Mathematics 586: Homework 7

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1. Implement the Thomas algorithm. Consider the Crank–Nicholson matrix

$$M = \begin{pmatrix} 1 & 0 & & \dots & & 0 \\ -\alpha/2 & (1+\alpha) & -\alpha/2 & 0 & \dots & 0 \\ & & & & & \\ & & \ddots & \ddots & & \\ & & & & & \\ 0 & \dots & 0 & -\alpha/2 & (1+\alpha) & -\alpha/2 \\ 0 & & \dots & & 0 & 1 \end{pmatrix} \in \mathbf{R}^{m\times m},$$

with $\alpha = 0.25$, and the vectors $\vec{x} = (\pi, \dots, \pi)^T \in \mathbf{R}^m$ and $\vec{b} = (\pi, \dots, \pi)^T \in \mathbf{R}^m$ (Note that these choices give us the solution $\vec{x} = M^{-1}\vec{b}$).

   (a) Solve the system $M\vec{x} = \vec{b}$ using the Thomas algorithm for $m = 10, 10^2, 10^3, 10^4, 10^5$. Report the relative errors

$$e_{rel} := \frac{\|x_{approx} - x_{exact}\|}{\|x_{exact}\|},$$

   in your computations.

   (b) Compute the execution time T_m for each of these simulations (in MATLAB this can be accomplished with the `tic` and `toc` commands). Verify via a least–squares fit of the data that $T_m = Cm^p$ where $p \approx 1$.

   (c) Repeat the experiments above with Gaussian elimination (in MATLAB this can be accomplished with the command `x = M\b`). Verify via a least–squares fit of the data that $T_m = Cm^p$ where $p \approx 3$.

2. Implement the Jacobi, Gauss–Seidel, and Successive Over–Relaxation (SOR) methods for solving tridiagonal systems of linear equations, $M\vec{x} = \vec{b}$. Use the stopping criteria:

$$\max\left\{\|\vec{x}^k - \vec{x}^{k-1}\|, \|A\vec{x}^k - \vec{b}\|\right\} < \tau.$$

Consider again the Crank–Nicholson matrix M, and the vectors $\vec{x}$ and $\vec{b}$ from the previous question.

   (a) Solve the system $M\vec{x} = \vec{b}$ using these three algorithms with $\tau = 10^{-10}$ and $\omega = 1.5$ (for SOR) for $m = 10, 10^2, 10^3, 10^4, 10^5$. Report the relative errors. How many iterations did each method require?

   (b) For

$$\omega_j = j/50, \quad j = 1, \dots, 99$$

   solve $M\vec{x} = \vec{b}$ ($m = 10^4$, $\tau = 10^{-10}$) using SOR. Report the number of iterations required for each ω_j. For which value(s) of ω_j is the number of iterations minimized?

3. The Crank–Nicholson scheme for the heat equation is:

$$-\frac{\alpha}{2}v_{j+1}^{n+1} + (1+\alpha)v_j^{n+1} - \frac{\alpha}{2}v_{j-1}^{n+1} = \frac{\alpha}{2}v_{j+1}^{n} + (1-\alpha)v_j^{n} + \frac{\alpha}{2}v_{j-1}^{n},$$

where $\alpha = k/h^2 > 0$.

   (a) Show that this scheme is consistent with the heat equation.
   (b) Show that this scheme is unconditionally stable.
   (c) Show that this scheme is convergent.

4. The θ–scheme for the heat equation is

$$-\theta\alpha v_{j+1}^{n+1} + (1+2\theta\alpha)v_j^{n+1} - \theta\alpha v_{j-1}^{n+1} = (1-\theta)\alpha v_{j+1}^{n} + (1-2(1-\theta)\alpha)v_j^{n} + (1-\theta)\alpha v_{j-1}^{n},$$

where $\alpha = k/h^2 > 0$, $0 \le \theta \le 1$. (Notice that if $\theta = 1/2$ you get Crank–Nicholson).

   (a) If $1/2 \le \theta \le 1$, show that this scheme is unconditionally stable.
   (b) If $0 < \theta < 1/2$, show that this scheme is stable if

$$\alpha \le \frac{1}{2(1-2\theta)}.$$

5. Consider the heat equation:

$$\begin{aligned} \partial_t u &= \partial_x^2 u \\ u(x,0) &= \sin(2x) \\ u(0,t) &= u(\pi,t) = 0. \end{aligned}$$

   (a) Implement the Time–Backward/Space–Centered Finite Difference scheme to approximate the solution to the heat equation above. Keeping the ratio $\lambda = k/h$ *fixed* at 1.0, choose four or five (k,h) pairs to demonstrate the convergence of your code at $T = 1$ by measuring

$$\varepsilon(k,h) := \frac{\left\|v^N - u_{exact}(\cdot,T)\right\|_h}{\left\|u_{exact}(\cdot,T)\right\|_h}.$$

   (b) Identify the order of convergence of this method. Why do you get this answer?

6. Consider the heat equation:

$$\begin{aligned} \partial_t u &= \partial_x^2 u \\ u(x,0) &= \sin(2x) \\ u(0,t) &= u(\pi,t) = 0. \end{aligned}$$

   (a) Implement the Crank–Nicholson Finite Difference scheme to approximate the solution to the heat equation above. Keeping the ratio $\lambda = k/h$ *fixed* at 1.0, choose four or five (k,h) pairs to demonstrate the convergence of your code at $T = 1$ by measuring

$$\varepsilon(k,h) := \frac{\left\|v^N - u_{exact}(\cdot,T)\right\|_h}{\left\|u_{exact}(\cdot,T)\right\|_h}.$$

   (b) Identify the order of convergence of this method. Why do you get this answer?