

MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago (Kashcheyeva, Nicholls)
December 13, 2007

Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Solve the initial value problems

(a) (10 points)

$$(x - 1)\frac{dy}{dx} = xy^2 + x, \quad y(0) = -1,$$

(b) (10 points)

$$\left(e^{x+y} - \frac{1}{x}\right)dx + (e^{x+y} - 1)dy = 0, \quad y(1) = 0.$$

2. (20 points) Consider the initial value problem

$$\frac{dx}{dt} = \sqrt{x} + 4t \quad x(2) = 4.$$

Use Euler's method with step size $h = .5$ to estimate $x(3)$.

3. (20 points) Consider the equation $y'' + 2y' + y = e^{-t} + 3$.

(a) (15 points) Find a particular solution (you must solve for the unknown coefficients).

(b) (5 points) Write down the general solution.

4. (20 points) A tank initially contains 40 L of pure water. A solution of salt and water is fed into the tank at a rate of 1 L/min with a concentration of .2 kg/L. The solution is drained from the tank at a rate of 1 L/min. Determine $x(t)$, the amount of salt in the tank at time t .

5. (20 points) Find the general solution of the system of ordinary differential equations:

$$x' + y' - 5y = 0$$

$$x + y' + 5y = 0.$$

6. (20 points) Find the inverse Laplace transform of:

$$F(s) = e^{-3s} \left(\frac{-2s + 42}{s^2 - 6s - 27} \right).$$

7. (20 points) Solve the initial value problem:

$$y''(t) + y'(t) - 12y(t) = 2\delta(t - 5), \quad y(0) = 1, \quad y'(0) = 10.$$

8. (20 points) Find *all* solutions of the boundary value problem:

$$\begin{aligned} y'' + 25y &= 0, & 0 < x < 5\pi \\ y(0) &= 0, \\ y(5\pi) &= 0. \end{aligned}$$

9. (20 points) Consider the function

$$f(x) = \begin{cases} 2\pi & 0 < x < 1 \\ -\pi & 1 < x < 2. \end{cases}$$

(a) (18 points) Find a Fourier sine series for $f(x)$ on $0 < x < 2$.

(b) (2 points) To what value does this series converge at $x = 1$?

10. (20 points) Solve the vibrating string problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= 4 \frac{\partial^2 u}{\partial x^2} & 0 < x < \pi, \quad t > 0 \\ u(0, t) &= u(\pi, t) = 0 \\ u(x, 0) &= 7 \sin x - 6 \sin 3x \\ \frac{\partial u}{\partial t}(x, 0) &= 2 \sin 3x + \sin 4x. \end{aligned}$$

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2 + b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
14. $\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f\}(s)$
16. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$
17. $\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$
18. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$
19. If f has period T then
$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1 - e^{-sT}} = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}$$
20. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$