MATHEMATICS 586: Homework 3
University of Illinois at Chicago (Professor Nicholls)
Spring 2024

Due Friday, March 15 by 2pm.

1. Find the explicit solution to the obstacle problem

$$
\begin{aligned}
& u(-1) = 0, && && \text{(1a)}\\
& u'' = 0, && -1 < x < A && \text{(1b)}\\
& u(A) = f(A), \quad u'(A) = f'(A) && && \text{(1c)}\\
& u(x) = f(x), && A < x < B && \text{(1d)}\\
& u(B) = f(B), \quad u'(B) = f'(B) && && \text{(1e)}\\
& u'' = 0, && B < x < 1 && \text{(1f)}\\
& u(1) = 0, && && \text{(1g)}
\end{aligned}
$$

   when the obstacle is

   (a) $f(x) = \frac{1}{2} - x^2$,
   (b) $f(x) = \frac{1}{2} - \sin^2(\pi x/2)$. (The free boundaries now have to be found numerically.)

2. Write a computer program to solve the obstacle problem (1) for an arbitrary function $f(x)$, using the Finite Difference method with the projected SOR algorithm. Compare the numerical solution *quantitatively* with the exact solution when

   (a) $f(x) = \frac{1}{2} - x^2$,
   (b) $f(x) = \frac{1}{2} - \sin^2(\pi x/2)$.

   For instance, if you set $\omega = 1.8$, $N = 1600$, and tolerance 10^{-6}, how many iterations are required for convergence? What is the (L^∞) norm of the difference between your approximate solution and the exact one?

3. Wilmott, Howison, and Dewynne: Chapter 9, #5. (Do not answer the questions in the last sentence.)

4. (Johnson: Exercise 1.1) Show that if w is continuous on $[0, 1]$ and

$$\int_0^1 wv \, \mathrm{d}x = 0, \quad \forall v \in V,$$

   then $w(x) \equiv 0$ for $x \in [0, 1]$.

5. (Johnson: Exercise 1.16) Show that the problem

$$
\begin{aligned}
-u''(x) &= f(x) && \text{on } I = (0,1)\\
u(0) &= u'(1) = 0,
\end{aligned}
$$

   can be given the following variational formulation: Find $u \in V$ such that

$$(u', v') = (f, v), \quad \forall v \in V,$$

where $V = \{v \mid v \text{ cont.}, v' \text{ p/w cont.}, v(0) = 0\}$. Formulate a Finite Element Method for this problem using piecewise linear functions. Determine the corresponding linear system of equations in the case of a uniform partition and study in particular how the boundary condition $u'(1) = 0$ is approximated by the method.

6. Solve the following one–dimensional Poisson problem by the Finite Element Method:

$$\begin{aligned} -u''(x) &= \pi^2 \sin(\pi x) \qquad\qquad 0 < x < 1 \\ u(0) &= u(1) = 0. \end{aligned}$$

Use piecewise linear functions and verify the second order accuracy by using 20, 40, and 80 uniform elements. Note that the exact solution is:

$$u_{exact}(x) = \sin(\pi x).$$