)p_v^T(c)p_w^T(c) = 8e_{uvw}^T \leq 9\omega/\Delta.$$

For each $c \in C(u) \cap C(v)$ and each pair $uv \in G_c^T$, let $B_{uv,c}$ denote the event that u and v both receive color c. By (P3), for each u,

$$\sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] \leq 4 \sum_{c \in C(u)} \sum_{ux \in G_c^T} p_u^T(c)p_x^T(c) = 4f_u^T \leq 64(1-\theta/4)^T\omega.$$

The event A_{uvw} depends on any event A_e or $B_{f,d}$, where u, v, or w is in the edge e or the edge f. Using (P5),

$$\begin{aligned}
&\sum_{e \in H^T : u \in e} \Pr[A_e] + \sum_{e \in H^T : v \in e} \Pr[A_e] + \sum_{e \in H^T : w \in e} \Pr[A_e] \\
&+ \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}] \sum_{c \in C(w)} \sum_{wx \in G_c^T} \Pr[B_{wx,c}] \\
&\leq 3(9\omega/\Delta)(1-\theta/3)^T\Delta + 3(64)(1-\theta/4)^T\omega \\
&\leq 219(1-\theta/4)^T\omega \\
&\leq 219e^{-\theta T/4}\omega \\
&= 219e^{-5\log\omega/4}\omega \\
&= 219(\frac{1}{\omega})^{5/4}\omega \\
&< 1/4.
\end{aligned}$$

The event $B_{uv,c}$ depends on any event A_e or $B_{f,d}$, where u or v is in e or f. Since

$$\sum_{e \in H^T : u \in e} \Pr[A_e] + \sum_{e \in H^T : v \in e} \Pr[A_e] + \sum_{c \in C(u)} \sum_{ux \in G_c^T} \Pr[B_{ux,c}] + \sum_{c \in C(v)} \sum_{vx \in G_c^T} \Pr[B_{vx,c}]$$

$$\leq 18(1-\theta/3)^T\omega + 128(1-\theta/4)^T\omega$$
$$\leq 1/4,$$

the Asymmetric Local Lemma implies that there exists a coloring where none of the events A_{uvw} or $B_{uv,c}$ occur. Since no color in $B^T(u)$ and no color with $p_c^T(u) = 0$ was assigned to u, this coloring, combined with the partial coloring from the algorithm, is a proper list coloring of $H \cup G$.

5 Triangle-free hypergraphs

We will derive Theorem 2 as a corollary of the following theorem:

Theorem 14. *Set* $c_0 = 1/259,200$*. Suppose* H *is a rank 3, triangle-free hypergraph with maximum* 3*-degree at most* Δ*, maximum 2-degree at most* $(c_0\Delta\log\Delta)^{1/2}$*, and maximum codegree at most* $\Delta^{6/10}$*. Then*

$$\chi_l(H) \leq (\frac{\Delta}{c_0\log\Delta})^{1/2}.$$

To prove this using Theorem 7, we need to find values for the parameters ω, ϵ, ω_0, and $\hat{p}$ which satisfy (R1)-(R7), (3.6), (3.7), and (3.8), and $\omega = c_0\log\Delta$. We will show that the following values satisfy these criteria:

$$\epsilon = 1/72 \qquad \omega = (1/24)(\epsilon/150)\log\Delta \qquad \hat{p} = \Delta^{-11/24} \qquad \omega_0 = 1/19\theta\hat{p}.$$

For (R1),

$$\theta\log(\hat{p}C) = \frac{\epsilon}{\omega}\log(\frac{\Delta^{1/24}}{\sqrt{\omega}}) = \frac{\epsilon}{\omega}(\frac{1}{24}\log\Delta - \frac{1}{2}\log\omega) = 86 - o(1) > 85.$$

The parameters clearly satisfy (R2)-(R7), so all that remains is to show inequalities (3.6), (3.7), and (3.8) hold. Fix a color c. In Claim 15, we first show that that hypergraph $H \cup G_c$ remains triangle-free throughout the algorithm. The next three claims then show that if the hypergraph remains triangle-free, we will have enough independence to derive (3.6), (3.7), and (3.8). Throughout the rest of this section, we will be taking intersections and unions over edges; when we do this, we use the notation e in place of $e \in E(H) \cup E(G_c)$.

Claim 15. *For iteration* i*, if* $H^i \cup G_c^i$ *is triangle-free, then* $H^{i+1} \cup G_c^{i+1}$ *is triangle-free.*

Proof. It suffices to show that when the algorithm creates G_c^{i+1} from G_c^i by adding an edge uv to G_c^i, no triangle is created. Toward a contradiction, suppose that a triangle is created with distinct edges $uv, e, f \in H^{i+1} \cup G_c^{i+1}$ and distinct vertices u, v, w such that $u \in e$, $v \in f$, $w \in e \cap f$, and $u \notin f$, $v \notin e$. Note that $u, v, w \in V(H^i \cup G_c^i)$ and $e, f \in H^i \cup G_c^i$. Since $w \in V(H^i \cup G_c^i)$, w has not been colored. Thus there exists a vertex $x \in V(H^i) - w$ and an edge $uvx \in H^i$ which gave rise to the edge uv. The edges uvx, e, and f form a triangle with vertices u, v, and w in $H^i + G_c^i$, a contradiction. □

In the rest of this section, we define

$$d(u, v) = |\{e \in H \cup G_c : u, v \in e\}|.$$

In addition, we drop the superscript from H^i and G_c^i.

Claim 16. *Suppose $uvw \in H$, $d(u, v) \geq 2$, and $d(w, v) \geq 2$. Then $d(u, w) = 1$.*

Proof. Since $d(u, v) \geq 2$ and $d(w, v) \geq 2$, there exist distinct edges $e, f \neq uvw$ such that $u, v \in e$ and $w, v \in f$. If there exists $x \neq v$ such that $uwx \in H$, then e, f, and uxw form a triangle with corresponding vertices u, v, and w. If $uw \in G_c$, then e, f, and uw form a triangle with vertices u, v, and w. □

Claim 17. *If uvw is an edge and $d(u, w) = 1$, then*

$$\Big(\bigcup_{e: u \in e; v \notin e} e - u\Big) \cap \Big(\bigcup_{e: w \in e; v \notin e} e - w\Big) = \emptyset, \tag{5.1}$$

$$\Big(\bigcup_{e: u \in e; v \notin e} e - u\Big) \cap \Big(\bigcup_{e: v \in e; u \notin e} e - v\Big) = \emptyset, \tag{5.2}$$

and

$$\Big(\bigcup_{e: w \in e; v \notin e} e - u\Big) \cap \Big(\bigcup_{e: v \in e; w \notin e} e - v\Big) = \emptyset. \tag{5.3}$$

Proof. Let $x \in U$, and let e be an edge such that $u \in e$, $v \notin e$, and $x \in e - u$. Then $e \neq uvw$, and since $d(u, w) = 1$, $x \notin \{u, v, w\}$.

Suppose f is an edge such that $w \in f$, $v \notin f$, and $x \in f - w$. Then, since $x \in f$, $f \neq uvw$. Using $d(u, w) = 1$, $u \in e$, $w \in f$ and $e, f \neq uvw$, we get $e \neq f$, $u \notin f$, and $w \notin e$. Since $x \notin uvw$, we obtain a triangle with edges e, f, and uvw and vertices u, w, and x.

Now suppose that $v, x \in f$ and $u \notin f$. Again, $f \neq uvw$. Because $u \in e$ and $u \notin f$, $e \neq f$. Since $u \notin f$, $v \notin e$, and $x \notin \{u, v, w\}$, e, f, and uvw form a triangle with vertices u, v, and x. By symmetry, this also gives (5.3). □

Claim 18. *If $uv \in G_c$, then*

$$(\bigcup_{e:u\in e; v\notin e} e - u) \cap (\bigcup_{e:v\in e; u\notin e} e - v) = \emptyset. \tag{5.4}$$

Proof. If there exist edges e and f and a vertex x such that $u \in e$, $v \notin e$, $v \in f$, $u \notin f$, and $x \in e - u \cap f - v$, then e, f, and uv form a triangle with vertices u, v, and x in $H \cup G_c$. □

For a set of vertices S, let $\gamma_S(c) = 1$ denote the event that $\gamma_v(c) = 1$ for all $v \in S$, and let $\gamma_S(c) \neq 1$ denote the event that $\gamma_v(c) = 0$ for some $v \in S$.

Claim 19. *For any three vertices x, y, and z,*

$$\Pr[\bigcap_{e:x\in e; y\notin e} \gamma_{e-x}(c) \neq 1] \leq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1] \leq q_x(c)(1 + 3\theta\hat{p}).$$

Proof. Note first that

$$\Pr[\bigcap_{e:x\in e; y\in e} \gamma_{e-x}(c) \neq 1] \geq \Pr[\gamma_y(c) = 0] \geq 1 - \theta\hat{p}.$$

Similarly,

$$\Pr[\bigcap_{e:x\in e; z\in e} \gamma_{e-x}(c)] \geq 1 - \theta\hat{p}.$$

Since the functions $\mathbf{I}[\bigcap_{x\in e; y\notin e} \gamma_{e-x}(c) \neq 1]$ and $\mathbf{I}[\bigcap_{x\in e; y\in e} \gamma_{e-x}(c) \neq 1]$ are monotone decreasing, the FKG inequality and then the previous two inequalities yield

$$\begin{aligned}
q_x(c) &= \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,y\in e} \gamma_{e-x}(c) \neq 1 \bigcap_{e:x,z\in e} \gamma_{e-x}(c) \neq 1] \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1] \Pr[\bigcap_{e:x\in e, y\in e} \gamma_{e-x}(c) \neq 1] \Pr[\bigcap_{e:x\in e, z\in e} \gamma_{e-x}(c) \neq 1] \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1](1 - \theta\hat{p})^2 \\
&\geq \Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1](1 - 2\theta\hat{p}).
\end{aligned}$$

Thus

$$\Pr[\bigcap_{e:x\in e; y,z\notin e} \gamma_{e-x}(c) \neq 1] \leq q_x(c)/(1 - 2\theta\hat{p}) \leq q_x(c)(1 + 3\theta\hat{p}).$$

□

We can now prove (3.6), (3.7), and (3.8). Suppose uvw is an edge. By Claim 16, we may assume $d(u,w) = 1$. The events $\bigcap_{u\in e; v\notin e} \gamma_{e-u}(c) \neq 1$, $\bigcap_{w\in e; v\notin e} \gamma_{e-w}(c) \neq 1$, and $\bigcap_{v\in e; u,w\notin e} \gamma_{e-v}(c) \neq 1$ depend only on the sets of random variables

$$\{\gamma_x(c) : x \in \bigcup_{e:u\in e; v\notin e} e-u\},$$

$$\{\gamma_x(c) : x \in \bigcup_{e:w\in e; v\notin e} e-w\},$$

and

$$\{\gamma_x(c) : x \in \bigcup_{e:v\in e; u,w\notin e} e-v\},$$

respectively. By (5.1), (5.2), and (5.3), these sets are pairwise disjoint, so the three events are independent of each other. Therefore, applying Claim 19,

$$\begin{aligned}
&\Pr[c \notin L(u) \cup L(v) \cup L(w)] \\
&= \Pr[\bigcap_{e:u\in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w\in e} \gamma_{e-w}(c) \neq 1] \\
&\leq \Pr[\bigcap_{e:u\in e; v\notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e; u,w\notin e} \gamma_{e-v}(c) \neq 1 \bigcap_{e:w\in e; v\notin e} \gamma_{e-w}(c) \neq 1] \\
&= \Pr[\bigcap_{e:u\in e; v\notin e} \gamma_{e-u}(c) \neq 1] \Pr[\bigcap_{e:v\in e; u,w\notin e} \gamma_{e-v}(c) \neq 1] \Pr[\bigcap_{e:w\in e; v\notin e} \gamma_{e-w}(c) \neq 1] \\
&\overset{\text{C.19}}{\leq} q_u(c)q_v(c)q_w(c)(1+3\theta\hat{p})^3 \\
&< q_u(c)q_v(c)q_w(c)(1+19\theta\hat{p}) \\
&= q_u(c)q_v(c)q_w(c)(1+1/\omega_0).
\end{aligned}$$

This proves (3.6). The proof of (3.7) is the same, except we start with any two vertices in uvw instead of all three.

Suppose now that $uv \in G_c$ for some color c. By (5.4) and Claim 19,

$$\begin{aligned}
\Pr[c \notin L(u) \cup L(v)] &= \Pr[\bigcap_{e:u\in e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e} \gamma_{e-v}(c) \neq 1] \\
&\leq \Pr[\bigcap_{e:u\in e; v\notin e} \gamma_{e-u}(c) \neq 1 \bigcap_{e:v\in e; u\notin e} \gamma_{e-v}(c) \neq 1] \\
&\overset{(5.4)}{=} \Pr[\bigcap_{e:u\in e; v\notin e} \gamma_{e-u}(c) \neq 1] \Pr[\bigcap_{e:v\in e; u\notin e} \gamma_{e-v}(c) \neq 1] \\
&\overset{\text{C.19}}{\leq} q_u(c)q_v(c)(1+3\theta\hat{p})^2 \\
&< q_u(c)q_v(c)(1+7\theta\hat{p}) \\
&< q_u(c)q_v(c)(1+1/\omega_0),
\end{aligned}$$

completing the proof of (3.8) and Theorem 14.

Proof of Theorem 2: Recall that $c_0 = 1/259,200$. Let H be a rank 3, triangle-free hypergraph with maximum 3-degree Δ and maximum 2-degree Δ_2. The original hypergraph H may have some pairs of vertices with codegree too large to apply Theorem 14, so we will work on a modified hypergraph instead. Let

$$K(u) = \{v \in N(u) : d(u,v) \geq \Delta^{6/10}\}.$$

Define a new hypergraph H' with $V(H') = V(H)$ and

$$E(H') = E(H) - (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{e : u, v \in e\}) + (\bigcup_{u \in V(H)} \bigcup_{v \in K(u)} \{u, v\})$$

Let Δ', Δ'_2, and δ' denote the maximum 3-degree, maximum 2-degree, and maximum codegree of H', respectively. Note that H' is still triangle-free, $\chi_l(H) \leq \chi_l(H')$, $\delta' \leq \Delta^{6/10}$, and $\Delta' \leq \Delta$.

Suppose $\Delta'_2 \leq \sqrt{\Delta}\sqrt{c_0 \log \Delta}$. Since $\Delta' \leq \Delta$ and $\delta' \leq \Delta^{6/10}$, Theorem 14 implies

$$\chi_l(H) \leq \chi_l(H') \leq (\frac{\Delta}{c_0 \log \Delta})^{1/2}.$$

On the other hand, suppose $\Delta'_2 > \sqrt{\Delta}\sqrt{c_0 \log \Delta}$. Then, since

$$\Delta \geq d_H(u) \geq \frac{1}{2} \sum_{v \in N_H(u)} d_H(u,v) \geq \frac{1}{2} \sum_{\substack{v \in N_H(u) \\ d_H(u,v) \geq \Delta^{6/10}}} d_H(u,v) \geq |K(u)|\Delta^{6/10}/2,$$

we have

$$\Delta'_2 \leq \Delta_2 + 2\Delta^{4/10} < \Delta_2 + \Delta'_2/2.$$

Choose Δ'' so that $\Delta'_2 = \sqrt{\Delta''}\sqrt{c_0 \log \Delta''}$. Since $\Delta'_2 > \sqrt{\Delta}\sqrt{c_0 \log \Delta}$, $\Delta'' > \Delta$. Then the maximum 3-degree of H' is at most $\Delta < \Delta''$, the maximum 2-degree of H' is at most $\Delta'_2 \leq \sqrt{\Delta''}\sqrt{c_0 \log \Delta''}$, and the maximum codegree of H' is at most $\Delta^{6/10} < \Delta''^{6/10}$, so Theorem 14 implies

$$\chi_l(H) \leq \chi_l(H') \leq (\frac{\Delta''}{c_0 \log \Delta''})^{1/2} = \frac{\Delta'_2}{c_0 \log \Delta''} < \frac{\Delta'_2}{c_0 \log \Delta'_2} < \frac{2\Delta_2}{c_0 \log 2\Delta_2}.$$

6 Acknowledgements

We would like to thank the referees for their feedback, particularly in helping us to improve the readability of the proof of (Q3).

References

[1] Miklós Ajtai, János Komlós, and Endre Szemerédi, *A note on Ramsey numbers*, J. Combin. Theory Ser. A **29** (1980), no. 3, 354–360. MR 600598 (82a:05064)

[2] Noga Alon, Michael Krivelevich, and Benny Sudakov, *Coloring graphs with sparse neighborhoods*, J. Combin. Theory Ser. B **77** (1999), no. 1, 73–82. MR 1710532 (2001a:05054)

[3] Tom Bohman, Alan Frieze, and Dhruv Mubayi, *Coloring $\mathcal{H}$-free hypergraphs*, Random Structures Algorithms **36** (2010), no. 1, 11–25. MR 2591044 (2011e:05081)

[4] R. L. Brooks, *On colouring the nodes of a network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197. MR 0012236 (6,281b)

[5] D. de Caen, *The current status of Turán's problem on hypergraphs*, Extremal problems for finite sets (Visegrád, 1991), Bolyai Soc. Math. Stud., vol. 3, János Bolyai Math. Soc., Budapest, 1994, pp. 187–197. MR 1319162 (95m:05127)

[6] Devdatt P. Dubhashi and Alessandro Panconesi, *Concentration of measure for the analysis of randomized algorithms*, Cambridge University Press, Cambridge, 2009. MR 2547432 (2011j:68176)

[7] Alan Frieze and Dhruv Mubayi, *On the chromatic number of simple triangle-free triple systems*, Electron. J. Combin. **15** (2008), no. 1, Research Paper 121, 27. MR 2443136 (2009j:05170)

[8] Venkatesan Guruswami and Ali Kemal Sinop, *The complexity of finding independent sets in bounded degree (hyper)graphs of low chromatic number*, Proceedings of the Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms (Philadelphia, PA), SIAM, 2011, pp. 1615–1626. MR 2858425 (2012j:68104)

[9] Wassily Hoeffding, *Probability inequalities for sums of bounded random variables*, J. Amer. Statist. Assoc. **58** (1963), 13–30. MR 0144363 (26 #1908)

[10] A Johansson, *Asymptotic choice number for triangle free graphs*, Tech. report, DIMACS, 1996.

[11] Jeong Han Kim, *On Brooks' theorem for sparse graphs*, Combin. Probab. Comput. **4** (1995), no. 2, 97–132. MR 1342856 (96f:05078)

[12] ———, *The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$*, Random Structures Algorithms **7** (1995), no. 3, 173–207. MR 1369063 (96m:05140)

[13] A. Kostochka, D. Mubayi, and J Verstraete, *Hypergraph ramsey numbers: Triangles versus cliques*, (submitted).

[14] Po-Shen Loh, *A note on embedding hypertrees*, Electron. J. Combin. **16** (2009), no. 1, Note 18, 4. MR 2515758 (2010i:05082)

[15] Colin McDiarmid, *On the method of bounded differences*, Surveys in combinatorics, 1989 (Norwich, 1989), London Math. Soc. Lecture Note Ser., vol. 141, Cambridge Univ. Press, Cambridge, 1989, pp. 148–188. MR 1036755 (91e:05077)

[16] ———, *Concentration*, Probabilistic methods for algorithmic discrete mathematics, Algorithms Combin., vol. 16, Springer, Berlin, 1998, pp. 195–248. MR 1678578 (2000d:60032)

[17] Michael Molloy and Bruce Reed, *Graph colouring and the probabilistic method*, Algorithms and Combinatorics, vol. 23, Springer-Verlag, Berlin, 2002. MR 1869439 (2003c:05001)

[18] Van H. Vu, *An upper bound on the list chromatic number of locally sparse graphs*, Tech. Report MSR-TR-2000-59, Microsoft Research, 2000.

[19] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996. MR 1367739 (96i:05001)