3

High-Order Perturbation of Surfaces Short Course: Analyticity Theory

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Abstract

In this contribution we take up the question of convergence of the classical High-Order Perturbation of Surfaces (HOPS) schemes we introduced in the first lecture. This is intimately tied to analyticity properties of the relevant fields and Dirichlet–Neumann Operators. We show how a straightforward approach cannot succeed. However, with a simple change of variables this very method delivers not only a clear and optimal analyticity theory, but also a stable and high-order numerical scheme. We justify this latter claim with representative numerical simulations involving all three of the HOPS schemes presented thus far.

3.1 Introduction

Over the past two lectures we have derived several High-Order Perturbation of Surfaces (HOPS) schemes for the numerical simulation of (i.) solutions to boundary value problems, (ii.) surface integral operators (e.g., the Dirichlet–Neumann Operator [DNO]), and (iii.) free and moving boundary problems (e.g., traveling water waves). These HOPS methods are rapid (amounting to surface formulations accelerated by FFTs), robust (for perturbation size sufficiently small), and simple to implement (Operator Expansions is a one-line formula, while Field Expansions (FE) is two lines!). The derivation of all of these schemes is based upon the *analyticity* of the unknowns, but we have not yet specified under what conditions this is true. In this contribution we describe a straightforward framework for addressing this question that can be extended to give the most generous hypotheses known.

The first result on analyticity properties of DNOs with respect to boundary perturbations is due to Coifman & Meyer [1]. In this work the DNO was shown to be analytic as a function of Lipschitz perturbations of a line in the plane. Using a different formulation Craig, Schanz, and Sulem [2] and Craig & Nicholls [3, 4] proved analyticity of the DNO for C^1 perturbations of a hyperplane in three and general d dimensions, respectively. All of these results

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depend upon delicate estimates of integral operators appearing in surface formulations of the problem defining the DNO.

Using a completely different approach (which we describe here), the author and F. Reitich produced a greatly simplified approach to establishing analyticity of DNOs and their related fields (at the cost of slightly less generous hypotheses, C^2 smoothness of the boundary perturbation) [5–7], which has the added benefit of generating a stabilized numerical approach. This has been generalized and extended in a number of directions, first to the Helmholtz equation by the author and Reitich [8, 9], and the author and Nigam [10, 11]. This was then extended to traveling water waves by the author and Reitich [12, 13] (see also the work of the author and Akers [14]) and the stability of these by the author [15] (see also [16–18]). On the theoretical side, the author and Hu [19, 20] showed how this very framework could be used to realize these analyticity results with the optimal smoothness requirements (Lipschitz perturbation) provided one is prepared to work with quite complicated function spaces for the field. The method was also extended to doubly perturbed domains by the author and Taber [21, 22], while the author and Fazioli generalized this to the case of *variations* of the DNO with respect to the boundary shape in [23, 24]. A rigorous numerical analysis of this algorithm for a wide array of problems was conducted by the author and Shen [25]. In this paper we revisit the proof presented in the original contribution [5], which, after many years of experience, has been further simplified and distilled.

The rest of the lecture is organized as follows. In § 3.2 we discuss a straightforward approach to an analyticity proof, which fails. In § 3.3 we outline an alternative formulation of the problem that requires a transparent boundary condition (§ 3.3.1) and a change of variables (§ 3.3.2), and results in the method of Transformed Field Expansions – TFE – (§ 4.2.1). In § 3.4 we produce an analyticity proof, which, as a side benefit, results in a stabilized numerical procedure described in § 3.5, which includes numerical tests (§ 3.5.1).

3.2 A Convergence Proof Fails

Inspired by the classical model for waves on the surface of an ideal, deep, two-dimensional fluid [5] we consider the laterally 2π-periodic Laplace problem with Dirichlet data at an irregular interface

$$\Delta v = 0 \qquad y < g(x), \tag{3.2.1a}$$

$$\partial_y v \to 0 \qquad y \to -\infty, \tag{3.2.1b}$$

$$v = \xi \qquad y = g(x). \tag{3.2.1c}$$

Supposing that $g(x) = \varepsilon f(x)$, the FE method is built upon the assumption that v depends *analytically* upon ε so that

$$v = v(x,y;\varepsilon) = \sum_{n=0}^{\infty} v_n(x,y)\varepsilon^n.$$

Following the FE philosophy, we insert this into the problem, (3.2.1), above yielding

$$\Delta v_n = 0 \qquad y < 0, \tag{3.2.2a}$$

$$\partial_y v_n \to 0 \qquad y \to -\infty, \tag{3.2.2b}$$

$$v_n = Q_n \qquad y = 0. \tag{3.2.2c}$$

In this latter system we have shown in the first lecture that

$$Q_n(x) = \delta_{n,0}\xi(x) - \sum_{m=0}^{n-1} F_{n-m}(x)\ \partial_y^{n-m} v_m(x,0), \quad F_m(x) := \frac{f^m(x)}{m!}, \tag{3.2.3}$$

where $\delta_{n,m}$ is the Kronecker delta. We can now wonder whether one can establish analyticity of v *directly* from these recursions, (3.2.3)? The “natural” approach would be to appeal to classical elliptic theory and use the triangle inequality. Recall that, under suitable conditions, the solution of the elliptic problem, (3.2.2), at order $n > 0$ should satisfy

$$\|v_n\|_X \leq C_e \|Q_n\|_Y,$$

for two function spaces (e.g., $X = H^2$ and $Y = H^{3/2}$ [26]). Now, we apply the estimate to our recursions, (3.2.3),

$$\begin{aligned} \|v_n\|_X &\leq C_e \left\| \delta_{n,0}\xi - \sum_{m=0}^{n-1} F_{n-m}(x)\ \partial_y^{n-m} v_m(x,0) \right\|_Y \\ &\leq C_e \left\{ \delta_{n,0} \|\xi\|_Y + \sum_{m=0}^{n-1} \left\| F_{n-m}(x)\ \partial_y^{n-m} v_m(x,0) \right\|_Y \right\}. \end{aligned} \tag{3.2.4}$$

We claim that this last estimate is *useless* as the sum is *unbounded*!

Proof. The boundary condition at order n is

$$v_n(x,0) = \delta_{n,0}\xi(x) - \sum_{m=0}^{n-1} F_{n-m}\partial_y^{n-m} v_m(x,0).$$

Recalling, from separation of variables, that

$$v_n(x,y) = \sum_{p=-\infty}^{\infty} a_{n,p} e^{|p|y} e^{ipx},$$

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we find

$$\sum_{p=-\infty}^{\infty} a_{n,p} e^{ipx} = \delta_{n,0} \sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{ipx} - \sum_{m=0}^{n-1} F_{n-m} \sum_{q=-\infty}^{\infty} |q|^{n-m} a_{m,q} e^{iqx}.$$

At wavenumber p we obtain

$$a_{n,p} = \delta_{n,0}\hat{\xi}_p - \sum_{m=0}^{n-1} \sum_{q=-\infty}^{\infty} \hat{F}_{n-m,p-q} |q|^{n-m} a_{m,q}.$$

To simplify our demonstration we make the choices:

$$f(x) = \xi(x) = 2\cos(x) = e^{ix} + e^{-ix}.$$

With this we discover a number of things. First, regarding the solution at order zero, we have

$$a_{0,p} = \begin{cases} 1 & p = \pm 1, \\ 0 & p \neq \pm 1 \end{cases}.$$

Second, the powers of f satisfy

$$\begin{aligned} f^0 &= 1, \\ f^1 &= e^{ix} + e^{-ix}, \\ f^2 &= e^{2ix} + 2 + e^{-2ix}, \\ f^3 &= e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}, \\ &\vdots \\ f^n &= e^{nix} + ne^{(n-2)ix} + \cdots + ne^{(2-n)ix} + e^{-nix}, \end{aligned}$$

so that $F_n = f^n/n! = e^{nix}/n! + \cdots + e^{-nix}/n!$. Upon defining $P_n := a_{n,n+1}$ we can show that

$$P_n = \delta_{n,0} - \sum_{m=0}^{n-1} \frac{(m+1)^{n-m}}{(n-m)!} P_m,$$

since $\hat{F}_{n,m} = 0$, $|m| > n$, and we can show (by induction) that

$$a_{n,m} = 0, \quad |m| > n+2.$$

We now appeal to the following theorem of Friedman and Reitich [27] which shows that, for this *subset* of Fourier coefficients, $P_n = a_{n,n+1}$, while the sum *converges* for ε sufficiently small, the relevant majorizing sequence (corresponding to the bound (3.2.4)) *diverges* for *any* non-zero choice of ε. □

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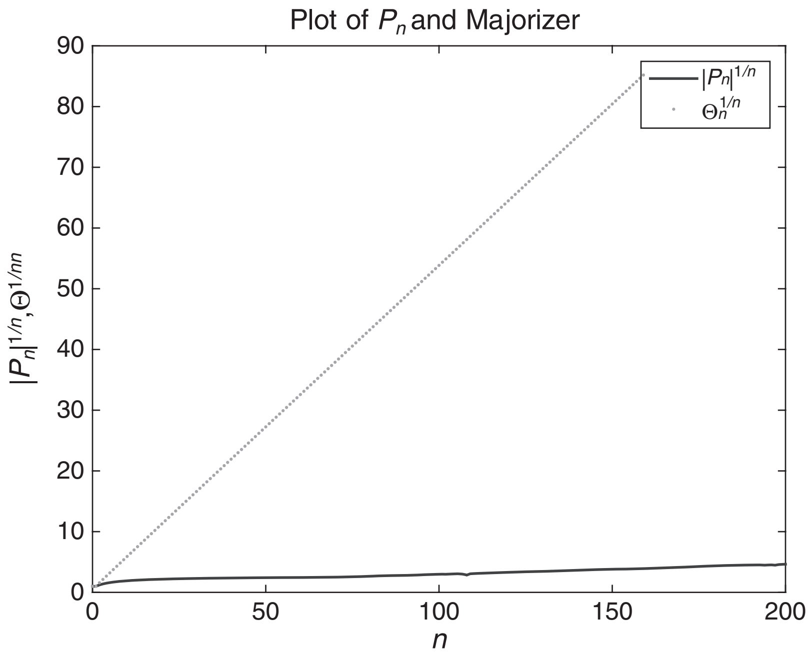

Figure 3.1. Plot of $|P_n|^{1/n}$ and $\Theta_n^{1/n}$ versus n.

Theorem 3.2.1 (Friedman & Reitich [27]). • *The sum* $\sum_{n=0}^{\infty} P_n \varepsilon^n$ *converges for* $\varepsilon < 1/e$.

- *Consider the majorizing sequence* Θ_n *(i.e.,* $|P_n| \le \Theta_n$*) defined by*

$$\Theta_n = \delta_{n,0} + \sum_{m=0}^{n-1} \frac{(m+1)^{n-m}}{(n-m)!} \Theta_m.$$

The sum $\sum_{n=0}^{\infty} \Theta_n \varepsilon^n$ *diverges for all* $\varepsilon > 0$.

A graphical depiction of this result is given in Figure 3.1, which demonstrates a non-zero radius of convergence for P_n and a nonexistent domain of convergence for Θ_n.

3.3 A Change of Coordinates

One answer to the question of why this proof fails is that the "classical" HOPS schemes require the differentiation of the field (e.g., in the case of FE) and/or field trace (e.g., in the case of Operator Expansions [OE]) *across* the boundary of the problem domain, $y = g(x)$. One classical technique for analyzing free- and moving-boundary problems such as these which addresses this concern is a simple change of variables mapping the domain from the deformed geometry

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$\{y < g(x)\}$ to a flat one $\{y' < 0\}$. We pursue a particular (non-conformal) choice, which, in the theory of gratings, is called the C–method [28–30] and, in atmospheric sciences, is known as σ-coordinates [31]. Now, the differentiations take place *within* the problem domain and/or at its boundary. With this strategy we realize a straightforward analyticity proof, and derive a stable and robust numerical algorithm.

3.3.1 A Transparent Boundary Condition

Before we state the change of variables we describe a domain decomposition that is very useful for HOPS methods. Once again, consider the boundary value problem (BVP) (3.2.1) generating the DNO

$$G(g)\xi = \left[\partial_y v - (\partial_x g)\partial_x v\right]_{y=g(x)}. \tag{3.3.1}$$

We choose $b > |g|_\infty$ and define the *artificial boundary* $\{y = -b\}$. It is easy to see that the BVP above, (3.2.1), is equivalent to:

$$\begin{aligned}
\Delta v &= 0 && -b < y < g(x),\\
v &= \xi && y = g(x),\\
\Delta w &= 0 && y < -b,\\
\partial_y w &\to 0 && y \to -\infty,\\
v &= w && y = -b,\\
\partial_y v &= \partial_y w && y = -b.
\end{aligned}$$

If we denote $\psi(x) := v(x, -b)$, we can solve the problem

$$\begin{aligned}
\Delta w &= 0 && y < -b, && (3.3.2a)\\
w &= \psi && y = -b, && (3.3.2b)\\
\partial_y w &\to 0 && y \to -\infty, && (3.3.2c)
\end{aligned}$$

and, defining the (second) DNO,

$$S[\psi] := \partial_y w(x, -b),$$

see that the original BVP, (3.2.1), is equivalent to

$$\begin{aligned}
\Delta v &= 0 && -b < y < g(x), && (3.3.3a)\\
v &= \xi && y = g(x), && (3.3.3b)\\
\partial_y v - S[v] &= 0 && y = -b. && (3.3.3c)
\end{aligned}$$

Such a strategy involves a “Transparent Boundary Condition” posed at the artificial boundary $\{y = -b\}$ and this strategy has been widely employed (see, e.g., [32–39]).

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All that remains is to specify S from (3.3.2). We note that the unique, bounded, periodic solution is given by

$$w(x,y) = \sum_{p=-\infty}^{\infty} \hat{\psi}_p e^{|p|(y+b)} e^{ipx}.$$

With this we can compute

$$\partial_y w(x,y) = \sum_{p=-\infty}^{\infty} |p| \, \hat{\psi}_p e^{|p|(y+b)} e^{ipx},$$

so that

$$S[\psi] = \partial_y w(x,-b) = \sum_{p=-\infty}^{\infty} |p| \, \hat{\psi}_p e^{ipx} = |D| \, \psi,$$

which defines the order-one Fourier multiplier $|D|$ (denoted "order-one" as this operator effectively takes one derivative).

3.3.2 The Change of Variables

To describe the idea behind our TFE approach we fix on the problem (3.3.3), and consider the "domain flattening" change of variables

$$x' = x, \qquad y' = b\left(\frac{y - g(x)}{b + g(x)}\right),$$

which maps $\{-b < y < g(x)\}$ to $\{-b < y' < 0\}$. Note that these can be inverted by

$$x = x', \qquad y = \frac{y'(b + g(x'))}{b} + g(x').$$

Defining the transformed field

$$u(x',y') := v(x', y'(b + g(x'))/b + g(x')),$$

since

$$\begin{aligned}\frac{\partial y'}{\partial x} &= b\left(\frac{(-\partial_x g)(b+g) - (y-g)(\partial_x g)}{(b+g)^2}\right) \\ &= b\left(\frac{(-\partial_x g)(b+g) - (y'(b+g)/b)(\partial_x g)}{(b+g)^2}\right) = -(\partial_{x'} g)\left(\frac{b+y'}{b+g}\right),\end{aligned}$$

it is not difficult to show that

$$\partial_x v = (\partial_{x'} u)(\partial_x x') + (\partial_{y'} u)(\partial_x y') = \partial_{x'} u - \frac{\partial_{x'} g}{b+g}(b+y')\partial_{y'} u$$

so that

$$(b+g)\partial_x = (b+g)\partial_{x'} - (\partial_{x'} g)(b+y')\partial_{y'}.$$

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Also,

$$\partial_y v = (\partial_{x'} u)(\partial_y x') + (\partial_{y'} u)(\partial_y y') = \frac{b}{b+g}\partial_{y'} u,$$

so that

$$(b+g)\partial_y = b\partial_{y'}.$$

We summarize these as

$$M(x)\partial_x = M(x')\partial_{x'} + N(x',y')\partial_{y'}, \quad M(x)\partial_y = b\partial_{y'},$$

where

$$M(x') := b + g(x'), \quad N(x',y') := -(\partial_{x'} g)(y' + b).$$

Laplace's equation, (3.3.3a), implies

$$\begin{aligned} 0 &= M^2 \Delta v = M^2 \partial_x^2 v + M^2 \partial_y^2 v \\ &= M\partial_x [M\partial_x v] - (\partial_x M) M \partial_x v + M\partial_y \left[M\partial_y v\right]. \end{aligned}$$

Applying the change of variables yields

$$\begin{aligned} 0 = &\left[M\partial_{x'} + N\partial_{y'}\right]\left[M\partial_{x'} u + N\partial_{y'} u\right] \\ &- (\partial_{x'} M)\left[M\partial_{x'} u + N\partial_{y'} u\right] + b\partial_{y'}\left[b\partial_{y'} u\right] \\ = &\, M\partial_{x'}\left[M\partial_{x'} u\right] + N\partial_{y'}\left[M\partial_{x'} u\right] + M\partial_{x'}\left[N\partial_{y'} u\right] + N\partial_{y'}\left[N\partial_{y'} u\right] \\ &- (\partial_{x'} M) M\partial_{x'} u - (\partial_{x'} M) N\partial_{y'} u + b\partial_{y'}\left[b\partial_{y'} u\right]. \end{aligned}$$

Attempting to put as many terms in "divergence form" as possible:

$$\begin{aligned} 0 = &\,\partial_{x'}\left[M^2\partial_{x'} u\right] - (\partial_{x'} M) M\partial_{x'} u + \partial_{y'}\left[NM\partial_{x'} u\right] - (\partial_{y'} N) M\partial_{x'} u \\ &+ \partial_{x'}\left[MN\partial_{y'} u\right] - (\partial_{x'} M) N\partial_{y'} u + \partial_{y'}\left[N^2\partial_{y'} u\right] - (\partial_{y'} N) N\partial_{y'} u \\ &- (\partial_{x'} M) M\partial_{x'} u - (\partial_{x'} M) N\partial_{y'} u + \partial_{y'}\left[b^2\partial_{y'} u\right]. \end{aligned}$$

This leads to

$$\operatorname{div}'\left[\begin{pmatrix} M^2 & MN \\ MN & b^2 + N^2 \end{pmatrix}\nabla' u\right] - (\partial_{x'} g)\begin{pmatrix} M \\ N \end{pmatrix}\cdot\nabla' u = 0,$$

where we have used

$$-2\partial_{x'} M - \partial_{y'} N = -\partial_{x'} g.$$

We write this as

$$\operatorname{div}'\left[A\nabla' u\right] - (\partial_{x'} g) B\cdot\nabla' u = 0,$$

where

$$A := \begin{pmatrix} M^2 & MN \\ MN & b^2 + N^2 \end{pmatrix}, \qquad B := \begin{pmatrix} M \\ N \end{pmatrix}.$$

We note that

$$A = b^2 I + A_1(g) + A_2(g), \quad B = B_0 + B_1(g),$$

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where

$$A_1(g) = \begin{pmatrix} 2bg & -b(y'+b)(\partial_{x'} g) \\ -b(y'+b)(\partial_{x'} g) & 0 \end{pmatrix},$$

$$A_2(g) = \begin{pmatrix} g^2 & -(y'+b)g(\partial_{x'} g) \\ -(y'+b)g(\partial_{x'} g) & (y'+b)^2(\partial_{x'} g)^2 \end{pmatrix},$$

and

$$B_0 = \begin{pmatrix} b \\ 0 \end{pmatrix}, \qquad B_1(g) = \begin{pmatrix} g \\ -(y'+b)(\partial_{x'} g) \end{pmatrix}.$$

Therefore, we can write

$$b^2 \Delta' u = F(x'; g, u), \tag{3.3.4}$$

where

$$F := -\operatorname{div}'\left[A_1 \nabla' u\right] - \operatorname{div}'\left[A_2 \nabla' u\right] + (\partial_{x'} g) B_0 \cdot \nabla' u + (\partial_{x'} g) B_1 \cdot \nabla' u, \tag{3.3.5}$$

and $F = \mathcal{O}(g)$.

The interfacial boundary condition, (3.3.3b), becomes

$$u(x', 0) = v(x, g(x)) = \xi(x) = \xi(x').$$

We write the transparent boundary condition, (3.3.3c), as

$$M\partial_y v - MS[v] = 0,$$

and, noting that $u(x', -b) = v(x, -b)$, we find

$$b\partial_{y'} u - MS[u] = 0,$$

so that

$$b\partial_{y'} u - bS[u] = J(x; g, u), \tag{3.3.6}$$

where

$$J = gS[u]. \tag{3.3.7}$$

To close, we write the definition of the DNO, (3.3.1), as

$$MG(g)[\xi] = \left[M\partial_y v - (\partial_x g) M \partial_x v\right]_{y=g}.$$

In our new coordinates we have

$$MG = \left[b\partial_{y'} u - (\partial_{x'} g) M \partial_{x'} u - (\partial_{x'} g) N \partial_{y'} u\right]_{y'=0},$$

which, since $N(x', 0) = -(\partial_{x'} g) b$, we rewrite as

$$bG = b\partial_{y'} u - b(\partial_{x'} g)\partial_{x'} u - g(\partial_{x'} g)\partial_{x'} u + b(\partial_{x'} g)^2 \partial_{y'} u - gG,$$

or

$$bG = b\partial_{y'} u + H(x; g, u), \tag{3.3.8}$$

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where

$$H(x;g,u) := -b(\partial_{x'} g)\partial_{x'} u - g(\partial_{x'} g)\partial_{x'} u + b(\partial_{x'} g)^2 \partial_{y'} u - gG. \qquad (3.3.9)$$

3.3.3 Transformed Field Expansions

Gathering all of these equations and dropping the primes we find that we must seek a ($L = 2\pi$) periodic solution of

$$\begin{aligned} b^2 \Delta u &= F && -b < y < 0, \\ u &= \xi && y = 0, \\ b\partial_y u - bS[u] &= J && y = -b, \end{aligned}$$

to compute the DNO

$$bG = b\partial_y u + H.$$

Supposing that $g = \varepsilon f$, $\varepsilon \ll 1$ for now, our HOPS approach, denoted the "Transformed Field Expansions" (TFE) method, seeks solutions of the form

$$u = u(x,y;\varepsilon) = \sum_{n=0}^{\infty} u_n(x,y)\varepsilon^n, \quad G = G(\varepsilon f)[\xi] = \sum_{n=0}^{\infty} G_n(f)[\xi]\varepsilon^n.$$

These can be shown to satisfy

$$\begin{aligned} b^2 \Delta u_n &= F_n && -b < y < 0 && (3.3.10a) \\ u_n &= \delta_{n,0}\xi && y = 0 && (3.3.10b) \\ b\partial_y u_n - bS[u_n] &= J_n && y = -b, && (3.3.10c) \end{aligned}$$

and

$$bG_n = b\partial_y u_n + H_n. \qquad (3.3.11)$$

In these

$$\begin{aligned} F_n = &-\text{div}\left[A_1(f)\nabla u_{n-1}\right] - \text{div}\left[A_2(f)\nabla u_{n-2}\right] \\ &+ (\partial_x f)B_0 \cdot \nabla u_{n-1} + (\partial_x f)B_1(f) \cdot \nabla u_{n-2}, \end{aligned}$$

and

$$\begin{aligned} J_n &= fS[u_{n-1}], \\ H_n &= -b(\partial_x f)\partial_x u_{n-1} - fG_{n-1} - f(\partial_x f)\partial_x u_{n-2} + b(\partial_x f)^2 \partial_y u_{n-2}. \end{aligned}$$

3.4 A Convergence Proof Succeeds

We now have a recursive sequence of elliptic BVPs, (3.3.10), to solve that *can* be successfully estimated (under generous hypotheses). One can ask why these recursions are "better." An answer is that all derivatives (tangential and normal) are taken at locations *within* the problem domain. Additionally, no more than *first* derivatives of f appear, and no more than *second* powers of f (and its derivative) appear. Our recursive estimation strategy requires two (classical) elements (i.) an "Algebra Lemma" to handle products of functions, and (ii.) an "Elliptic Estimate" to bound solutions of the BVP, (3.3.10), in terms of the inhomogenous terms. The proofs of each can be found in classical texts; see, e.g., Ladyzhenskaya & Ural'tseva [26] or Evans [40].

Lemma 3.4.1. *Given an integer $s \geq 0$ and any $\sigma > 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s([0,2\pi])$, $w \in H^s[0,2\pi] \times [-b,0]$ then*

$$\|f w\|_{H^s} \leq \mathcal{M}(s)\, |f|_{C^s}\, \|w\|_{H^s},$$

and if $\tilde{f} \in C^{s+1/2+\sigma}([0,2\pi])$, $\tilde{w} \in H^{s+1/2}([0,2\pi])$ then

$$\left\|\tilde{f}\tilde{w}\right\|_{H^{s+1/2}} \leq \mathcal{M}(s) \left|\tilde{f}\right|_{C^{s+1/2+\sigma}} \|\tilde{w}\|_{H^{s+1/2}}.$$

Theorem 3.4.2. *Given an integer $s \geq 0$, if $F \in H^s([0,2\pi] \times [-b,0])$, $\xi \in H^{s+3/2}([0,2\pi])$, $J \in H^{s+1/2}([0,2\pi])$, then the unique solution of*

$$\begin{aligned} &b^2 \Delta w = F && -b < y < 0, \\ &w = \xi && y = 0, \\ &b\partial_y w - bS[w] = J && y = -b, \end{aligned}$$

satisfies

$$\|w\|_{H^{s+2}} \leq C_e \left\{ \|F\|_{H^s} + \|\xi\|_{H^{s+3/2}} + \|J\|_{H^{s+1/2}} \right\},$$

for some constant $C_e = C_e(s)$.

3.4.1 The Analyticity Result

In order to establish our desired result, we begin by demonstrating that the field, u, depends analytically upon ε.

Theorem 3.4.3. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,2\pi])$ and $\xi \in H^{s+3/2}([0,2\pi])$ then $u_n \in H^{s+2}([0,2\pi] \times [-b,0])$ and*

$$\|u_n\|_{H^{s+2}} \leq K B^n,$$

for constants $K, B > 0$.

With this in hand we are able to show the following.

Theorem 3.4.4. *Given any integer $s \geq 0$, if $f \in C^{s+2}([0,2\pi])$ and $\xi \in H^{s+3/2}([0,2\pi])$ then $G_n \in H^{s+1/2}([0,2\pi])$ and*

$$\|G_n\|_{H^{s+1/2}} \leq \tilde{K}B^n,$$

for constants $\tilde{K}, B > 0$.

For these we require the following inductive lemma.

Lemma 3.4.5. *Given an integer $s \geq 0$, if $f \in C^{s+2}([0,2\pi])$ and*

$$\|u_n\|_{H^{s+2}} \leq KB^n, \quad \forall\, n < \bar{n},$$

for constants $K, B > 0$, then there exists a constant $\bar{C} > 0$ such that

$$\max\left\{\|F_{\bar{n}}\|_{H^s}, \|J_{\bar{n}}\|_{H^{s+1/2}}\right\} \leq K\bar{C}\left\{|f|_{C^{s+2}} B^{\bar{n}-1} + |f|_{C^{s+2}}^2 B^{\bar{n}-2}\right\}.$$

Proof. For simplicity we focus on the term

$$F_{\bar{n}} = -\mathrm{div}\left[A_1 \nabla u_{\bar{n}-1}\right] + \cdots = -\partial_x\left[A_1^{xx}\partial_x\left[u_{\bar{n}-1}\right]\right] + \cdots,$$

where $A_1^{xx}(x) = 2bf(x)$. We can estimate

$$\begin{aligned}
\|F_{\bar{n}}\|_{H^s} &\leq \left\|A_1^{xx}\partial_x\left[u_{\bar{n}-1}\right]\right\|_{H^{s+1}} + \cdots \leq \mathcal{M}\left|A_1^{xx}\right|_{C^{s+1}} \|u_{\bar{n}-1}\|_{H^{s+2}} + \cdots \\
&\leq \mathcal{M}\left(2b\,|f|_{C^{s+1}}\right) KB^{\bar{n}-1} + \cdots
\end{aligned}$$

So, if we choose

$$\bar{C} > 2b\mathcal{M},$$

then we are done. □

Now we are in a position to prove Theorem 3.4.3.

Theorem 3.4.3. We work by induction in n: At order $n = 0$ we use the elliptic estimate to deduce

$$\|u_0\|_{H^{s+2}} \leq C_e \|\xi\|_{H^{s+3/2}},$$

so we set $K := C_e \|\xi\|_{H^{s+3/2}}$. Now, we suppose that the inductive estimate is valid for all $n < \bar{n}$ and use the elliptic estimate at order $\bar{n}$:

$$\|u_{\bar{n}}\|_{H^{s+2}} \leq C_e\left\{\|F_{\bar{n}}\|_{H^s} + \|J_{\bar{n}}\|_{H^{s+1/2}}\right\}.$$

Lemma 3.4.5 tells us that

$$\|u_{\bar{n}}\|_{H^{s+2}} \leq C_e 2\bar{C}K\left\{|f|_{C^{s+2}} B^{\bar{n}-1} + |f|_{C^{s+2}}^2 B^{\bar{n}-2}\right\}.$$

So, we are done if we choose

$$B > \max\left\{4C_e\bar{C}, 2\sqrt{C_e\bar{C}}\right\}|f|_{C^{s+2}}.$$

□

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We can now prove the analyticity of the DNO.

Theorem 3.4.4. Once again, we use induction in n: At order $n = 0$ we have

$$G_0(x) = \partial_y u_0(x,0),$$

so, from the previous theorem,

$$\|G_0\|_{H^{s+1/2}} = \left\|\partial_y u_0\right\|_{H^{s+1/2}} \leq \|u_0\|_{H^{s+2}} \leq K,$$

and we choose $\tilde{K} \geq K$. Now, we assume the inductive estimate is true for all $n < \bar{n}$ and estimate

$$\|G_{\bar{n}}\|_{H^{s+1/2}} = \left\|\partial_y u_{\bar{n}}\right\|_{H^{s+1/2}} + (1/b)\|H_{\bar{n}}\|_{H^{s+1/2}},$$

A lemma similar to Lemma 3.4.5 delivers

$$\|H_{\bar{n}}\|_{H^{s+1/2}} \leq \tilde{K}\bar{C}\left\{|f|_{C^{s+2}} B^{\bar{n}-1} + |f|^2_{C^{s+2}} B^{\bar{n}-2}\right\}.$$

Again, we are done if we choose

$$B > \max\left\{4C_e\bar{C}, 2\sqrt{C_e\bar{C}}\right\}|f|_{C^{s+2}}.$$

□

Remark We note that the theory can be extended in a number of ways. In particular to finite depth ($h < \infty$), three dimensions (by simply viewing $p \in \mathbf{Z}^2$) [5], and joint analyticity with respect to multiple boundaries [21]. Furthermore, analytic continuation can be justified by considering perturbations about a generic real-valued function $f_0(x)$ [7]. Finally variations of the DNO with respect to boundary deformations can also be shown to be analytic [23].

3.5 Stable Numerics

In light of the powerful and straightforward proof we were able to deliver for the analyticity of both the field and DNO, one can wonder about a numerical simulation based upon these TFE recursions. We recall the recursions for the u_n, (3.3.10), and G_n, (3.3.11), and the first thing we notice is the inhomogeneous nature of these, i.e. $F_n, J_n, H_n \not\equiv 0$ in general. Consequently, a method based upon separation of variables (such as FE and OE) is no longer available, thus a *volumetric* approach is mandated.

As we specify this method, we note that we have already expanded in a Taylor series in the perturbation parameter ε and this is our first "discretization." We work recursively at each perturbation order, so that once we know $\{u_0,\ldots,u_{n-1}\}$ and $\{G_0,\ldots,G_{n-1}\}$ we can form $\{F_n, J_n, H_n\}$. The periodic lateral boundary conditions suggest a Fourier expansion in the x variable.

Finally, at each perturbation order and each wavenumber one must solve a two-point boundary value problem. For this we choose a Chebyshev collocation method [41, 42], and thus we approximate u by

$$u^{N,N_x,N_y}(x',y';\varepsilon) := \sum_{n=0}^{N} \sum_{p=-N_x/2}^{N_x/2-1} \sum_{\ell=0}^{N_y} \hat{u}_n(p,\ell) e^{ipx'} T_\ell(2y'/b+1)\varepsilon^n,$$

where the $\hat{u}_n(p,\ell)$ are determined by a collocation approach [7, 9].

3.5.1 Numerical Tests

We now seek to validate and evaluate this third HOPS scheme, TFE, for approximating DNOs, and compare its performance against that of the previous approaches, FE and OE. Recall the exact solution we used in a previous lecture to test the FE and OE algorithms. If we choose a wavenumber, say r, and a profile $f(x)$, for a given $\varepsilon > 0$, it is easy to see that the Dirichlet data

$$\xi_r(x;\varepsilon) := v_r(x,\varepsilon f(x)) = e^{|r|\varepsilon f(x)} e^{irx},$$

generates Neumann data

$$\begin{aligned} \nu_r(x;\varepsilon) &:= \left[\partial_y v_r - \varepsilon(\partial_x f)\partial_x v_r\right](x,\varepsilon f(x)) \\ &= \left[|r| - \varepsilon(\partial_x f)(ir)\right] e^{|r|\varepsilon f(x)} e^{irx}. \end{aligned}$$

We consider a problem with geometric and numerical parameters

$$\begin{aligned} L = 2\pi, \quad \varepsilon = 0.5, \quad b = 1, \quad f(x) = \exp(\cos(x)), \\ N_x = 256, \quad N_y = 64, \quad N = 16. \end{aligned} \tag{3.5.1}$$

In Figure 3.2 we display results of our numerical experiments with the FE, OE, and TFE algorithms as N is refined from 0 to 16 using Taylor summation. We repeat this with Padé approximation in Figure 3.3 for this moderate perturbation. We notice that, while this deformation is not small, the TFE computation demonstrates it is *within* the disk of analyticity of the DNO. The instabilities in the FE and OE methods (first reported in [5]) render these methods useless with Taylor summation. We do point out that the analytic continuation technique of Padé approximation not only enhances the TFE results, but also renders the FE and OE simulations useful.

We now reconsider these calculations in the context of a much larger deformation size $\varepsilon = 1.0$ (twice as large). In Figures 3.4 and 3.5 we show results generated by the FE, OE, and TFE algorithms as N is refined from 0 to 16 using Taylor and Padé summation, respectively. As is evident from the TFE simulation, here the deformation is large enough so that we are no longer within the disk of analyticity of the DNO. While none of the HOPS algorithms

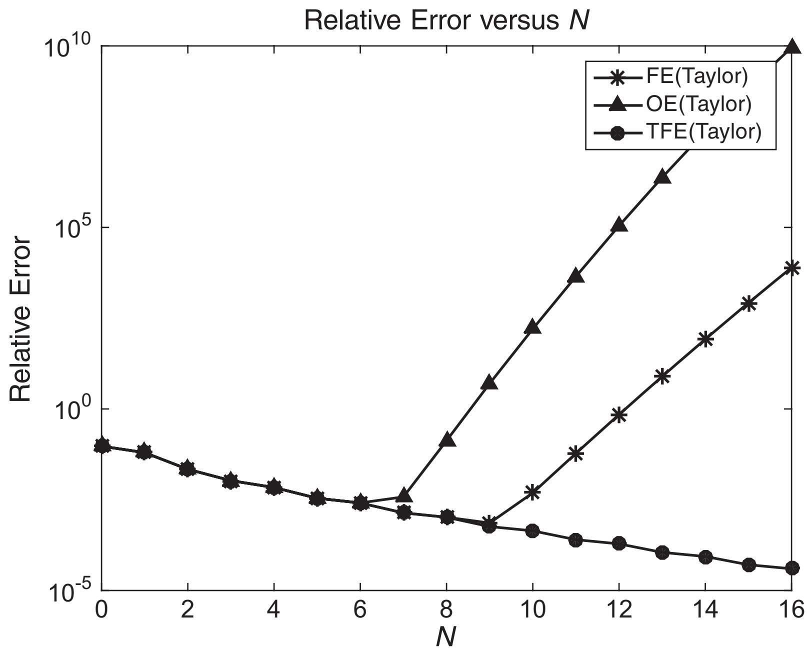

Figure 3.2. Relative error in FE, OE, and TFE algorithms with Taylor summation versus perturbation order N for configuration, (3.5.1), with moderate deformation $\varepsilon = 0.5$.

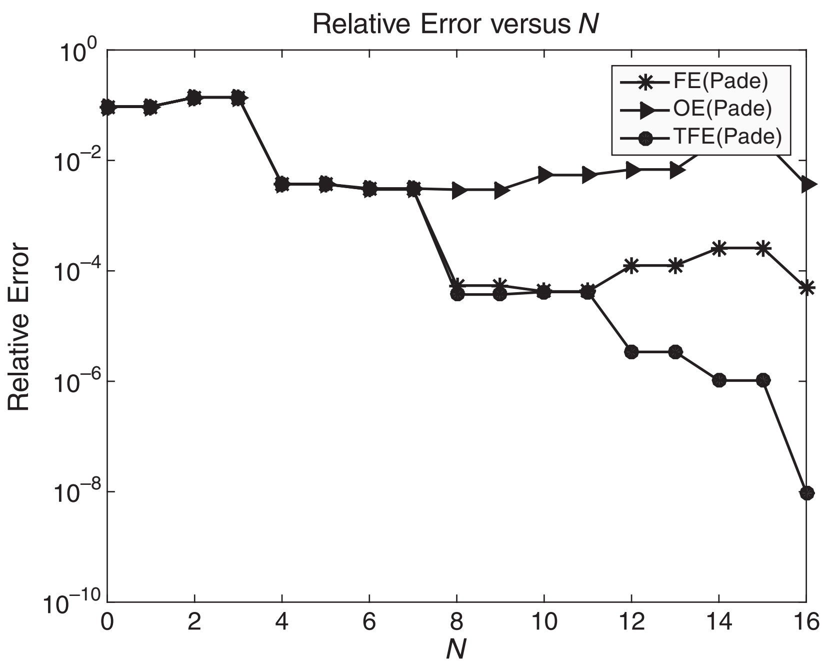

Figure 3.3. Relative error in FE, OE, and TFE algorithms with Padé summation versus perturbation order N for configuration, (3.5.1), with moderate deformation $\varepsilon = 0.5$.

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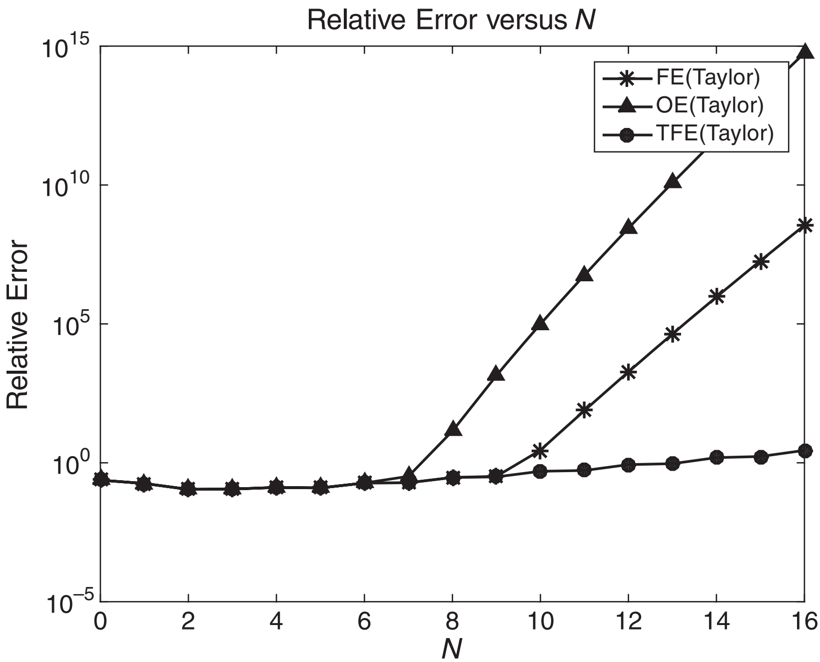

Figure 3.4. Relative error in FE, OE, and TFE algorithms with Taylor summation versus perturbation order N for configuration, (3.5.1), with large deformation $\varepsilon = 1.0$.

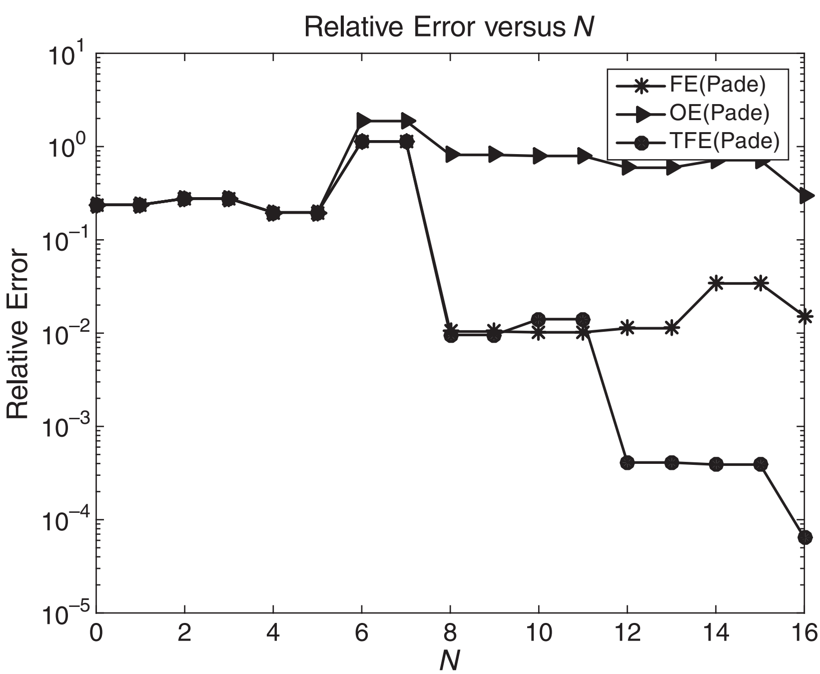

Figure 3.5. Relative error in FE, OE, and TFE algorithms with Padé summation versus perturbation order N for configuration, (3.5.1), with large deformation $\varepsilon = 1.0$.

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deliver any accuracy with Taylor summation (as they are prohibited by the theory), Padé approximation allows one to access the domain of analyticity *outside* the disk of convergence.

Remark As with FE and OE, the TFE algorithm can be extended to finite depth ($h < \infty$) and three dimensions (using $p \in \mathbf{Z}^2$) [6].

Dedication. The author would like to dedicate this contribution to his wonderful daughter, Emma. Without her love and enthusiasm, life would not be near as much fun. He learned many things at the "Theory of Water Waves" programme at the Isaac Newton Institute, but none was more surprising than Emma's demonstration of how exciting it is to ride around Cambridge on the upper level (at the front, of course) of a double-decker bus.

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