MATHEMATICS 220: FINAL EXAM
University of Illinois at Chicago (Cabrera, Nicholls)
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Please read the exam carefully and follow all instructions. **SHOW ALL OF YOUR WORK.** Please put a box around your final answer.

1. (20 points) Consider the differential equation:

$$(2xy + \cos x)\,dx + x^2\,dy = 0.$$

   (a) (6 points) Show that the equation is linear and exact.

   (b) (14 points) Find the general solution.

2. (20 points) Solve the initial value problem:

$$y'' + 4y' + 5y = 0; \quad y(0) = 1, \quad y'(0) = 0.$$

3. (20 points) Consider the initial value problem:

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}; \quad y(0) = 1.$$

   (a) (10 points) Estimate $y(2)$ using 2 steps of Euler's Method.

   (b) (10 points) Find the exact solution, $y(x)$.

4. (20 points) Find the general solution of the differential equation:

$$y''(t) + 4y(t) = \sec(2t).$$

5. (20 points) Consider the system of ODE:

$$\begin{aligned} x' &= x^2 + \cos(y) - e^{-2x}y^{-2} \\ y' &= 2e^{-2x}y^{-1} - 2xy; \end{aligned}$$

   find the integral curves (i.e., solve the "phase plane equation" giving an implicit solution for $y(x)$).

6. (25 points) Find the inverse Laplace transform of:

$$F(s) = e^{-4s}\left(\frac{9s^2 - 16s + 4}{s(s^2 - 3s + 2)}\right).$$

7. (25 points) Consider the initial value problem:

$$x'' - 4x = 3\delta(t-2); \quad x(0) = 0, \quad x'(0) = 1.$$

(a) (15 points) Find $X(s) = \mathscr{L}\{x(t)\}$, the Laplace transform of the solution.

(b) (10 points) Use the inverse Laplace transform to find the solution $x(t)$.

8. (25 points) Find the solution $u(x,t)$ to:

$$\begin{aligned} &u_t = u_{xx} \\ &u(x,0) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases} \\ &u(0,t) = u(\pi,t) = 0. \end{aligned}$$

9. (25 points) Find the real values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also determine the nontrivial solutions (eigenfunctions).

$$\begin{aligned} &y'' + \lambda y = 0, && 0 < x < 1 \\ &y'(0) = 0 \\ &y(1) = 0. \end{aligned}$$

List of Laplace Transforms

1. $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$
2. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
3. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$
4. $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad s > 0$
5. $\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \quad s > 0$
6. $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \quad s > a$
7. $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \quad s > a$
8. $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \quad s > a$
9. $\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$
10. $\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$
11. $\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f\}(s-a)$
12. $\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$
13. $\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - sf(0) - f'(0)$
14. $\mathcal{L}\left\{f^{(n)}\right\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
15. $\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}(s)$
16. $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$
17. $\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$
18. $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$
19. If f has period T then

$$\mathcal{L}\{f\}(s) = \frac{F_T(s)}{1-e^{-sT}} = \frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

20. $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$

List of PDE Formulae

1. The solution of the homogeneous heat equation with Dirichlet boundary conditions is:
$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(\beta n\pi/L)^2 t} \sin\left(\frac{n\pi}{L}x\right).$$

2. The solution of the homogeneous heat equation with Neumann boundary conditions is:
$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-(\beta n\pi/L)^2 t} \cos\left(\frac{n\pi}{L}x\right).$$

3. The inhomogeneous heat equation has a solution of the form $u(x,t) = v(x) + w(x,t)$, where v is the steady–state solution and w solves a homogeneous heat equation.

4. The solution of the homogeneous wave equation with Dirichlet boundary conditions is:
$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\alpha\frac{n\pi}{L}t\right) + c_n \sin\left(\alpha\frac{n\pi}{L}t\right)\right\} \sin\left(\frac{n\pi}{L}x\right).$$