Turán problems and shadows II: trees

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Abstract

The expansion G^+ of a graph G is the 3-uniform hypergraph obtained from G by enlarging each edge of G with a vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\mathrm{ex}_r(n,F)$ denote the maximum number of edges in an r-uniform hypergraph with n vertices not containing any copy of F. The authors [11] recently determined $\mathrm{ex}_3(n,G^+)$ namely when G is a path or cycle, thus settling conjectures of Füredi-Jiang [9] (for cycles) and Füredi-Jiang-Seiver [10] (for paths).

Here we continue this project by determining the asymptotics for $\mathrm{ex}_3(n,G^+)$ when G is any fixed forest. This settles a conjecture of Füredi [8]. Using our methods, we also show that for any graph G, either $\mathrm{ex}_3(n,G^+) \le \left(\frac{1}{2}+o(1)\right)n^2$ or $\mathrm{ex}_3(n,G^+) \ge (1+o(1))n^2$, thereby exhibiting a jump for the Turán number of expansions.

1 Introduction

An r-uniform hypergraph F, or simply *r-graph*, is a family of r-element subsets of a finite set. We associate an r-graph F with its edge set and call its vertex set $V(F)$. Given a set of r-graphs $\mathcal{F}$, let $\mathrm{ex}_r(n,\mathcal{F})$ denote the maximum number of edges in an r-graph on n vertices that does not contain any r-graph from $\mathcal{F}$. When $\mathcal{F}=\{F\}$ we write $\mathrm{ex}_r(n,F)$. We will omit the subscript r in this notation if it is obvious from context, and this paper deals exclusively with the case $r=3$. Let G be a graph, and for each edge $e\in G$ let X_e be a set of $r-2$ vertices so that $X_e\cap V(G)=\emptyset$ and $X_e\cap X_f=\emptyset$ when $e\neq f$. The r-uniform *expansion* G^+ of a graph G is the r-graph $G^+=\{e\cup X_e : e\in G\}$.

Expansions of graphs include many important hypergraphs whose extremal functions have been investigated, for instance when G is a triangle and more generally a clique [8, 9, 10, 11, 13, 14, 15]. Even the simplest case of the expansion of a path with two edges is non-trivial, in this case we are asking for two hyperedges intersecting in exactly one point. Here the extremal function was determined by Frankl [6], settling a conjecture of Erdős and Sós. If a graph is not r-colorable then its r-uniform expansion G^+ is not r-partite, so $\mathrm{ex}_r(n,G^+)=\Omega(n^r)$. We focus on $\mathrm{ex}_r(n,G^+)$ when G is r-partite, where a well-known result of Erdős [2] yields $\mathrm{ex}(n,G^+)=O(n^{r-\epsilon_G})$ for some $\epsilon_G>0$.

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The authors [11] had previously determined $\mathrm{ex}_3(n, G^+)$ exactly (for large n) when G is a path or cycle of fixed length $k \geq 3$, thereby answering questions of Füredi-Jiang-Seiver [10] and Füredi-Jiang [9].

1.1 Results

A set of vertices in a hypergraph F containing exactly one vertex from every edge of F is called a *crosscut*, following Frankl and Füredi [7]. Let $\sigma(F)$ be the minimum size of a crosscut of F if it exists, i.e.,

$$\sigma(F) := \min\{|X| : X \subset V(F), \forall e \in F, |e \cap X| = 1\}$$

if such an X exists. Note that crosscuts always exist for expansions, since one can pick a vertex in X_e for every edge $e \in G$ and the resulting set of vertices is a crosscut of G^+ of size $|G|$. In the case that G is a tree, one can obtain a smaller crosscut by choosing some vertices in $V(G)$ that form an independent set in G, and vertices in X_e for those edges e not covered by the independent set.

Since the r-graph on n vertices consisting of all edges containing exactly one vertex from a fixed subset of size $\sigma(F) - 1$ does not contain F, we have

$$\mathrm{ex}_r(n, F) \geq (\sigma(F) - 1)\binom{n - \sigma(F) + 1}{r - 1} \sim (\sigma(F) - 1 + o(1))\binom{n}{r - 1}. \tag{1}$$

An intriguing open question is when asymptotic equality holds above and this is one of our motivations for this project. Indeed, it appears that the parameter $\sigma(F)$ often plays a crucial role in determining the extremal function for F. The value of $\mathrm{ex}_3(n, G^+)$ was determined precisely by the authors [11] when G is a path or cycle. Füredi [8] determined the asymptotics when G is a forest and $r \geq 4$, by showing that $\mathrm{ex}_r(n, G^+) = (\sigma(G^+) - 1 + o(1))\binom{n}{r-1}$. Füredi's proof involved extensive use of the delta system method but the method does not work for $r = 3$. Determining $\mathrm{ex}_r(n, G^+)$ when G is a tree seems to get harder as r gets smaller, for example, when $r = 2$ it becomes the Erdős-Sós Conjecture [5]. Füredi conjectured [8] that $\mathrm{ex}_3(n, G^+) \sim (\sigma(G^+) - 1)\binom{n}{2}$ when G is a forest, and our main result verifies this conjecture:

Theorem 1.1. (Main Result) *Let G be a forest. Then*

$$\mathrm{ex}_3(n, G^+) \sim (\sigma(G^+) - 1)\binom{n}{2}.$$

Our next result concerns $\mathrm{ex}_3(n, G^+)$ for any graph G with $\sigma(G^+) = 2$. Note that all such graphs are subgraphs of either $K_{2,t}$ for some $t \geq 2$ or S_t^* which is the graph obtained from a star with $t \geq 2$ edges by adding an edge not incident to the highest degree vertex.

Theorem 1.2. *For every fixed graph G with $\sigma(G^+) = 2$,*

$$\mathrm{ex}_3(n, G^+) \sim \binom{n}{2}.$$

A straightforward consequence of Theorem 1.2 is that for any graph G, we have either

$$\mathrm{ex}_3(n, G^+) \leq \left(\frac{1}{2} + o(1)\right) n^2 \quad \text{or} \quad \mathrm{ex}_3(n, G^+) \geq (1 + o(1))n^2.$$

This paper is organized as follows: in Section 2 we prove some preliminary lemmas. In Section 3, we give a bipartite version of the canonical Ramsey theorem of Erdős and Rado [3], which is one of the main tools for Theorem 1.1. We prove Theorem 1.1 in Section 6 and Theorem 1.2 in Section 7.

Notation and terminology. A 3-graph is called a *triple system*. The edges will be written as unordered lists, for instance, xyz represents $\{x, y, z\}$. For a set X of vertices of a hypergraph H, let $H - X = \{e \in H : e \cap X = \emptyset\}$. If $X = \{x\}$, then we write $H - x$ instead of $H - X$. The *codegree* of a pair $\{x, y\}$ of vertices in H is $d_H(x, y) = |\{e \in H : S \subset e\}|$ and for a set S of vertices, $N_H(S) = \{x \in V(H) : S \cup \{x\} \in H\}$ so that $|N_H(S)| = d_H(S)$ when $|S| = 2$. The *shadow* of H is the graph $\partial H = \{xy : \exists e \in H, \{x, y\} \subset e\}$. The edges of ∂H will be called the *sub-edges* of H. A triple system is *linear* if every pair of its edges intersect in at most one point. For an edge e in a triple system H, let $\delta_H(e)$ and $\triangle_H(e)$ respectively denote the smallest and largest codegree among the three pairs in e.

2 Full hypergraphs

In this section we state and prove a basic result about hypergraphs that generalizes the fact that a graph with average degree d contains a subgraph of minimum degree at least $d/2$.

Definition 2.1. *A triple system H is d-*full* if every sub-edge of H has codegree at least d.*

Thus H is d-full is equivalent to saying that the minimum non-zero codegree in H is at least d. The following lemma extends the well known fact that any graph G has a subgraph of minimum degree at least $d + 1$ with at least $|G| - d|V(G)|$ edges.

Lemma 2.2. *For $d \geq 1$, every n-vertex triple system H has a $(d + 1)$-full subgraph F with*

$$|F| \geq |H| - d|\partial H|.$$

Proof. A *d-sparse sequence* is a maximal sequence $e_1, e_2, \ldots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, e_i is contained in at most d edges of H which contain none of $e_1, e_2, \ldots, e_{i-1}$. The 3-graph F obtained by deleting all edges of H containing at least one of the e_i is $(d+1)$-full. Since a d-sparse sequence has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$. □

3 Colors, lists, and a canonical Ramsey theorem

One of our main new ideas is to use the canonical Ramsey theorem of Erdős and Rado [3]. We need a bipartite version of this classical result.

Definition 3.1. *Let F be a bipartite graph with parts X and Y and an edge-coloring χ. Then*

1. χ *is* X-canonical *if for each* $x \in X$*, all edges of* F *on* x *have the same color and edges on different vertices in* X *have different colors*
2. χ *is* canonical *if* χ *is* X*-canonical or* Y*-canonical*
3. χ *is* rainbow *if the colors of all the edges of* F *are different and*
4. χ *is* monochromatic *if the colors of all the edges of* F *are the same.*

Recall that a *sunflower* or Δ*-system* is a collection of sets such that the intersection of any two of them is equal to the intersection of all of them. This common intersection is called the *core* of the sunflower. A key result on sunflowers is the Erdős-Rado Sunflower Lemma [4]:

Lemma 3.2. (Erdős-Rado Sunflower Lemma) *If* F *is a collection of sets of size at most* k *and* $|F| \geq k!(s-1)^k$*, then* F *contains a sunflower with* s *sets.*

If χ is an edge-coloring of a graph F and $G \subset F$, then $\chi|_G$ denotes the edge-coloring of G obtained by restricting χ to the edge-set of G. A bipartite version of the canonical Ramsey theorem is as follows:

Theorem 3.3. *For each* $s > 0$ *there exists* $t > 0$ *such that for any edge-coloring* χ *of* $G = K_{t,t}$*, there exists* $K_{s,s} \subset G$ *such that* $\chi|_{K_{s,s}}$ *is monochromatic or rainbow or canonical.*

Proof. Let X and Y be the parts of G and let $S = \{y_1, y_2, \dots, y_{2s^2}\} \subset Y$. Let W be the set of vertices $x \in X$ contained in at least s edges of the same color connecting x with S. If $|W| > m := s^2\binom{2s^2}{s}$, then there is a set $Y' \subset S$ of size s and a set $X' \subset W$ of size s^2 such that for every $x \in X'$, the edges xy with $y \in Y'$ all have the same color. In this case we recover either a monochromatic $K_{s,s}$ or an X'-canonical $K_{s,s}$. Now suppose $|W| \leq m$. For $x \in X_0 := X \backslash W$, let $C(x)$ be a set of $2s$ distinct colors on edges between x and S. By Lemma 3.2, if $|X_0| > (2s)!(s!m)^{2s}$, then there exists $X_1 \subset X_0$ such that $\{C(x) : x \in X_1\}$ is a Δ-system of size $s!m$. Let C be the core of this Δ-system. First suppose $|C| \geq s$. Take a subset $C' \subset C$ with $|C'| = s$. To each vertex x in X_1, associate an s-subset S_x in S such that the edges $\{xy : y \in S_x\}$ have all the colors from C' appearing on them. There are $\binom{2s^2}{s} = m/s^2$ s-subsets of S, and $|X_1| = s!m$, so we have a set X_2 of at least $s^2 \cdot s! > s \cdot s!$ vertices in X_1 which each sends s edges with colors from C' into a fixed subset Y_3 of S of size s. This implies that for some set $X_3 \subset X_2$ of size s, the $K_{s,s}$ between X_3 and Y_3 is Y_3-canonical. Finally, suppose $|C| < s$. Pick $C'(x) \subset C(x) \backslash C$ of size s for $x \in X_0$. Since $|X_0| = s!m > s\binom{2s^2}{s}$, we find a set $Y^* \subset S$ of size s as well as a set $X^* \subset X$ of s vertices $x \in X_0$ such that the edges between x and Y^* have colors from $C'(x)$. Since the sets $C'(x)$ are disjoint, this is a rainbow copy of $K_{s,s}$. □

We now link this to hypergraphs via the following definition.

Definition 3.4. *Let* H *be a* 3*-graph. For* $G \subset \partial H$ *and* $e \in G$*, let*

$$L_G(e) = N_H(e) \setminus V(G).$$

The set $L_G(e)$ *is called the* list *of* e *and the elements of* $L_G(e)$ *are called* colors.

Let $L_G = \bigcup_{e \in G} L_G(e)$ – this is the set of colors in the lists of edges of G.

Definition 3.5. *A* list edge coloring *of G is a map $\chi : G \to L_G$ with $\chi(e) \in L_G(e)$ for all $e \in G$. List-edge-colorings $\chi_1, \chi_2 : G \to L_G$ are* disjoint *if $\chi_1(e) \neq \chi_2(f)$ for all $e, f \in G$.*

If χ is an injection – the coloring is rainbow – then clearly $G^+ \subset H$. We require one more definition:

Definition 3.6. *Let H be a 3-graph and $m \in \mathbb{N}$. An m-*multicoloring *of $G \subset \partial H$ is a family of list-edge-colorings $\chi_1, \chi_2, \dots, \chi_m : G \to L_G$ such that $\chi_i(e) \neq \chi_j(e)$ for every $e \in G$ and $i \neq j$.*

A necessary and sufficient condition for the existence of an m-multicoloring of G is that all edges of G have codegree at least m in H. We stress here that the definitions are all with respect to the fixed host 3-graph H. The following result will be key to the proofs of Theorem 1.1 and Theorem 1.2.

Theorem 3.7. *Let $m, s \in \mathbb{N}$, let H be a 3-graph, and let $G = K_{t,t} \subset \partial H$. Suppose G has an m-multicoloring. If t is large enough, then there exists $F = K_{s,s} \subset G$ such that F has either a rainbow list-edge-coloring or an m-multicoloring such that the colorings are pairwise disjoint, and each coloring is monochromatic or canonical.*

Proof. Set $s = t_m/m^2$ and $t_m < t_{m-1} < \cdots < t_1 < t_0 = t$ where Theorem 3.3 with input t_i has output t_{i-1}. Pick a color $c_1(e)$ on each edge $e \in G$ and apply Theorem 3.3 to G. We obtain a rainbow, monochromatic or canonical subgraph G_1 of G where $G_1 = K_{t_1,t_1}$. If it is rainbow, then we are done, so assume it is monochromatic or canonical. For every $e \in G_1$, remove $c_1(e)$ from its list. Now pick another color on each edge of G_1 and repeat. We obtain subgraphs $G_m \subset G_{m-1} \subset \cdots \subset G_1$ such that each G_i is monochromatic or canonical where $G_i = K_{t_i,t_i}$ has parts X_i, Y_i. In particular, each coloring of the m-multicoloring of G restricted to G_m is monochromatic or canonical.

Let us assume that we have a monochromatic colorings, b X_m-canonical colorings, and c Y_m-canonical colorings of G_m where $a + b + c = m$. It suffices to ensure that these colorings are pairwise disjoint. A color $\chi(xy)$ in an X_i-canonical coloring of G_i cannot appear in a $Y_{i'}$-canonical coloring of $G_{i'}$ for $i' > i$ as $\chi(xy)$ was deleted from all edges incident to x when forming G_{i+1}. A similar statement holds with X and Y interchanged, so every X_m-canonical coloring of G_m is disjoint from every Y_m-canonical coloring of G_m. The same argument shows that no color in a monochromatic coloring appears in a canonical coloring. It suffices to show that colors on different X_m-canonical colorings are disjoint (and the same for Y_m-canonical).

Let the b X_m-canonical colorings be $\chi_1, ..., \chi_b$. Construct an auxiliary graph K with $V(K) = X_m$ where $xx' \in K$ if there exist $i \neq i'$ and a color α that is canonical for x in χ_i and canonical for x' in $\chi_{i'}$. Then $\Delta(K) \leq b(b-1)$, so K has an independent set of size $s = t_m/m^2 \leq |X_m|/b^2$. Let us restrict X_m to this independent set. We repeat this procedure for Y_m and finally obtain a subgraph $F = K_{s,s}$ with an m-multicoloring that satisfies the requirement of the theorem. □

4 Cleaning lemmas

The lemmas in this section will allow us to find for an appropriate triple system H a large dense graph $G \subset \partial H$ that possesses an m-multicoloring with the colors outside of $V(G)$. Using such substructures, we will embed expansions of graphs into H.

Lemma 4.1. *Let $m, t \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and H be an n-vertex triple system. Suppose that $F \subset \partial H$ and for each $f \in F$ let $S_f \subset V(H) \setminus f$ with $|S_f| = m$. If $|F| \geq \delta n^2$ and n is large enough, then there exists $K \subset F$ such that $K \cong K_{t,t}$ and $S_f \cap V(K) = \emptyset$ for each $f \in K$.*

Proof. Let T be a random subset of $V(H)$ obtained by picking each vertex independently with probability $p = 1/2$. Let $G = \{f \in F : f \subset T, S_f \cap T = \emptyset\}$. Then

$$\mathbb{E}(|G|) \geq |F|p^2(1-p)^m \geq \frac{\delta}{2^{m+2}}n^2.$$

So there is a $T \subset V(H)$ with $|G|$ at least this large. If n is large enough, then the Kövari-Sós-Turán Theorem implies that there exists a complete bipartite graph $K \subset G \subset F$ with parts of size t. Due to the definition of G, the subgraph K satisfies the requirements of the lemma. □

Lemma 4.2. *Let $A_1, \dots, A_m$ be disjoint subsets of a set V and $a_1, \dots, a_m$ be distinct elements of V. Then there are $\lceil \frac{m}{3} \rceil$ pairwise disjoint sets of the kind $A_i + a_i := A_i \cup \{a_i\}$.*

Proof. Note that the statement of the lemma allows $a_i \in A_i$. Since all $a_1, \dots, a_m$ are distinct, if $(A_i + a_i) \cap (A_j + a_j) \neq \emptyset$, then $a_i \in A_j$, or $a_j \in A_i$, or both. Let F be the digraph with vertex set $\{A_1 + a_1, \dots, A_m + a_m\}$ and $A_iA_j \in F$ if $a_i \in A_j$ and $i \neq j$. Since the outdegree of every vertex in F is at most 1, F is 3-colorable and thus has an independent I of size $\lceil \frac{m}{3} \rceil$. By definition, the members of I are pairwise disjoint. □

5 Trees and crosscuts

In this section we produce a structural decomposition of a tree T that will be used later to embed T^+ in a hypergraph. We will also prove some basic lemmas about this decomposition.

Let G be a graph and consider a minimum crosscut X of G^+. For an edge $e \in G$, let v_e be the unique vertex such that $e \cup v_e \in G^+$; say that v_e is the *enlargement* of e. Partition X into $I \cup J$ where J comprises the vertices of X that are used for enlargement of the edges of G. Let $R \subset G$ be the set of edges e such that $v_e \in J$. Then $I \subset V(G)$ is an independent set in G and $R \subset G - I$. Furthermore, $\sigma(G^+) = |X| = |I| + |R|$. On the other hand for any independent set $I \subset V(G)$ and subgraph $R \subset G - I$, such that every edge of $G - I$ is in R, we obtain a crosscut $X = I \bigcup \cup_{e \in R}\{v_e\}$ of G^+. Consequently,

$$\sigma(G^+) = \min\{|I| + |G - I| : I \subset V(G) \text{ is an independent set}\}.$$

In the ensuing proof, it is more convenient to work with the pair (I, R) rather than a crosscut X of G^+.

Definition 5.1. *A* crosscut pair *of a graph G is a pair (I, R) where*
- *$I \subset V(G)$ is an independent set,*
- *$R = \{e \in G : e \cap I = \emptyset\}$.*

The crosscut pair (I, R) is optimal *if $|I| + |R| = \sigma(G^+)$.*

Given a crosscut pair (I, R), let

$$L = \{v \in V(G) \backslash (V(R) \cup I) : d_G(v) = 1\} \quad and \quad D = \{v \in V(G) \backslash (V(R) \cup I) : d_G(v) > 1\}$$

so that $D = V(G) \setminus (V(R) \cup I \cup L)$.

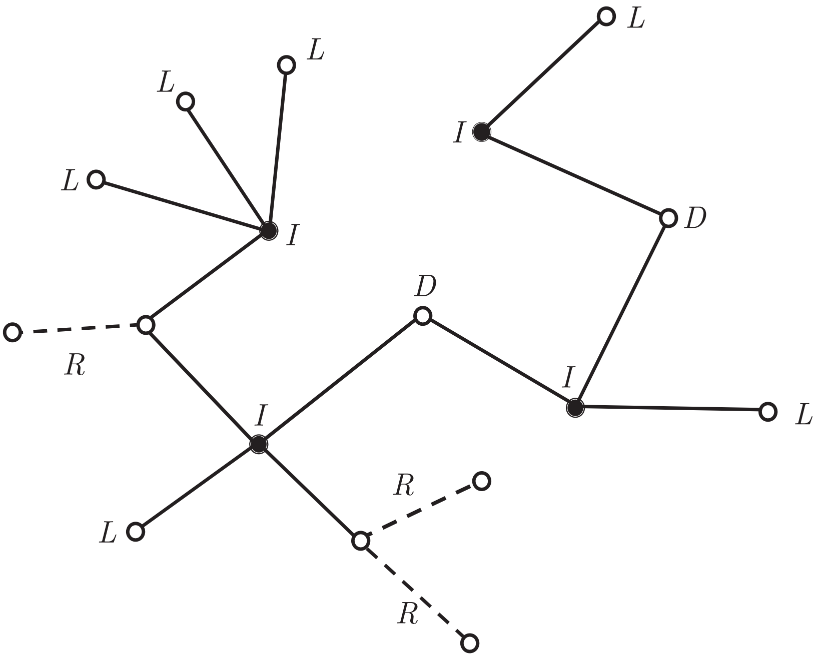

Figure 1 : Optimal crosscut decomposition $\sigma(G^+) = 7 = |I| + |R|$

Lemma 5.2. *Let T be a tree with $\sigma(T^+) = \ell + 1 > 0$. Consider an optimal crosscut pair (I, R) of T that maximizes $|I|$. Then (a) $|R| \leq \ell/2$ and (b) no pendant edge of T belongs to R.*

Proof. Suppose $yz \in R$ is a pendant edge of T with $d_T(z) = 1$. Then z is not adjacent to any vertex of I. The crosscut pair (I', R') where $I' = I \cup \{z\}$ and $R' = R \backslash \{yz\}$ contradicts the maximality of I, proving (b). Since T has no cycles, the number of edges induced by $I \cup V(R)$ is at most $|I| + |V(R)| - 1$. On this other hand, each vertex r of R must be connected to I by one of these edges, else we could move r into I, contradicting the maximality of I. Consequently,

$$|V(R)| + |R| \leq |I| + |V(R)| - 1.$$

This yields $|I| \geq 1 + |R|$. Since (I, R) is optimal, $|I| + |R| = \ell + 1$ and this gives (a). $\square$

Lemma 5.3. *Let F be a k-vertex forest. Then there exists a k-vertex tree $T \supset F$ with $\sigma(T^+) = \sigma(F^+)$.*

Proof. Let F have components $T_1, T_2, \ldots, T_s$. For each $j = 1, \ldots, s$, let (I_j, R_j) be an optimal crosscut pair of T_j with $|I_j|$ a maximum. If $I_j = \emptyset$, then any pendant edge of T_j is in R, contradicting Lemma 5.2.(b). Therefore $I_j \neq \emptyset$ for $j = 1, 2, \ldots, s$. If $I = \bigcup_{j=1}^{s} I_j$ and $R = \bigcup_{j=1}^{s} R_j$, then clearly (I, R) is an optimal crosscut pair of F, and $\sigma(F^+) = \sum_{j=1}^{s} \sigma(T_j^+)$. For each $j : 1 \leq j \leq s-1$, let us add an edge between T_j and T_{j+1} as follows: pick $u \in V(T_j) - I_j$ and $v \in I_{j+1}$ and add the edge uv. This results in a tree $T \supset F$ with $\sigma(T^+) = \sigma(F^+)$. $\square$

In what follows, we will be thinking of trees as bipartite graphs.

Definition 5.4. *Let T be a tree with parts P and Q, with $|P| \leq |Q|$. Let*

$$\lambda(T) = \begin{cases} |P| - 1 & \text{if some leaf of } T \text{ is in } P \\ |P| & \text{otherwise.} \end{cases}$$

If F is a forest with components $S_1, \ldots, S_h$, then we define $\lambda(F) = \sum_{i=1}^{h} \lambda(S_i)$.

Lemma 5.5. *For every forest F, $\lambda(F) \leq |F|/2$.*

Proof. It is an easy exercise to show that if T is a tree with parts P and Q and $|P| = |Q|$, then each of P and Q contains a leaf. This shows $\lambda(T) \leq |T|/2$ for every tree T, and applying this to the components of F, we obtain the lemma. $\square$

A *cycle* in a hypergraph is a sequence of vertices $v_1, \ldots, v_t$ and a sequence of distinct edges $e_1, \ldots, e_t$ where e_i contains both v_i and v_{i+1} for $i = 1, \ldots, t-1$ and e_t contains both v_t and v_1. A hypergraph is a *linear forest* if it has no cycles.

Lemma 5.6. *Let T be a tree with an optimal crosscut pair (I, R), $\sigma(T^+) = \ell + 1 > 0$ and $\lambda := \lambda(T)$. Then*

$$d_T(r) \leq \ell - \lambda \quad \text{for each } r \in V(R). \tag{2}$$

Proof. Suppose that R consists of h components $R_1, \ldots, R_h$ and $r \in V(R_1)$. The second end of every edge $rv \in G - R$ must be in I. Also every vertex of R has a neighbor in I, because otherwise we could move the vertex into I and obtain a crosscut pair of the same size and larger $|I|$. Moreover, as T is acyclic, every $V(R_j)$ has at least $|V(R_j)| = 1 + |R_j|$ neighbors in I and the neighborhoods of sets R_j in I form a hypergraph linear forest. Thus r is not adjacent to at least $\sum_{j=1}^{h} |R_j| = \ell_2$ vertices in I. Let $\ell_1 = |I|$. By definition, r is not adjacent to at least $\lambda(R_1)$ vertices in $V(R_1)$ (the smaller partite set of $V(R_1)$). So, since $|R| - \lambda \geq |R_1| - \lambda(R_1)$,

$$d_T(r) \leq (|I| - \ell_2) + (|V(R_1)| - \lambda(R_1)) \leq \ell_1 - \ell_2 + (1 + |R| - \lambda) = \ell_1 + 1 - \lambda. \tag{3}$$

The last expression is at most $\ell - \lambda$ unless $\ell_2 = 1$. Suppose $\ell_2 = 1$. Then $h = 1$, $\lambda = 0$, r has exactly one neighbor in $V(R)$, and instead of (3), we have

$$d_T(r) \leq (|I| - \ell_2) + 1 = \ell_1 = \ell - \lambda.$$

So, (2) holds in this case, as well. $\square$

6 Proof of Theorem 1.1

Let G be a forest with k vertices, and $\ell = \sigma(G^+) - 1$. We are to show $\mathrm{ex}_3(n, G^+) \le (\ell + o(1))\binom{n}{2}$. Let H be a triple system on n vertices with $|H| > (\ell + \epsilon)\binom{n}{2}$ where $\epsilon > 0$. By Lemma 5.3, $G^+ \subset T^+$ for some tree T with k vertices and $\sigma(T^+) = \sigma(G^+)$, so it is enough to show $T^+ \subset H$. for $n > n_0(\epsilon, k)$. Suppose for a contradiction that $T^+ \not\subset H$.

6.1 Finding a rich triple system

Recall that $\delta_H(e) = \min_{uv \subset e} d_H(uv)$. In this section we show how to find hypergraphs $H_3 \subset H_1 \subset H$ such that $\delta_{H_1}(e) \ge \ell + 1$ for every $e \in H_1$, H_3 has quadratically many edges, and $\delta_H(f) \le 3k$ for all $f \in H_3$. We will later use H_1 and H_3 to embed T^+.

Let H_1 be obtained from H by consecutive deletion of edges having a pair of codegree at most ℓ in the current 3-graph, so that $\delta_{H_1}(e) \ge \ell + 1$ for all $e \in H_1$. Let F_1 denote the set of deleted edges. Since we deleted at most ℓ edges at each step and the number of steps is at most $\binom{n}{2}$, we have $|F_1| \le \ell n^2/2$ and $|H_1| = |H_0| - |F_1| \ge (\ell + \epsilon)n^2/2 - \ell n^2/2 \ge \epsilon n^2/2$. By definition,

$$\delta_{H_1}(e) \ge \ell + 1 \qquad \text{for every } e \in H_1. \tag{4}$$

Let

$$H_2 = \{e \in H_1 : \delta_H(e) > 3k\} \quad \text{and} \quad H_3 = \{e \in H_1 : \delta_H(e) \le 3k\},$$

so that $H_1 = H_2 \cup H_3$. Suppose for a contradiction that $|H_2| > 3k^2 n$. If $|\partial H_2| > kn$, then $T \subset \partial H_2$, and we greedily extend T to $T^+ \subset H$. Otherwise, $|\partial H_2| \le kn$, in which case by Lemma 2.2 H_2 has a $3k$-full subgraph of size at least $|H_2| - 3k|\partial H_2| > 0$. This subgraph clearly contains a copy of T^+. This contradiction shows $|H_2| \le 3k^2 n$, and therefore $|H_3| = |H_1| - |H_2| \ge \epsilon n^2/4$ for large enough n.

By the definition of H_3, in each $e \in H_3$ we can fix some $f_e \in \binom{e}{2}$ with $d_H(f_e) \le 3k$. Let $F = \{f_e : e \in H_3\}$. Then $|F| \ge |H_3|/3k > \epsilon n^2/12k$. For each $f \in F$, let $S_f \subset N_{H_1}(f)$ with $|S_f| = \ell + 1$. Applying Lemma 4.1 to $F \subset \partial H_3$ we find a copy K of $K_{t,t}$ for large t such that each edge f of K is contained in $\ell + 1$ edges $f \cup \{v\} \in H_1$ with $v \in S_f$.

The $\ell + 1$ edges $f \cup \{v\}$ with $v \in S_f$ containing f for every $f \in K$ give an $(\ell + 1)$-multicoloring of K, so by Theorem 3.7 there is $G_0 = K_{s,s} \subset K$ (s large) with an $(\ell + 1)$-multicoloring $M_1, \ldots, M_{\ell+1}$ such that

- some M_i is rainbow, or
- the M_i's are pairwise disjoint and each M_i is either monochromatic or canonical.

Let X and Y be the partite sets of G_0 and

$$Z = \bigcup_{x \in X, y \in Y} N_{H_1}(xy) - V(G_0).$$

We will often think of M_i as a 3-graph comprising the edges xyw where $x \in X, y \in Y$ and w is the color of xy.

6.2 Canonical colorings and embeddings

In this section we prove a series of claims using Theorem 3.7 that allow us to embed T^+ within H_1 in certain situations.

Claim 1. *No M_i is rainbow.*

Proof: Suppose M_1 is rainbow. Since $s > 3k$, there is an embedding $\psi(T)$ of T into G_0. Since M_1 is rainbow, its edges containing the edges of $\psi(T)$ form $T^+ \subset H_1$. $\square$

Claim 2. *If some M_i is Y-canonical then there are no X-canonical M_j.*

Proof: Suppose M_1 is Y-canonical and M_2 is X-canonical. Then for every $y \in Y$ there is $w(y)$ such that $xyw(y) \in H_1$ for all $x \in X, y \in Y$ and for every $x \in X$ there is $u(x)$ such that $xyu(x) \in H_1$ for all $x \in X, y \in Y$. Let $\hat{T}$ be a directed out-rooted tree obtained from T with any root v. We embed it into G_0, and expand each edge as follows: if the image of a directed edge of $\hat{T}$ is xy, then expand it to $xyw(y)$ and if the image is yx, then expand it to $yxu(x)$. $\square$

Choose an optimal crosscut pair of T with maximum $|I|$. Let $\ell_1 = |I|$ and $\ell_2 = |R|$. By Lemma 5.2.(b), the pendant edges of T are not in R.

Claim 3. *At most $\ell_1 - 1$ of the M_i are monochromatic.*

Proof: Suppose, without loss of generality, that for $i = 1, 2, \ldots, \ell_1$, each M_i is monochromatic and w_i is the common vertex of all edges in M_i. If $I = \{a_1, \ldots, a_{\ell_1}\}$, then for $i = 1, \ldots, \ell_1$, we place a_i onto w_i, and then embed $T - I$ into G_0. Since each of $w_1, \ldots, w_{\ell_1}$ is adjacent in ∂H_1 with each vertex of G_0, this yields an embedding of T into ∂H_1. Next we extend the ℓ_2 edges of R using for each of them an edge from one of the ℓ_2 sets $M_{\ell_1+1}, \ldots, M_{\ell_1+\ell_2}$ (one edge from each set). Every other edge of the embedded T is incident with one of w_i. If such an edge has the form $w_i x$ (respectively, $w_i y$) then we take any unused $y \in Y$ (respectively, $x \in X$) and extend it to $\{w_i, x, y\}$. $\square$

Claim 4. $R \neq \emptyset$.

Proof: Suppose $R = \emptyset$ and U, U' are partite sets of T. Then all vertices of I are in the same partite set, say U, of T (in fact, $I = U$ as I covers all edges of T). By Claims 1, 2 and 3 and symmetry, we may assume that M_1 is Y-canonical. For every $y \in Y$ there is $w(y)$ such that $xyw(y) \in H_3$ for all $x \in X, y \in Y$. Place the vertices of T into $X \cup Y$ so that $U \subset X$ and $U' \subset Y$. Since G_0 is a complete bipartite graph, this yields an embedding of T into G_0. Since T is a tree, $|T| = |U'| + |U| - 1 = |U'| + \ell$. For every $y \in Y$ which is the image of some $b \in U'$ we expand one edge xy by adding $w(y)$. For the remaining ℓ edges we use edges of $M_2, \ldots, M_{\ell+1}$, from distinct M_j for distinct edges. $\square$

We recall from the last section the definition of $\lambda(F)$ for a forest F.

Claim 5. *If some M_i is Y-canonical, then at most $\lambda(R) - 1$ of the M_j are monochromatic.*

Proof: Suppose that M_ℓ is Y-canonical (we may assume this by Claim 3) and suppose, for a contradiction, that for $i = 1, \dots, \lambda(R)$, each M_i is monochromatic and w_i is the common vertex of all edges in M_i. Also for every $y \in Y$ there is $w(y)$ such that each edge in M_ℓ containing y also contains $w(y)$. We embed R into the subgraph of ∂H_1 induced by $Y \cup \{w(y) : y \in Y\} \cup \{w_1, \dots, w_{\lambda(R)}\}$ as follows. Suppose the components formed by the edges of R are $R_1, \dots, R_h$ with smaller partite sets $P_1, \dots, P_h$ and if P_j contains leaves, then b_j is one of them. We choose arbitrary $y_1, \dots, y_h \in Y$, and for j such that b_j exists, place b_j onto $w(y_j)$ and the neighbor in R_j of b_j onto y_j. Then place the remaining $\lambda(R)$ vertices of $P_1 \cup \dots \cup P_h$ onto vertices in $\{w_1, \dots, w_{\lambda(R)}\}$ and the remaining vertices of $V(R)$ (which comprise $\bigcup_i V(R_i) \setminus (P_i \cup N_R(b_i))$) onto arbitrary free vertices in Y. Since each $w_i y \in \partial H_1$ for all $y \in Y$ this yields an embedding of R in ∂H_1. Next, place the vertices of $D \cup L$ into new free vertices of Y, and finally, place all vertices of I onto distinct vertices in X.

This gives an embedding of T into ∂H_1. We expand it to an embedding of T^+ into H_1 as follows. Since $xyw(y) \in H_1$ for all $x \in X$ and $|X| \geq s$, we can expand the edges of the form $yw(y)$ at the end. Expand all edges of the form $w_i y$ and $w_i x$ by adding a free vertex from X and Y, respectively. This allows us to expand all edges of T except those that contain some vertex of D as an endpoint. We now focus on these edges which connect D to I.

For every y onto which we placed a vertex $a \in D$, we expand one edge of the kind xy by adding $w(y)$ and all other such edges using some M_j (distinct for distinct edges of T). To prove that we have enough free M_j first observe that the number of edges in T connecting D to I is $|D| + \ell_1 - 1$ (because I is an independent set and each edge joins precisely two components). Of these edges, $|D|$ will be expanded by expanding pairs of the form $yw(y)$ as mentioned earlier, so we must only expand $\ell_1 - 1$ more edges. The number of M_j that have already been used is at most $\lambda(R) + 1$ and so the number of unused M_j is at least $\ell + 1 - (\lambda(R) + 1) = \ell - \lambda(R)$. We finally show $\ell - \lambda(R) \geq \ell_1 - 1$ to complete the embedding. By Lemma 5.5, $\lambda(R) \leq |R|/2$. Since $|I| + |R| = \ell + 1$ and $|R| \leq \ell/2$ from Lemma 5.2, we have $\ell - |R|/2 \geq \ell_1 - 1$, and therefore

$$\ell - \lambda(R) \geq \ell - \frac{|R|}{2} \geq \ell_1 - 1.$$

This shows that $T^+ \subset H$, a contradiction. □

Let $\lambda = \lambda(R)$. By Claims 1, 2, 3 and symmetry, we may order the colorings so that $M_1, \dots, M_p$ are Y-canonical and the remaining are monochromatic. Furthermore, by Claim 5,

$$p \geq \ell + 2 - \lambda.$$

6.3 Constructing the digraph D_g

In this section we construct a digraph D_g whose underlying edges lie in ∂H_1 and it will be obtained iteratively from a sequence of digraphs $D_1, D_2, \dots$. The digraph D_g will be the vertex disjoint union of homomorphic images of directed out-trees each with height at most $\ell_2 + 1$. The rich structure of D_g encodes edges of H_1 and will later be used to embed T^+ in H_1.

For each $i \in \{1, \ldots, p\}$ and every $y \in Y$, let $w_i(y)$ be the vertex such that each edge in M_i containing y contains also $w_i(y)$ and let $W_i = \{w_i(y) : y \in Y\}$. Also for every $y \in Y$, let $W(y) = \{w_i(y) : i = 1, \ldots, p\}$. Let $Q = \{\alpha_{p+1}, \ldots, \alpha_{\ell+1}\}$ be the set of the colors used in the monochromatic colorings M_i.

By definition, for each $i \in \{1, \ldots, p\}$, the subgraph of ∂H_1 induced by $X \cup W_i$ contains the complete bipartite graph with partite sets X and W_i. By Theorem 3.7, all W_i are mutually disjoint and disjoint from Y. By the same theorem, we also have $W_i \cap Q = \emptyset$ where Q is the set of vertices/colors in the monochromatic colorings $M_{p+1}, \ldots, M_{\ell+1}$.

Basic cleaning procedure: By the codegree condition (4), for each $x \in X, w_1(y) \in W_1$, we can choose a set $S(x, w_1(y)) \subset N_{H_1}(xw_1(y))$ with $y \in S(x, w_1(y))$ and $|S(x, w_1(y))| = \ell + 1$. Define the 3-graph

$$H_1' = \{xwz \in H_1 : x \in X, w \in W_1, z \in S(x, w)\}$$

with $V(H_1') = \cup_{e \in H_1'} e$ so that

$$|V(H_1')| \leq |X| + |W_1| + (\ell + 1)|X||W_1| < (\ell + 3)s^2.$$

Let F' be the complete bipartite graph with parts X and W_1 so that $F' \subset \partial H_1'$. Then $|F'| = |X||W_1| = s^2 \geq \delta |V(H_1')|^2$ for $\delta = 1/(\ell + 3)$. Since s is large, we may apply Lemma 4.1 to $F' \subset \partial H_1'$ to obtain a large complete bipartite subgraph $G_{1,1} \subset F'$ such that $S(x, w) \cap V(G_{1,1}) = \emptyset$ for all $xw \in G_{1,1}$. Since $|S(x, w)| \geq \ell + 1$ for all xw, we can view $G_{1,1}$ as being multicolored with $\ell + 1$ colors, with one of the color classes corresponding to the vertices y. Moreover, all colors lie outside $V(F')$. The reason we need this is to apply Claim 1 below. This is the *basic cleaning procedure.*

By Theorem 3.7, we obtain subsets $X_{1,1}' \subset X, W_{1,1}' \subset W_1$, such that the $(\ell + 1)$-multicoloring restricted to $X_{1,1}' \times W_{1,1}'$ comprises rainbow, monochromatic, or canonical colorings. Let $Y_{1,1}' = \{y \in Y : w_1(y) \in W_{1,1}'\}$. None of the colorings is rainbow by Claim 1. Due to the colors corresponding to $Y_{1,1}'$, one of these colorings is $W_{1,1}'$-canonical, so by Claim 2 none of the colorings is $X_{1,1}'$-canonical. Consequently, Claims 3-5 imply that there is an integer p_1, and a set $\{w_{1,1}(y) \ldots, w_{1,p_1}(y)\}$ for each $y \in Y_{1,1}'$, whose vertices correspond to the $W_{1,1}'$-canonical colors of $w_1(y)$. Moreover, $w_{1,1}(y) = y$,

$$w_{1,j}(y) \neq w_{1,j}(y') \text{ for } y \neq y' \quad \text{and} \quad w_{1,j}(y) \neq w_{1,j'}(y) \text{ for } j \neq j'. \tag{5}$$

In other words the $(j + 1)$st canonical color class contains all edges of the form $xw_1(y)w_{1,j}(y)$ for $x \in X_{1,1}'$ and $y \in Y_{1,1}'$. Let $|X_{1,1}'| = |Y_{1,1}'| = s_{1,1}'$ and $Y_{1,1}' = \{y_1, \ldots, y_{s_{1,1}'}\}$. Add the colors of monochromatic colorings to Q.

Type-1 cleaning: Recall that $w_{1,1}(y_h) = y_h$ for all $1 \leq h \leq s_{1,1}'$. By Lemma 4.2 with $A_h = \{y_h, w_1(y_h)\}$ and $a_h = w_{1,2}(y_h)$, we can renumber y_h so that the sets

$$\{y_1, w_1(y_1), w_{1,2}(y_1)\}, \ldots, \{y_{s_{1,1}'/3}, w_1(y_{s_{1,1}'/3}), w_{1,2}(y_{s_{1,1}'/3})\}$$

are pairwise disjoint. Applying Lemma 4.2 with $A_h = \{y_h, w_1(y_h), w_{1,2}(y_h)\}$ and $a_h = w_{1,3}(y_h)$, then with $A_h = \{y_h, w_1(y_h), w_{1,2}(y_h), w_{1,3}(y_h)\}$ and $a_h = w_{1,4}(y_h)$, and so on, we obtain that for $s_{1,1}'' = \left\lceil \frac{s_{1,1}'}{3^{p_1}} \right\rceil$ we can renumber y_h so that the sets

$$\{y_1, w_1(y_1), w_{1,2}(y_1), \ldots, w_{1,p_1}(y_1)\}, \ldots, \{y_{s_{1,1}''}, w_1(y_{s_{1,1}''}), w_{1,2}(y_{s_{1,1}''}), \ldots, w_{1,p_1}(y_{s_{1,1}''})\}$$

are pairwise disjoint. Let $Y''_{1,1} = \{y_1, \ldots, y_{s''_{1,1}}\}$ and $X''_{1,1}$ be any subset of $X'_{1,1}$ of size $s''_{1,1}$. This is the *type-1 cleaning.*

Type-2 cleaning: Note that we automatically have $w_{1,j}(y) \cap X''_{1,1} = \emptyset$, since $xw_1(y)w_{1,j}(y) \in H_1$ for all $x \in X''_{1,1}$ so in particular, these three vertices are distinct. Since for every $1 \le j \le p_1$ all vertices $w_{1,j}(y_h)$ are distinct, at most $|Q| \le \ell$ of them are in Q. Deleting from $Y''_{1,1}$ the at most $p_1|Q|$ vertices y_h such that

$$\{y_1, w_1(y_1), w_{1,2}(y_1), \ldots, w_{1,p_1}(y_1)\} \cap Q \neq \emptyset,$$

we obtain a $Y_{1,1} \subset Y''_{1,1}$ such that for distinct $y \in Y_{1,1}$ the sets $\{y, w_1(y), w_{1,2}(y), \ldots, w_{1,p_1}(y)\}$ are disjoint from each other and from Q and $X''_{1,1}$. Then we choose any $X_{1,1} \subset X''_{1,1}$ with $|X_{1,1}| = |Y_{1,1}|$. This is the *type-2 cleaning.*

Recall that s is taken sufficiently large so that the bipartite graphs we are considering are also large. Now define $G_{1,2}$ to be the complete bipartite graph with parts $X_{1,1}$ and W_2 and repeat the cleaning procedures above to obtain the integer p_2, subsets $X_{1,2} \subset X_{1,1}$ and $Y_{1,2} \subset Y_{1,1}$ and vertices $w_{2,j}(y)$ that are distinct for distinct y and also distinct from $w_{1,j'}(y')$ if $y \neq y'$. Continuing in this way we obtain sets $X_{1,1} \supset X_{1,2} \supset \cdots \supset X_{1,p} := X_2$ and $Y_{1,1} \supset Y_{1,2} \supset \cdots \supset Y_{1,p} := Y_2$, $\mathbf{V}_2 = \{(i, j_i) : i \in [p], j_i \in [p_i]\} \subset [\ell]^2$ and vertices $w_{\mathbf{v}}(y)$ for $\mathbf{v} \in \mathbf{V}_2$ and $y \in Y_2$ with $w_{\mathbf{v}}(y) \notin \{w_{\mathbf{v}}(y'), w_i(y')\}$ for $y \neq y'$.

Given a vector $\mathbf{x}$ let $\mathbf{x} * j$ be the vector obtained from $\mathbf{x}$ by adding a new last coordinate with entry j (for example if $\mathbf{x} = (3, 7)$ then $\mathbf{x} * 4 = (3, 7, 4)$). For $\mathbf{v} \in \mathbf{V}_2$, set $W_{\mathbf{v}} = \cup_{y \in Y_2} w_{\mathbf{v}}(y)$. Let us also construct the auxiliary digraph D_2 with vertex set $Y_2 \cup \bigcup_{i=1}^{p} W_i \cup \bigcup_{\mathbf{v} \in \mathbf{V}_2} W_{\mathbf{v}}$ with edges of the form $yw_i(y)$ for all y, i and $w_i(y)w_{i,j}(y)$ for $i \in [p]$ and $j \in [p_i]$. Because of cleanings, D_2 is the vertex disjoint union of homomorphic images of trees of height at most two, one for each $y \in Y_2$.

Claim 6. $|Q| \le k$.

Proof: Suppose for contradiction that $|Q| < k$. By the definition of monochromatic colorings and by construction, for each $x \in X_2$ and each $w \in Q$, $xw \in \partial H_1$ and the codegree of xw is at least $|Y_2| \ge 3k$ (since s and hence $|Y_2|$ are large). So we simply embed T into the complete bipartite graph with partite sets X_2 and Q, and then expand it. □

To summarize, we have a set of (one dimensional) vectors $\mathbf{V}_1 = \{(1), \ldots, (p)\}$, nonnegative integers $p_{\mathbf{v}} \le \ell$ for each $\mathbf{v} \in \mathbf{V}_1$ and

- $\mathbf{V}_2 = \cup_{\mathbf{v} \in \mathbf{V}_1} \{\mathbf{v} * i : i \in [p_{\mathbf{v}}]\} \subset [\ell + 1]^2$,
- $X_2 \subset X$ and $Y_2 \subset Y$,
- vertices $w_{\mathbf{v}}(y)$ with $w_{\mathbf{v}}(y) \neq w_{\mathbf{v}'}(y')$ if $y \neq y'$ and $\mathbf{v}, \mathbf{v}' \in \mathbf{V}_1 \cup \mathbf{V}_2$,
- edges $xw_{\mathbf{v}}(y)w_{\mathbf{v}*i}(y) \in H_1$ for all $x \in X_2, y \in Y_2, \mathbf{v} \in \mathbf{V}_1, i \in [p_{\mathbf{v}}]$ (so $\mathbf{v} * i \in \mathbf{V}_2$),
- a digraph D_2 with vertex set $Y_2 \cup \bigcup_{y \in Y_2, \mathbf{v} \in \mathbf{V}_1 \cup \mathbf{V}_2} w_{\mathbf{v}}(y)$ and edges $yw_{\mathbf{v}}(y)$ for $y \in Y_2, \mathbf{v} \in \mathbf{V}_1$ and $w_{\mathbf{v}}(y)w_{\mathbf{v}'}(y)$ as long as $\mathbf{v}' = \mathbf{v} * j$ for some $j \in [p_{\mathbf{v}}]$,
- the set Q of all "central" vertices in monochromatic colorings of $X_2 \times Y_2$, and $|Q| \le k$.

General Setup: Let $t \leq \ell_2 + 1$ and suppose we have the following:

- $\mathbf{V}_t \subset [\ell + 1]^t$,
- $X_t \subset X$ and $Y_t \subset Y$,
- for all $\mathbf{v} \in \cup_{i=1}^{t} \mathbf{V}_t$ and $y \in Y_t$ a vertex $w_{\mathbf{v}}(y)$ such that for $y \neq y'$, $w_{\mathbf{v}}(y) \neq w_{\mathbf{v}'}(y')$,
- edges $xw_{\mathbf{v}}(y)w_{\mathbf{v}*i}(y) \in H_1$ for all $x \in X_t, y \in Y_t, \mathbf{v} \in \cup_{j=1}^{t-1} \mathbf{V}_j, i \in [p_{\mathbf{v}}]$,
- a digraph D_t with vertex set
$$Y_t \cup \bigcup_{y \in Y_t, \mathbf{v} \in \cup_{i=1}^{t} \mathbf{V}_t} w_{\mathbf{v}}(y)$$
and edges $w_{\mathbf{v}}(y)w_{\mathbf{v}'}(y)$ as long as $\mathbf{v}' = \mathbf{v} * j$ for some $j \in [p_{\mathbf{v}}]$ (define $y := w_{\emptyset}(y)$),
- the set Q of all "central" vertices in monochromatic colorings of $X_t \times Y_t$, and $|Q| \leq k$.

We will now show how to construct the same setup with $t + 1$.

Let $\mathbf{V}_t = \{\mathbf{v}(1), \ldots, \mathbf{v}(m(t))\}$. Consider the complete bipartite subgraph $G_{t,1}$ of ∂H_1 with parts X_t and $W_{\mathbf{v}(1)} = \{w_{\mathbf{v}(1)}(y) : y \in Y_t\}$. We apply the basic cleaning procedure to $G_{t,1}$ and obtain subsets $X'_{t,1} \subset X_t$ and $Y'_{t,1} \subset Y_t$ and colorings $M_1, \ldots, M_{\ell+1}$ of the edges of $G_{t,1}$ that are rainbow, canonical, or monochromatic. By Claim 1, no coloring is rainbow. By construction, we already have one $W_{\mathbf{v}(1)}$-canonical coloring obtained by considering the in-neighbors of $w_{\mathbf{v}(1)}(y)$ in D_t. By Claim 4, $R \neq \emptyset$ and thus $\ell_1 \leq \ell$. We may assume that M_1 is $W_{\mathbf{v}(1)}$-canonical. Hence by Claim 2 no M_i is $X'_{t,1}$-canonical. By Claim 3, the number of monochromatic colorings is at most $\ell_1 - 1 \leq \ell - 1$, which means that the number of $W_{\mathbf{v}(1)}$-canonical colorings is at least $(\ell+1)-(\ell-1) = 2$. Consequently, there is a positive integer $p_{\mathbf{v}(1)}$ and $p_{\mathbf{v}(1)}$ colorings (excluding the $W_{\mathbf{v}(1)}$-canonical coloring given by the in-neighbors of $W_{\mathbf{v}(1)}$) that are $W_{\mathbf{v}(1)}$-canonical and the remaining $\ell+1-(p_{\mathbf{v}(1)}+1)$ colorings are monochromatic. We also have vertices $w_{\mathbf{v}(1)*i}(y)$ for all $i = 1, \ldots, p_{\mathbf{v}(1)}$ which are distinct for distinct y and distinct i. As before, for each $j \in [p_{\mathbf{v}(1)}]$, the jth canonical color class consists of all (hyper)edges of the form $xw_{\mathbf{v}(1)}(y)w_{\mathbf{v}(1)*j}(y)$ for all $x \in X'_{t,1}, y \in Y'_{t,1}$.

Next we perform the type-1 cleaning procedure (using Lemma 4.2) to make sure that if $y \neq y'$ then $w_{\mathbf{v}(1)*i}(y) \notin \{w_{\mathbf{v}}(y'), y'\}$ for any $\mathbf{v} \in \mathbf{V}_1 \cup \ldots \cup \mathbf{V}_t$. This results in subsets $X''_{t,1} \subset X'_{t,1}$ and $Y''_{t,1} \subset Y'_{t,1}$. Finally, we perform the type-2 cleaning procedure to obtain $X_{t,1} \subset X''_{t,1}$ and $Y_{t,1} \subset Y''_{t,1}$ so that these sets do not contain any vertices that correspond to monochromatic colorings in any previous round. Add the central vertices of the monochromatic colorings to Q. Repeating the proof of Claim 6, we still have $|Q| \leq k$.

Now we repeat these procedure with $\mathbf{v}(2)$ to obtain $X_{t,2} \subset X_{t,1}$ and $Y_{t,2} \subset Y_{t,1}$. Finally we perform this procedure with $\mathbf{v}(m(t))$ to obtain $X_{t+1} = X_{t,m(t)}$ and $Y_{t+1} = Y_{t,m(t)}$ and $\mathbf{V}_{t+1} = \cup_{\mathbf{v} \in \mathbf{V}_t} \{\mathbf{v} * i : i \in [p_{\mathbf{v}}]\}$. We also have vertices $w_{\mathbf{v}}(y)$ for every $y \in Y_{t+1}$ and $\mathbf{v} \in \mathbf{V}_{t+1}$ that are distinct for distinct y and a digraph D_{t+1} defined in the obvious way which consists of the vertex disjoint union of homomorphic image of trees of height $t + 1$, one for each $y \in Y_{t+1}$. Edges of the digraph encode the canonical colorings, as in the case $t = 1, 2$.

We repeat this procedure till we obtain sets X_g, Y_g, D_g, for $g := \ell_2+1$. By Claim 5, the outdegree of vertex $w_{\mathbf{v}}(y) \in V(D_g)$ is

$$p_{\mathbf{v}} \geq (\ell + 2 - \lambda) - 1 = \ell + 1 - \lambda.$$

Note that this is one less than the bound for p because we have one in-neighbor that accounts for one canonical coloring.

6.4 Embedding T^+ using D_g

In this section we use the properties of D_g to embed T^+ in H_1. Our plan is to place the edges of R on the edges of D_g and to place the vertices of I onto some vertices in X_g. Let $T_1 = T - L$. Consider every tree in the forest R as a (directed) rooted tree R_i with root r_i which is a vertex in $V(R_i)$ of the largest degree in T_1. Suppose we have h such trees. By Lemma 5.6,

$$d_{T_1}(r_i) \leq \ell - \lambda \quad \text{for all} \quad 1 \leq i \leq h.$$

For each $y \in Y_g$, let $D_g(y)$ be the component of D_g containing y. Choose h vertices $y_1, \ldots, y_h \in Y_g$ arbitrarily, and for $1 \leq i \leq h$ we will embed R_i into $D_g(y_i) - w_1(y_i)$ as follows (we exclude $w_1(y_i)$ because we will use $w_1(y_i)$ later in the embedding of T^+). Place r_i on y_i. Suppose r_i has u out-neighbors in R_i. By construction, y_i has $p \geq \ell + 2 - \lambda$ outneighbors in $D_g(y_i)$. So by (2), we can place the outneighbors in R_i of r_i on outneighbors of y_i in $D_g(y_i)$. Then we place the outneighbors of placed vertices and so on. The general situation is that some $v \in V(R_i)$ is placed on some $w_{\mathbf{v}}(y)$ and has u outneighbors in R_i. By Lemma 5.2, $\ell \geq 2\ell_2$. By Lemma 5.5, $\lambda = \lambda(R) \leq |R|/2 = \ell_2/2$. So $w_{\mathbf{v}}(y)$ has $p_{\mathbf{v}} \geq \ell + 1 - \lambda \geq \frac{3\ell_2}{2} + 1$ outneighbors in $D_g(y)$. At most $\ell_2 - u$ of them are already occupied by previously embedded vertices. This leaves more than u available outneighbors of $w_{\mathbf{v}}(y)$ to place the outneighbors in R_i of v on them.

After placing all vertices in $V(R)$, we call a vertex of H_1 *free*, if it is not occupied by vertices in $V(R)$ and is not the outneighbor of any occupied vertex in D_g. By construction, there are at most $|V(R)|\ell_2\ell \leq \ell^3$ non-free vertices. We now place the vertices of I on arbitrary distinct vertices in X_g (they are all free at this moment by construction). Then we place the vertices of D on distinct free vertices in Y_g. Let φ be the embedding we are producing. Let each $a \in I$ be placed on $\varphi(a) \in X_g$. This yields an embedding of T_1 into ∂H_1. In what follows, say that a pair xy is *expanded* to a triple xyz. Our next goal will be to expand the edges of T_1. After that we will embed the edges of $T - T_1$ and expand them (these are the edges incident to L).

Since the codegree of every edge in D_g is at least $|X_g|$, we do not worry about expanding the edges in R: we can do it greedily at the end. Recall that vertices in D are adjacent only to I. We need to expand the $|I| + |D| + h - 1$ edges connecting I with $D \cup V(R)$. For every host y of a vertex $a \in D$ and the host $x = \varphi(a')$ of a neighbor $a' \in I$ of a, we expand the edge yx to $\{x, y, w_1(y)\}$. So the number of edges of $T_1 - R$ not yet expanded is $|I| + h - 1 = \ell_1 + h - 1$. Since the sets $V(D_g(y))$ are disjoint for distinct $y \in Y_g$, expanding the edges incident with $\varphi(a)$ for $a \in D$ is easy: we simply use the vertices $w_2(\varphi(a)), w_3(\varphi(a))$ and so on. Since the number of such edges is at most $\ell_1 - 1 \leq p - 2$, no problem arises.

When we expand an edge yx where x is the host of some $a \in I$ and y is the host of some $b \in V(R_i) \subset V(R)$, we need some more care, since some outneighbors of y in $D_g(y)$ can be occupied. For $i = 1, \ldots, h$ let $U(i) = |R_i| + |E_{T_1}(I, V(R_i))|$. Then

$$\sum_{i=1}^{h} U(i) = |T_1 - A_0| \leq |I| + |R| + h - 1 = \ell + h. \tag{6}$$

Order the R_is so that $U(1) \geq U(2) \geq \ldots \geq U(h)$ and expand the edges incident to R_is in the reverse order. Since each $b \in V(R)$ is adjacent to some $a \in I$, $U(i) \geq 3$ for every i. Suppose that it is now the turn to expand the edges incident to R_i and $i \geq 2$. Then $U(i) \leq U(2) \leq \frac{\ell+h-3(h-2)}{2} \leq \frac{\ell+2}{2}$. We expand the edges one by one. Suppose we need now to expand $w_{\mathbf{v}}(y)x$, where $w_{\mathbf{v}}(y)$ is the host of a vertex $b \in V(R_i)$ (possibly $\mathbf{v} = \emptyset$ in which case by convention $w_{\mathbf{v}}(y) = y$ and $p_{\mathbf{v}} = p$). The outdegree in D_g of $w_{\mathbf{v}}(y)$ is $p_{\mathbf{v}} \geq \ell + 1 - \lambda$. At most $|R_i|$ of the outneighbors of $w_{\mathbf{v}}(y)$ are occupied. If we already expanded some edges incident with R_i, they block at most $|E_{T_1}(I, V(R_i))| - 1$ outneighbors of y. Consequently, we have at least

$$(\ell + 1 - \lambda) - (U(i) - 1) \geq \frac{\ell}{2} - \lambda + 1 \geq \frac{\ell}{2} - \frac{\ell_2}{2} + 1 > 0$$

free outneighbors of y, and any free outneighbor may be used to expand $w_{\mathbf{v}}(y)x$.

Finally, we work with R_1. It is possible that $U(1)$ is as large as $\ell_2 + \ell_1$. On the other hand, we have not yet used the universal vertices for monochromatic multicolorings, and this is the time to use them. Now for each $a \in V(R_1)$ and $x \in X_g$, the pair $\varphi(a)x$ has $1 + \ell = \ell_1 + \ell_2$ different colors in the canonical multicoloring (including any universal vertices), which means $1 + \ell$ possibilities to expand $\varphi(a)x$. Since the number of edges $U(1)$ to be embedded when we embed R_1 is at most $\ell + 1$, we can perform the embedding greedily.

Having embedded and expanded T_1, we work with L. Since Y_g is large, one by one, take $c \in L$, place it on a free $y \in Y_g$ and expand the obtained edge yx via $w_1(y)$. $\square$

7 Proof of Theorem 1.2

Suppose $\sigma(G^+) = 2$ and $|V(G)| = k$. Since the n-vertex triple system of all edges containing a fixed vertex does not contain G^+ with $\sigma(G^+) = 2$ (by definition), $\mathrm{ex}(n, G^+) \geq \binom{n-1}{2}$. Also if $\sigma(G^+) = 2$, then either some vertex of G covers all but one edge in G (and this edge connects two leaves) or two non-adjacent vertices of G cover all edges of G. In the former case, G is contained in the star-plus-one-edge graph S^*_{k-1} and in the latter, G is contained in $K_{2,k-2}$. Thus it is enough to consider the cases $G = K_{2,k-2}$ and $G = S^*_{k-1}$.

Suppose we have an n-vertex 3-graph H not containing G^+ for $G \in \{S^*_{k-1}, K_{2,k-2}\}$ with $|H| = (1 + \varepsilon)\binom{n}{2}$ where $\varepsilon > 0$ and n is sufficiently large. It is enough to assume $k \geq 5$. Let H' be obtained from H by consecutive deletion of edges having a pair of codegree one, so that the minimum codegree of edges in H' is at least two. If we deleted m edges, then $|\partial H'| \leq \binom{n}{2} - m$. Let E be the set of edges of H' in which the codegrees of all pairs (in H') are at most 3 or at least two pairs have codegree (in H') exactly two. We claim that

$$|E| \leq |\partial H'|. \tag{7}$$

To see this, define $\omega = \sum_{e \in H'} \sum_{f \subset e} 1/d(f)$, where $d(f)$ is the codegree of f in H'. By definition of E, for every $e \in E$ we have $\sum_{f \subset e} \frac{1}{d(f)} \geq 1$. Since $E \subset H'$, we get $\omega \geq |E|$. By interchanging the sums, we see $\omega = |\partial H'|$:

$$\omega = \sum_{f \in \partial H'} \sum_{e \supset f} \frac{1}{d(f)} = \sum_{f \in \partial H'} 1 = |\partial H'|.$$

Therefore $|E| \leq |\partial H'|$ as claimed.

Let $H'' = H' \backslash E$. By (7),

$$|H''| \geq |H'| - |\partial H'| = |H| - m - \left(\binom{n}{2} - m\right) = |H| - \binom{n}{2} \geq \varepsilon \binom{n}{2}.$$

By the definition of E, if $e \in H''$ and the codegrees in H' of the vertex pairs in e are $c_1 \leq c_2 \leq c_3$, then $c_1 \geq 2$, $c_2 \geq 3$ and $c_3 \geq 4$. Then for each $e \in H''$, there is an expansion of a triangle T_e in H' such that

$$\text{every edge of } T_e \text{ shares 2 vertices with } e. \tag{8}$$

We partition H'' into three triple systems. Let H_1 be the set of $e \in H''$ containing a pair $f = f_e \subset e$ with $3 \leq d_{H'}(f) \leq 3k$, H_2 be the set of $e \in H''$ with one pair $f_e \subset e$ having $d_{H'}(f_e) = 2$ and two pairs of codegree (in H') at least $3k+1$, and $H_3 = \{e \in H'' : \delta_{H'}(e) \geq 3k+1\}$. By the definition of H'' we have $H_1 \cup H_2 \cup H_3 = H''$ and one of the three cases below must hold.

Case 1: $|H_1| \geq \frac{\varepsilon n^2}{9}$. Let $F = \{f_e : e \in H_1\}$ (f_e is defined above) so that $|F| \geq |H_1|/3k \geq \varepsilon n^2/27k$. For every $f \in F$, choose $S_f \subset N_{H'}(f)$ with $|S_f| = 3$ such that $S_f \cap N_{H_1}(f) \neq \emptyset$ (we can do it, since by definition, each $f \in F$ is f_e for some $e \in H_1$). By Lemma 4.1 applied to F, for a large t there exists $K \subseteq F$ such that $K \cong K_{t,t}$ and for every $f \in K$, $S_f \cap V(K) = \emptyset$. By Theorem 3.7, if t is large enough, there exists $K' \cong K_{2k,2k} \subset K$ and three disjoint list-edge-colorings $\chi_i : K' \to L_{K'}$ such that each χ_i is monochromatic or canonical, or some χ_i is rainbow. Let $X = \{x_1, x_2, \ldots, x_{2k}\}$ and $Y = \{y_1, y_2, \ldots, y_{2k}\}$ be the parts of K'. If say coloring χ_1 is rainbow, then clearly $K^+_{2,k-2} \subset K^+_{2k,2k} \subset H_1$ and we are done when $G = K_{2,k-2}$. Suppose $G = S^*_{k-1}$. By the construction of K, there is an edge $zx_1y_1 \in H_1$ such that $z \notin V(K)$. By (8), H' contains a triangle $\{x_1y_1u_1, x_1zu_2, y_1zu_3\}$. For at most four values of $2 \leq i \leq 2k$, $\{z, u_1, u_2, u_3\} \cap \{y_i, \chi_1(x_1y_i)\} \neq \emptyset$. So, H_1 contains $(S^*_{2k-5})^+$ with the center x_1. Since $k \geq 5$, we are done.

Suppose now that no coloring is rainbow. We have three possibilities.

Case 1.1. $G = S^*_{k-1}$. If some coloring χ_i is monochromatic, say, $\chi_1(e) = \alpha$ for all $e \in K_{2k,2k}$, then the edges $x_iy_i\alpha$ for $1 \leq i \leq k-1$ and the edge $x_1y_2\chi_2(x_1y_2)$ form a $(S^*_{k-1})^+ \subset H'$ with the center α. Otherwise, we may assume that χ_1 is X-canonical. Let α_i be the color in χ_1 common to every edge containing x_i. Since $d_{H'}(y_1\alpha_1) \geq 2$, there is a vertex $w \neq x_1$ such that $wy_1\alpha_1 \in H'$. By symmetry, we may assume that $w \notin \{x_2, y_2, \ldots, x_k, y_k\}$. Then the edges $y_1\alpha_ix_i$ for $2 \leq i \leq k-2$, $wy_1\alpha_1$, $x_1\alpha_1y_2$ and $y_1x_1\chi_2(x_1y_1)$ form a $(S^*_{k-1})^+ \subset H'$ with the center y_1.

Case 1.2. $G = K_{2,k-2}$ *and some coloring* χ_i *is monochromatic.* If two or more of the colorings are monochromatic, say, $\chi_1(e) = \alpha$ and $\chi_2(e) = \beta$ for all $e \in K_{2k,2k}$ with $\alpha \neq \beta$, then the edges $x_iy_i\alpha$ and $x_iy_{i+k}\beta$ for $1 \leq i \leq k$ form a $K^+_{2,k} \subset H'$. If only one coloring is monochromatic, then the other two are canonical. We may assume $\chi_1(e) = \alpha$ for $e \in K_{2k,2k}$ and χ_2 is X-canonical. Let

α_i be the color common in χ_2 to every edge containing x_i. Then the edges $\alpha x_i y_i$ and $\alpha_i x_i y_{k+i}$ for $1 \leq i \leq k$ form a $K_{2,k}^+ \subset H'$.

Case 1.3: *$G = K_{2,k-2}$ and no coloring is monochromatic.* This means all of the χ_i are canonical. In particular, by symmetry, we can assume χ_1 and χ_2 are both X-canonical. If α_i is the common color of every edge on x_i under χ_1, and β_i is the common color of every edge on x_i under χ_2, then the edges $y_1 x_i \alpha_i$ and $y_2 x_i \beta_i$ for $1 \leq i \leq k$ form a $K_{2,k}^+ \subset H'$. This finishes Case 1.

Case 2: $|H_2| \geq \frac{\varepsilon n^2}{9}$. By the Kövari-Sós-Turán Theorem, for every k there is $s(k)$ such that every subgraph M of $K_{s(k),s(k)}$ with at least $s(k)^2/2$ edges contains a $K_{2k,2k}$. Similarly to Case 1, let $F = \{f_e : e \in H_2\}$, where $d_{H'}(f_e) = 2$. For every $f \in F$, let $S_f = N_{H'}(f)$. By definition, $|S_f| = 2$ and $S_f \cap N_{H_2}(f) \neq \emptyset$. Then $|F| \geq |H_2|/2 \geq \varepsilon n^2/18$. By Lemma 4.1 applied to F, for a large t there exists $K \subseteq F$ such that $K \cong K_{t,t}$ and for every $f \in K$, $S_f \cap V(K) = \emptyset$. By Theorem 3.7, if t is large enough, there exists $K_0 \cong K_{s(k),s(k)} \subset K$ and disjoint list-edge-colorings χ_1 and χ_2 of K_0 such that each χ_i is monochromatic or canonical, or some χ_i is rainbow. Since each of the lists contains a color corresponding to an edge in H_2, we may assume that for at least of half of the edges $f \in K_0$, $f \cup \{\chi_1(f)\} \in H_2$. Then by the definition of $s(k)$, there exists $K' \cong K_{2k,2k} \subset K_0$ such that for every $f \in K'$, $f \cup \{\chi_1(f)\} \in H_2$. Now we repeat the proof of Case 1 word by word till (and including) Case 1.2, since in these subcases we have used only two colorings. In Case 1.3, the problem arises only when χ_1 is X-canonical and χ_2 is Y-canonical (or vice-versa). Let $\chi_2(x_1 y_i) = \alpha_i$ for $1 \leq i \leq k$. Since χ_1 is X-canonical, we have edges $y_i x_1 \gamma \in H'$ for $1 \leq i \leq k$, where γ is the common color of all edges on x_1 in χ_1. By construction, for every $1 \leq i \leq k$, edge $x_1 y_i \gamma$ is in H_2 and hence $d_{H'}(y_i \gamma) \geq 3k + 1$. Therefore we may choose vertices $\beta_1, \beta_2, \ldots, \beta_k \in L_K \backslash \{y_1, y_2, \ldots, y_k, \alpha_1, \alpha_2, \ldots, \alpha_k\}$ such that $\gamma \beta_i y_i$ are all edges of H'. These edges together with the edges $x_1 y_i \alpha_i$ form $K_{2,k}^+ \subset H'$.

Case 3: $|H_3| \geq \frac{\varepsilon n^2}{9}$. If $|\partial H_3| > \frac{\varepsilon}{200k} n^2$, then similarly to Case 1, for every $f \in \partial H_3$, choose $S_f \subset N_{H'}(f)$ with $|S_f| = 3$ such that $S_f \cap N_{H_3}(f) \neq \emptyset$. By Lemma 4.1 applied to $F = \partial H_3$, for a large t there exists $K \subseteq F$ such that $K \cong K_{t,t}$ and for every $f \in K$, $S_f \cap V(K) = \emptyset$. From this point, we just repeat the proof of Case 1.

So $|\partial H_3| \leq \frac{\varepsilon}{200k} n^2$. Then by Lemma 2.2, H_3 contains an $8k$-full subgraph H^* with at least $|H_3| - 8k|\partial H_3| \geq \frac{\epsilon}{20} n^2$ edges. Since $|H^*| \geq \frac{\epsilon}{20} n^2$ and $|\partial H^*| \leq |\partial H_3| \leq \frac{\varepsilon}{200k} n^2$, we have $d_{H^*}(xy) \geq 2k$ for some $xy \in \partial H^*$. This means that the edge xy in the graph ∂H^* is in at least $2k$ triangles by (8). So, ∂H^* contains S_k^* with the center x and $K_{2,k}$ with the small partite set $\{x, y\}$. This means ∂H^* contains a copy of G. Since H^* is $8k$-full, our copy of G greedily extends to $G^+ \subset H^*$. This finishes the main proof.

The jump in the Turán number follows immediately by observing that if $\sigma(G^+) \geq 3$, then we may apply (1) and obtain $\mathrm{ex}_3(n, G^+) \geq (2 - o(1))\binom{n}{2}$. □

8 Concluding Remarks

• Our methods can be used to determine the order of magnitude of the Turán number of expansions of other bipartite graphs like the 3-dimensional cube and complete bipartite graphs. These will be presented in a forthcoming paper.

• Our approach may also be suitable for other extremal problems on trees and forests in hypergraphs including the following conjecture of Kalai (see Frankl and Füredi [7]), extending the Erdős-Sós Conjecture to r-graphs. An r-tree is an r-graph with edges $e_1, \ldots, e_q$ where for each i, $e_i \cap (\cup_{j<i} e_j) \subset e_k$ for some $k < i$.

Conjecture 8.1. (Erdős-Sós for graphs and Kalai 1984 for $r \geq 3$) *Let $r \geq 2$ and T be an r-tree on v vertices. Then*

$$ex_r(n, T) \leq \frac{v-r}{r}\binom{n}{r-1}.$$

This conjecture has been solved for certain classes of trees (see [7]).

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