MATHEMATICS 586: Homework 4
University of Illinois at Chicago (Professor Nicholls)
Spring 2024

Due Friday, April 19 by 2pm.

1. (Higham Exercise 15.2) Show that
$$\hat{b}_M^2 := \frac{1}{M}\sum_{i=1}^{M}(X_i - a_M)^2,$$
satisfies
$$\mathbf{E}\left[\hat{b}_M^2\right] = \left(\frac{M-1}{M}\right) b^2.$$
This confirms that $\hat{b}_M^2$ is not an unbiased estimator of $\mathrm{Var}(X)$. Conclude from this that b_M^2 *is* an unbiased estimator of $\mathrm{Var}(X)$.

2. (Higham Exercise P15.1) Adapt `ch15.m` to produce a picture like that in Figure 15.2.

3. (Higham Exercise 16.2) Starting from
$$\log\left(\frac{S(n\delta t)}{S_0}\right) = n\log d + \log(u/d)\sum_{i=1}^{n} R_i,$$
show that
$$\mathbf{E}\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right] = n\log d + \log(u/d)np.$$
and
$$\mathrm{Var}\left[\log\left(\frac{S(n\delta t)}{S_0}\right)\right] = \left(\log(u/d)\right)^2 np(1-p).$$

4. (Higham Exercise P16.1) Alter `ch16.m` so that the choice
$$u = e^{r\delta t}\left(1 + \sqrt{e^{\sigma^2\delta t} - 1}\right), \quad d = e^{r\delta t}\left(1 - \sqrt{e^{\sigma^2\delta t} - 1}\right),$$
are used. Reproduce Table 16.1 and Figure 16.2.

5. (Higham Exercise 21.7) Show that the antithetic estimators
$$\hat{I}_M = \frac{1}{M}\sum_{i=1}^{M}\frac{f(U_i) + f(1-U_i)}{2}, \quad U_i \sim U(0,1),$$
and
$$\hat{I}_M = \frac{1}{M}\sum_{i=1}^{M}\frac{f(U_i) + f(-U_i)}{2}, \quad U_i \sim N(0,1),$$
are exact in the case where f is linear, that is, $f(x) = \alpha x + \beta$, for $\alpha, \beta \in \mathbf{R}$.

6. (Higham Exercise 22.2) Show that $\mathrm{Var}(Z_\theta) < \mathrm{Var}(X)$ if and only if θ lies between 0 and $2\theta_{min}$, where
$$\theta_{min} := \mathrm{Cov}(X,Y)/\mathrm{Var}(Y).$$

7. (Higham Exercise P22.1) Alter `ch22.m` so that the θ version is used. Reproduce Table 22.3 for $\theta = 0.9$ and $\theta = 1.1$.