1. (Higham Exercise 13.2) Consider the following approach to computing a sequence of approximations \( x_0, x_1, x_2, \ldots \) to \( x^\ast \), the solution of \( F(x^\ast) = 0 \). Given \( x_n \), let \( x_{n+1} \) be the solution of \( p_n(x) = 0 \), where \( p_n(x) \) is an approximation to \( F(x) \) determined by the three conditions: (a.) \( p_n(x) \) is linear, (b.) \( p_n(x_n) = F(x_n) \), and (c.) \( p'(x_n) = F'(x_n) \). Draw a picture to illustrate this construction and then show that \( x_{n+1} \) is given by
\[
x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.
\]

2. (Higham Exercise P13.1) Investigate the convergence of the bisection method on the problem solved by \texttt{ch13.m}.

3. (Higham Exercise 14.2) Verify the identity
\[
\frac{\partial^2 C}{\partial \sigma^2} = \frac{T - t}{4\sigma^3} (\hat{\sigma}^4 - \sigma^4) \frac{\partial C}{\partial \sigma}.
\]

4. (Higham Exercise 15.2) Show that
\[
\hat{b}_M^2 := \frac{1}{M} \sum_{j=1}^{M} (X_j - a_M)^2
\]
satisfies
\[
\mathbb{E} \left[ \hat{b}_M^2 \right] = \left( \frac{M-1}{M} \right) b^2. \tag{1}
\]
This confirms that \( \hat{b}_M^2 \) is not an unbiased estimator of \( \text{var}(X) \).
Conclude from (1) that
\[
b_M^2 := \frac{1}{M-1} \sum_{j=1}^{M} (X_j - a_M)^2
\]
is an unbiased estimator of \( \text{var}(X) \).

5. (Higham Exercise P15.1) Adapt \texttt{ch15.m} to produce a picture like that in Figure 15.2 (A plot of “Option Value Approximation” versus “Number of Samples” with 95 % confidence interval bars for 12–15 values of \( M \) between 10 and \( 10^5 \)).

6. (Higham Exercise 16.2) Starting from
\[
\log \left( \frac{S(n\delta t)}{S_0} \right) = n \log(d) + \log \left( \frac{u}{d} \right) \sum_{j=1}^{n} R_j,
\]
show that
\[
\mathbb{E} \left[ \log \left( \frac{S(n\delta t)}{S_0} \right) \right] = n \log(d) + \log \left( \frac{u}{d} \right) np,
\]
and
\[
\text{var} \left[ \log \left( \frac{S(n\delta t)}{S_0} \right) \right] = \left( \log \left( \frac{u}{d} \right) \right)^2 np(1-p).
\]

7. (Higham Exercise 20.3) Let \( Z \sim N(0, 1) \) and \( Y = \alpha + \beta Z \), for \( \alpha, \beta \in \mathbb{R} \). Show that
\[
\text{var} \left[ (Y - \mathbb{E}(Y))^2 \right] = 2\beta^2.
\]

8. (Higham Exercise 20.5) Show that maximizing
\[
\prod_{j=1}^{M} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\hat{U}_{n+1-j}^2/(2\sigma^2)}, \quad \hat{U}_{n+1-j} = U_{n+1-j}/\sqrt{\Delta t}, \quad U_j := \log(S(t_j)/S(t_{j-1})),
\]
with respect to \( \sigma \) leads to the estimate
\[
\sigma^* = \sqrt{\frac{1}{\Delta t} \frac{1}{M} \sum_{j=1}^{M} U_{n+1-j}^2}.
\]

Hints: (1) Take logs—maximizing a positive quantity is equivalent to maximizing its log, and (2) regard \( \sigma^2 \) as the unknown parameter, rather than \( \sigma \).