MATHEMATICS 586: Homework 4
University of Illinois at Chicago (Professor Nicholls)
Spring 2010

Due Friday, April 16 by 2pm.

1. (Higham Exercise 13.2) Consider the following approach to computing a sequence of approximations $x_0, x_1, x_2, \ldots$ to x^*, the solution of $F(x^*) = 0$. Given x_n, let x_{n+1} be the solution of $p_n(x) = 0$, where $p_n(x)$ is an approximation to $F(x)$ determined by the three conditions: (a.) $p_n(x)$ is linear, (b.) $p_n(x_n) = F(x_n)$, and (c.) $p'(x_n) = F'(x_n)$. Draw a picture to illustrate this construction and then show that x_{n+1} is given by

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}.$$

2. (Higham Exercise P13.1) Investigate the convergence of the bisection method on the problem solved by `ch13.m`.

3. (Higham Exercise 14.2) Verify the identity

$$\frac{\partial^2 C}{\partial \sigma^2} = \frac{T-t}{4\sigma^3}\left(\hat{\sigma}^4 - \sigma^4\right)\frac{\partial C}{\partial \sigma}.$$

4. (Higham Exercise 15.2) Show that

$$\hat{b}_M^2 := \frac{1}{M}\sum_{j=1}^{M}(X_j - a_M)^2$$

satisfies

$$\mathbf{E}\left[\hat{b}_M^2\right] = \left(\frac{M-1}{M}\right) b^2. \tag{1}$$

This confirms that $\hat{b}_M^2$ is not an unbiased estimator of $\text{var}(X)$.

Conclude from (1) that

$$b_M^2 := \frac{1}{M-1}\sum_{j=1}^{M}(X_j - a_M)^2$$

is an unbiased estimator of $\text{var}(X)$.

5. (Higham Exercise P15.1) Adapt `ch15.m` to produce a picture like that in Figure 15.2 (A plot of "Option Value Approximation" versus "Number of Samples" with 95 % confidence interval bars for 12–15 values of M between 10 and 10^5).

6. (Higham Exercise 16.2) Starting from

$$\log\left(\frac{S(n\delta t)}{S_0}\right) = n\log(d) + \log\left(\frac{u}{d}\right)\sum_{j=1}^{n} R_j,$$

show that

$$\mathbf{E}\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right] = n\log(d) + \log\left(\frac{u}{d}\right)np,$$

and

$$\text{var}\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right] = \left(\log\left(\frac{u}{d}\right)\right)^2 np(1-p).$$

7. (Higham Exercise 20.3) Let $Z \sim N(0,1)$ and $Y = \alpha + \beta Z$, for $\alpha, \beta \in \mathbf{R}$. Show that

$$\text{var}\left[(Y - \mathbf{E}(Y))^2\right] = 2\beta^4.$$

8. (Higham Exercise 20.5) Show that maximizing

$$\prod_{j=1}^{M} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\hat{U}^2_{n+1-j}/(2\sigma^2)}, \quad \hat{U}_{n+1-j} = U_{n+1-j}/\sqrt{\Delta t}, \quad U_j := \log(S(t_j)/S(t_{j-1})),$$

with respect to σ leads to the estimate

$$\sigma^* = \sqrt{\frac{1}{\Delta t}\frac{1}{M}\sum_{j=1}^{M} U^2_{n+1-j}}.$$

Hints: (1) Take logs–maximizing a positive quantity is equivalent to maximizing its log, and (2) regard σ^2 as the unknown parameter, rather than σ.