

Mathematics 586: Homework 6 Supplement

1. Consider the function $f(x) = e^x$.

(a) Verify computationally that both the “forward difference” approximation

$$(\delta_+ f)(x; h) := \frac{f(x+h) - f(x)}{h}$$

and the “backward difference” approximation

$$(\delta_- f)(x; h) := \frac{f(x) - f(x-h)}{h}$$

simulate $f'(x)$ to order one in h at $x = 1$. In particular, pick several (four or five) values of $h = h_j$, compute $(\delta_+ f)(1; h_j)$ and $(\delta_- f)(1; h_j)$, compute the errors

$$\varepsilon_+(h_j) = \frac{|(\delta_+ f)(1; h_j) - f'(1)|}{|f'(1)|}, \quad \varepsilon_-(h_j) = \frac{|(\delta_- f)(1; h_j) - f'(1)|}{|f'(1)|},$$

and then do a least squares fit to the error relation $\varepsilon = Ch^r$; you should see $r \approx 1$. To do this fit, use the fact that

$$\log(\varepsilon) = \log(C) + r \log(h),$$

so that you can do a *linear* least squares fit to the data $\{\log(h_j), \log(\varepsilon_j)\}$.

(b) Verify computationally that the “centered difference” approximation

$$(\delta_0 f)(x; h) := \frac{f(x+h) - f(x-h)}{2h}$$

simulates $f'(x)$ to order two in h at $x = 1$.

2. Consider the heat equation:

$$\begin{aligned} \partial_t u &= \partial_x^2 u \\ u(x, 0) &= \sin(2x) \\ u(0, t) &= 0 \\ u(\pi, t) &= 0. \end{aligned}$$

- Find the exact solution using the method of separation of variables.
- Implement the Time–Forward/Space–Centered Finite Difference scheme to approximate the solution to the heat equation above. Keeping the ratio $\alpha = k/h^2$ fixed at 0.25, choose four or five (k, h) pairs to demonstrate the convergence of your code.
- Using the method outlined in the previous problem computationally identify the order of convergence. Why do you get this answer?