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The domain of analyticity of Dirichlet–Neumann operators

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Dirichlet–Neumann operators arise in many applications in the sciences, and this has inspired a number of studies on their analytical properties. In this paper we further investigate the analyticity properties of Dirichlet–Neumann operators as functions of the boundary shape. In particular, we study the size of the disc of convergence of their Taylor-series representation. For this we use a complexification technique which requires a novel reformulation of the problem, coupled with methods for systems of elliptic partial differential equations. Numerical results to illustrate our theoretical conclusions are presented.

1. Introduction

Dirichlet–Neumann operators (DNOs) arise in a diverse array of physically relevant problems. From the boundary-value problems of linear acoustics [14,18] and electromagnetics [7] to the free-boundary problems of solid [16] and fluid mechanics [1,20], these operators (and their higher-order analogues) permit the equivalent restatement of the governing partial differential equations in terms of *boundary* quantities. This not only typically simplifies the statement of the problem (e.g. by incorporating far-field boundary conditions in the unknown solution), but also reduces the dimension of the governing equations, delivering huge savings in a numerical simulation. For these reasons the detailed analytic study of DNOs and their effective numerical simulation have received a great deal of interest in the literature [6,10,17,26].

In the applications listed above, the governing equation is linear with constant coefficients (e.g. the Laplace or the Helmholtz equation) so that the difficulty in computing the DNO lies in the geometric complications. For a simple, separable geometry the DNO is easy to compute and this observation has been used by several groups [11,13–15,19] to specify 'transparent' boundary conditions at artificial boundaries for problems of unbounded extent (e.g. linear acoustics, electromagnetics or linear elasticity). For more complicated geometries, if the domain shape is a small deformation of a separable geometry, then the observations above suggest that a perturbative approach would be fruitful. In fact, this line of enquiry has been followed by several groups with great success [8,9,12,17,21–28,30]. The current contribution compares most closely with the previous work of the present authors [17] on the most general conditions under which the DNO depends *analytically* upon boundary perturbations. Here we use the method of complexification to re-establish the analyticity result. However, this method will additionally provide a radius of convergence (equal to the distance to the closest singularity in the complex plane). More specifically, we show that if the domain is given by $\{(x,y) \mid -h < y < \lambda f(x)\}$, then the domain of analyticity is given by

$$\left\{\lambda \in \mathbb{R} \;\middle|\; \lambda < \left(\left(\frac{|f|_{L^\infty}}{h}\right)^2 + |\nabla_x f|^2\right)^{-1/2}\right\}$$

(cf. theorem 4.5). Actually, if we allow λ to vary in $\mathbb{C}$, after a change of variable (see (2.6)), then the above domain of analyticity is extended to complex λ in the open disc with radius

$$\left(\left(\frac{|f|_{L^\infty}}{h}\right)^2 + |\nabla_x f|^2\right)^{-1/2}.$$

Note that the radius is a combination of the physical obstruction at depth h and the supremum norm of the gradient of the domain shape. This new complexification method not only delivers this crucial information on the location of singularities of the expansion of the DNO, but is also much simpler than the 'method of majorants' used in our previous work [17].

To establish analyticity we show first-order complex differentiability, which we accomplish with a finite difference of our complexified equation. To show that this difference converges, we require elliptic estimates which, in the absence of the imaginary part, are straightforward. However, this ellipticity *fails* if the imaginary part is too large and, thus, we impose conditions on the domain shape and its gradient (see lemma 2.1) to ensure that the complexified system retains its ellipticity. In fact, the domain of ellipticity is given by

$$\left\{\lambda \in \mathbb{C} \;\middle|\; |\operatorname{Re}\lambda|\frac{|f|_{L^\infty}}{h} + |\operatorname{Im}\lambda||\nabla_x f| \leqslant 1-\delta\right\} \tag{1.1}$$

(cf. (2.7)), where $f(x)$ characterizes the domain shape and h denotes its depth. Thus, the real part of the singular set is restricted only by the physical obstruction at depth h, while the imaginary part is constrained by the supremum norm of the gradient of the domain shape.

The elliptic estimates we establish guarantee the convergence of subsequences. However, to show the convergence for the full limit we need uniqueness of solutions to the limit system. This is not obvious as we have an elliptic system posed on an infinite domain. To simplify our approach, we use a particular complexification so that the resulting system does not contain any lower-order terms (see (2.4)). For this system we are able to establish uniqueness rigorously (see theorem 3.1).

The organization of the paper is as follows. In § 2 we recall the defining equations for the DNO associated with the Laplace equation on a deformed strip which arises in ideal free-surface fluid flows [20]. In § 2.1 we review a change of variables which we have found useful in studies of analyticity properties of DNOs, and in § 2.2 we prove a coercivity result which is the key to our theory. In § 3 we give existence and uniqueness results for the complexified equations defining the DNO. Using these, in § 4, we show the analyticity of the solutions of the complexified system together with the DNO itself. In § 5 we display illustrative numerical examples which demonstrate the conclusions of our theorems.

2. Governing equations

As mentioned in § 1, DNOs arise in a wide array of applications. One problem which has generated a considerable amount of interest recently is that of the simulation of ocean waves via the Euler equations of ideal fluid, free-surface fluid mechanics [20]. For an ocean of depth h, the problem domain is the perturbed strip

$$S_{h,\eta} = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid -h < y < \eta(x, t)\},$$

where $d = 2, 3$. The well-known Euler equations governing the motion of the ideal fluid are

$$\begin{aligned} \Delta\varphi &= 0 && \text{in } S_{h,\eta}, \\ \partial_y\varphi &= 0 && \text{at } y = -h, \\ \partial_t\eta &= \partial_y\varphi - \nabla_x\eta \cdot \nabla_x\varphi && \text{at } y = \eta, \\ \partial_t\varphi &= -g\eta - \tfrac{1}{2}|\nabla\varphi|^2 && \text{at } y = \eta, \end{aligned}$$

where φ is the velocity potential ($\boldsymbol{u} = \nabla\varphi$) and g is the gravitational constant. Zakharov [31] showed that this water-wave problem admits a Hamiltonian formulation in terms of *surface* quantities provided that one chooses the canonical variables $(\eta(x, t), \xi(x, t))$, where $\xi(x, t) := \varphi(x, \eta(x, t), t)$ is the velocity potential at the free surface. Craig and Sulem [9] later made this formulation more explicit with the introduction of the DNO, $G(\eta)$, which maps the Dirichlet data ξ to the Neumann data:

$$\nu(x, t) = G(\eta)[\xi(x, t)] := [\partial_y\varphi - \nabla_x\eta \cdot \nabla_x\varphi]_{y=\eta(x,t)}.$$

Our purpose in this paper is to study the analyticity properties of just such a DNO although, for clarity, we remove the specific notation of water waves.

Our starting point is the elliptic boundary-value problem which defines our DNO:

$$\Delta v = 0 \quad \text{in } S_{h,g}, \tag{2.1 a}$$
$$v(x, g(x)) = \xi(x), \tag{2.1 b}$$
$$\partial_y v(x, -h) = 0, \tag{2.1 c}$$

which is posed on the domain

$$S_{h,g} = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid -h < y < g(x)\}.$$

With this notation the DNO, G, is defined by

$$G(g)[\xi] = [\partial_y v - \nabla_x g \cdot \nabla_x v]_{y=g(x)}. \tag{2.2}$$

2.1. A change of variables

We have found the change of variables

$$x' = x, \qquad y' = h\left(\frac{y - g(x)}{h + g(x)}\right)$$

to be very useful in our previous theoretical investigations [17] and we use these again here. Note that this change of variables transforms the domain $S_{h,g}$ to $S_{h,0}$. Defining

$$u(x', y') := v\left(x', y'\left[\frac{h + g(x')}{h}\right] + g(x')\right),$$

we now seek differential equations for u from (2.1) and the transformed differential operators:

$$\begin{aligned}
(h + g(x))\nabla_x &= (h + g(x'))\nabla_{x'} - (h + y')\nabla_{x'} g(x')\partial_{y'},\\
(h + g(x))\operatorname{div}_x &= (h + g(x'))\operatorname{div}_{x'} - (h + y')\nabla_{x'} g(x') \cdot \partial_{y'},\\
(h + g(x))\partial_y &= h\partial_{y'}.
\end{aligned}$$

Clearly, the most challenging transformation is the Laplace equation, (2.1 a), and there are many possible ways to write the resulting equations (see, for example, [26, 30]). However, we have found it necessary in this work to maintain the divergence form of the differential operator. For this we begin with

$$\begin{aligned}
0 &= (h + g)\operatorname{div}_x \nabla_x v + (h + g)\partial_y \partial_y v\\
&= \operatorname{div}_x[(h + g)\nabla_x v] - \nabla_x g \cdot \nabla_x v + \partial_y[(h + g)\partial_y v].
\end{aligned}$$

Using the transformation formulae above, it is not difficult to see that

$$\begin{aligned}
&\operatorname{div}_{x'}[(h + g)\nabla_{x'} u] - \operatorname{div}_{x'}[(h + y')\nabla_{x'} g \partial_{y'} u]\\
&\qquad - \partial_{y'}[(h + y')\nabla_{x'} g \cdot \nabla_{x'} u] + \partial_{y'}\left[\frac{h^2 + (h + y')^2|\nabla_{x'} g|^2}{h + g}\partial_{y'} u\right] = 0,
\end{aligned}$$

which, upon dropping the primes, we write abstractly as

$$\operatorname{div}[\mathcal{A}\nabla u] = 0, \tag{2.3}$$

where

$$\mathcal{A} := \begin{pmatrix} (h + g)I & -(h + y)\nabla_x g\\ -(h + y)\nabla_x g\cdot & Q \end{pmatrix}$$

and

$$Q(x, y) := \frac{h^2 + (h + y)^2|\nabla_x g|^2}{h + g}.$$

We point out that we interpret the term $|\nabla_x g|^2$ as $(\nabla_x g) \cdot (\nabla_x g)$ throughout the paper. For real-valued g the two are equivalent but, of course, for complex-valued functions the two are quite different. It is not difficult to transform (2.1 b) and

(2.1 c) so that (2.1) reads, in the transformed coordinates:

$$\operatorname{div}[\mathcal{A}\nabla u] = 0 \quad \text{in } S_{h,0}, \tag{2.4 a}$$

$$u(x,0) = \xi(x), \tag{2.4 b}$$

$$\partial_y u(x,-h) = 0. \tag{2.4 c}$$

2.2. The strong Legendre condition

As stated in § 1, we take a perturbative approach to this problem. Where we depart from previous treatments of this problem is that we allow $g(x)$ to vary in the complex plane. While this may not make sense physically, (2.4) makes sense as long as $h+g(x)$ does not vanish. However, a second, crucial concern for our analysis is the ellipticity of (2.4): for $g(x)$ and $u(x,y)$ real functions, this system is elliptic, but this is *not* guaranteed when $g(x)$ is complex. To investigate this further we define

$$g = g_1 + \mathrm{i}g_2, \qquad \xi = \xi_1 + \mathrm{i}\xi_2, \qquad u = u_1 + \mathrm{i}u_2, \tag{2.5}$$

and insert these into (2.4). It is straightforward to show that this gives rise to the two-equation system

$$\left.\begin{aligned} \operatorname{div}[\mathcal{A}_1\nabla u_1] - \operatorname{div}[\mathcal{A}_2\nabla u_2] = 0, \\ \operatorname{div}[\mathcal{A}_1\nabla u_2] + \operatorname{div}[\mathcal{A}_2\nabla u_1] = 0, \end{aligned}\right\} \tag{2.6}$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the real and imaginary parts of $\mathcal{A}$:

$$\mathcal{A}_1 := \begin{pmatrix} (h+g_1)I & -(h+y)\nabla_x g_1 \\ -(h+y)\nabla_x g_1 \cdot & Q_1 \end{pmatrix},$$

$$\mathcal{A}_2 := \begin{pmatrix} g_2 I & -(h+y)\nabla_x g_2 \\ -(h+y)\nabla_x g_2 \cdot & Q_2 \end{pmatrix},$$

and

$$\begin{aligned} Q_1 := \frac{1}{(h+g_1)^2+(g_2)^2}\{\{h^2+(h+y)^2(|\nabla_x g_1|^2-|\nabla_x g_2|^2)\}(h+g_1) \\ + \{2(h+y)^2\nabla_x g_1\cdot\nabla_x g_2\}g_2\}, \\ Q_2 := \frac{1}{(h+g_1)^2+(g_2)^2}\{-\{h^2+(h+y)^2(|\nabla_x g_1|^2-|\nabla_x g_2|^2)\}g_2 \\ + \{2(h+y)^2\nabla_x g_1\cdot\nabla_x g_2\}(h+g_1)\}. \end{aligned}$$

To establish our analyticity theorem we appeal to Schauder's theory for systems in divergence form. The relevant function spaces are

$$C^{1+\alpha}(\mathbb{R}^{d-1}) = \{g \mid |g|_{C^{1+\alpha}} = |g|_{C^{1+\alpha}(\mathbb{R}^{d-1})} < \infty\},$$

where

$$|g|_{C^{1+\alpha}} = |g|_{L^\infty} + \sup_{x_1\neq x_2}\frac{|\nabla_x g(x_1)-\nabla_x g(x_2)|}{|x_1-x_2|^\alpha}.$$

For Schauder's theory we need the following 'strong' Legendre condition (cf. [5]).

LEMMA 2.1. *Assume that* $g_1, g_2 \in W^{1,\infty}(\mathbb{R}^{d-1})$ *and*

$$\left|(h+g_1)\nabla_x\left(\frac{g_2}{h+g_1}\right)\right|_{L^\infty} \leqslant 1-\delta, \quad h+g_1(x) \geqslant \delta \tag{2.7}$$

for some $\delta > 0$. *Then the strong Legendre condition is satisfied, i.e.*

$$\langle \mathcal{A}_1\xi^{(1)}, \xi^{(1)}\rangle - \langle \mathcal{A}_2\xi^{(2)}, \xi^{(1)}\rangle + \langle \mathcal{A}_2\xi^{(1)}, \xi^{(2)}\rangle + \langle \mathcal{A}_1\xi^{(2)}, \xi^{(2)}\rangle \geqslant \tau(\|\xi^{(1)}\|^2 + \|\xi^{(2)}\|^2)$$

for all $\xi^{(1)}, \xi^{(2)} \in \mathbb{R}^d$, $\tau > 0$ *depends on* $\delta, h, \|g_1\|_{W^{1,\infty}(\mathbb{R}^{d-1})}, \|g_2\|_{W^{1,\infty}(\mathbb{R}^{d-1})}$.

Proof. We define

$$\begin{aligned} J &:= \langle \mathcal{A}_1\xi^{(1)}, \xi^{(1)}\rangle - \langle \mathcal{A}_2\xi^{(2)}, \xi^{(1)}\rangle + \langle \mathcal{A}_2\xi^{(1)}, \xi^{(2)}\rangle + \langle \mathcal{A}_1\xi^{(2)}, \xi^{(2)}\rangle \\ &= \langle \mathcal{A}_1\xi^{(1)}, \xi^{(1)}\rangle + \langle \mathcal{A}_1\xi^{(2)}, \xi^{(2)}\rangle \end{aligned}$$

by the self-adjointness of $\mathcal{A}_2$. Thus, the lemma rests upon the coercivity of $\mathcal{A}_1$ and the calculation

$$\begin{aligned} J_1 &:= \langle \mathcal{A}_1\xi, \xi\rangle \\ &= (h+g_1)\xi_x \cdot \xi_x + Q_1\xi_y^2 - 2(h+y)\xi_y\nabla_x g_1 \cdot \xi_x, \end{aligned}$$

where $\xi = (\xi_x, \xi_y) \in \mathbb{R}^{d-1} \times \mathbb{R}$. To estimate this we focus on the final term

$$\begin{aligned} |2(h+y)\xi_y\nabla_x g_1 \cdot \xi_x| &= \left|\left\{\frac{\sqrt{2}(h+y)\xi_y}{\sqrt{h+g_1-\tau}}\nabla_x g_1\right\} \cdot \{\sqrt{2}\sqrt{h+g_1-\tau}\xi_x\}\right| \\ &\leqslant \frac{(h+y)^2\xi_y^2}{h+g_1-\tau}|\nabla_x g_1|^2 + (h+g_1-\tau)|\xi_x|^2, \end{aligned}$$

where we need to choose τ small enough such that $h+g_1-\tau > 0$. With this we have

$$\begin{aligned} J_1 &\geqslant (h+g_1)|\xi_x|^2 - (h+g_1-\tau)|\xi_x|^2 + Q_1\xi_y^2 - \frac{(h+y)^2|\nabla_x g_1|^2}{h+g_1-\tau}\xi_y^2 \\ &= \tau|\xi_x|^2 + \left\{Q_1 - \frac{(h+y)^2|\nabla_x g_1|^2}{h+g_1-\tau}\right\}\xi_y^2, \end{aligned}$$

and we are done if the second term is bounded from below by τ. We write this term as $A/D - B/(C-\tau)$, where

$$\begin{aligned} A &= \{h^2 + (h+y)^2(|\nabla_x g_1|^2 - |\nabla_x g_2|^2)\}(h+g_1) + \{2(h+y)^2\nabla_x g_1 \cdot \nabla_x g_2\}g_2, \\ D &= (h+g_1)^2 + (g_2)^2, \\ B &= (h+y)^2|\nabla_x g_1|^2, \\ C &= h+g_1, \end{aligned}$$

so that we need

$$\frac{A}{D} - \frac{B}{C-\tau} \geqslant \tau$$

or

$$AC - BD \geqslant \tau[(A+CD) - D\tau]. \tag{2.8}$$

After some simplifications, we can show that

$$\begin{aligned} AC - BD &= h^2(h+g_1)^2 - (h+y)^2|(h+g_1)\nabla_x g_2 - g_2 \nabla_x g_1|^2 \\ &= h^2(h+g_1)^2 - (h+y)^2(h+g_1)^4 \left|\nabla_x \left(\frac{g_2}{h+g_1}\right)\right|^2, \end{aligned}$$

so that (noticing that $-h < y < 0$), using our hypotheses,

$$\begin{aligned} AC - BD &\geqslant h^2(h+g_1)^2 - h^2(h+g_1)^4 \left|\nabla_x \left(\frac{g_2}{h+g_1}\right)\right|^2 \\ &= h^2(h+g_1)^2 \left\{1 - (h+g_1)^2 \left|\nabla_x \left(\frac{g_2}{h+g_1}\right)\right|^2\right\} \\ &\geqslant h^2\delta^3, \end{aligned}$$

and we find that (2.8) is valid if we choose sufficiently small τ. □

Given this coercivity result we can easily establish the following inequality, which will be of use several times later in the paper.

Lemma 2.2. *Assume that the hypotheses of lemma 2.1 hold and that*

$$\phi(x) \in C^\infty(\mathbb{R}^{d-1}).$$

Suppose that a solution $W = (W_1, W_2)$ of

$$\operatorname{div}[\mathcal{A}_1 \nabla W_1] - \operatorname{div}[\mathcal{A}_2 \nabla W_2] = F_1, \tag{2.9 a}$$

$$\operatorname{div}[\mathcal{A}_2 \nabla W_1] + \operatorname{div}[\mathcal{A}_1 \nabla W_2] = F_2, \tag{2.9 b}$$

$$W_1(x,0) = W_2(x,0) = 0, \tag{2.9 c}$$

$$\partial_y W_1(x,-h) = \partial_y W_2(x,-h) = 0 \tag{2.9 d}$$

satisfies

$$\int_{S_{h,0}} \phi^2(x)|W|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{2.10}$$

Then, for any $\varepsilon_1, \varepsilon_2, \mu > 0$, $0 < \alpha < 1$ with $\tau - C_1\varepsilon_1 - C_2\varepsilon_2 > 0$,

$$\begin{aligned} K_1 \int_{S_{h,0}} \phi^2 |\nabla W|^2 \,\mathrm{d}x\,\mathrm{d}y &+ K_0 \int_{S_{h,0}} |W|^2\phi^2 \,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \tilde{K}_0 \int_{S_{h,0}} |W|^2 |\nabla\phi|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{C_0}{\mu} \int_{S_{h,0}} |F|^2\phi^2 \,\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} K_1 &= \alpha(\tau - C_1\varepsilon_1 - C_2\varepsilon_2), \\ K_0 &= \frac{2(1-\alpha)(\tau - C_1\varepsilon_1 - C_2\varepsilon_2)}{h^2} - C_0\mu, \\ \tilde{K}_0 &= \left(\frac{C_1}{\varepsilon_1} + \frac{C_2}{\varepsilon_2}\right) \end{aligned}$$

and $F = (F_1, F_2)$. Here τ is defined in lemma 2.1, and the $C_j = C_j(\mathcal{A}_j) > 0$ come from the boundedness of $\mathcal{A}_j$.

Proof. We shall first assume that

$$\phi(x) \equiv 0 \quad \text{for } |x| \gg 1. \tag{2.12}$$

We begin with an elementary Poincaré estimate providing a lower bound on the gradient of W. Since $W(x,0) \equiv 0$, we can write

$$W(x,y) = -\int_y^0 \partial_y W(x,s)\,\mathrm{d}s.$$

Using this and the Cauchy–Schwarz inequality,

$$|W(x,y)|^2 \leqslant \biggl(\int_y^0 |\partial_y W(x,s)|^2\,\mathrm{d}s\biggr)\biggl(\int_y^0 1\,\mathrm{d}s\biggr) \leqslant |y|\biggl(\int_y^0 |\partial_y W(x,s)|^2\,\mathrm{d}s\biggr).$$

Multiplying by $\phi(x)^2$ and integrating over $S_{h,0}$, we obtain the estimate

$$\int_{S_{h,0}} \phi^2 |W|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant \tfrac{1}{2}h^2 \int_{S_{h,0}} \phi^2 |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{2.13}$$

Proceeding to the estimate (2.11), we multiply (2.9 a) by $\phi^2 W_1$ and, upon rearranging, we find

$$\begin{aligned}\phi^2 W_1 F_1 &= \phi^2 W_1\{\operatorname{div}[\mathcal{A}_1 \nabla W_1] - \operatorname{div}[\mathcal{A}_2 \nabla W_2]\}\\ &= \operatorname{div}[\phi^2 W_1 \mathcal{A}_1 \nabla W_1] - \phi^2 \nabla W_1 \cdot (\mathcal{A}_1 \nabla W_1) - 2W_1 \phi \nabla\phi \cdot (\mathcal{A}_1 \nabla W_1)\\ &\quad - \operatorname{div}[\phi^2 W_1 \mathcal{A}_2 \nabla W_2] + \phi^2 \nabla W_1 \cdot (\mathcal{A}_2 \nabla W_2) + 2W_1 \phi \nabla\phi \cdot (\mathcal{A}_2 \nabla W_2).\end{aligned}$$

Integrating over $S_{h,0}$ and using the divergence theorem gives

$$\begin{aligned}&\int_{S_{h,0}} \phi^2\{\nabla W_1 \cdot (\mathcal{A}_1 \nabla W_1) - \nabla W_1 \cdot (\mathcal{A}_2 \nabla W_2)\}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = 2\int_{S_{h,0}} \phi\{-W_1 \nabla\phi \cdot (\mathcal{A}_1 \nabla W_1) + W_1 \nabla\phi \cdot (\mathcal{A}_2 \nabla W_2)\}\,\mathrm{d}x\,\mathrm{d}y - \int_{S_{h,0}} \phi^2 W_1 F_1\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

A similar calculation with (2.9 b) multiplied by $\phi^2 W_2$ yields

$$\begin{aligned}&\int_{S_{h,0}} \phi^2\{\nabla W_2 \cdot (\mathcal{A}_2 \nabla W_1) + \nabla W_2 \cdot (\mathcal{A}_1 \nabla W_2)\}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = 2\int_{S_{h,0}} \phi\{-W_2 \nabla\phi \cdot (\mathcal{A}_2 \nabla W_1) - W_2 \nabla\phi \cdot (\mathcal{A}_1 \nabla W_2)\}\,\mathrm{d}x\,\mathrm{d}y - \int_{S_{h,0}} \phi^2 W_2 F_2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Summing these, and using lemma 2.1 to bound the left-hand side from below, we obtain

$$\begin{aligned}\tau \int_{S_{h,0}} \phi^2 |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant 2\int_{S_{h,0}} \phi\{&-W_1 \nabla\phi \cdot (\mathcal{A}_1 \nabla W_1) + W_1 \nabla\phi \cdot (\mathcal{A}_2 \nabla W_2)\\ &- W_2 \nabla\phi \cdot (\mathcal{A}_2 \nabla W_1) - W_2 \nabla\phi \cdot (\mathcal{A}_1 \nabla W_2)\}\,\mathrm{d}x\,\mathrm{d}y\\ &- \int_{S_{h,0}} \phi^2 (W \cdot F)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Taking advantage of the boundedness of $\mathcal{A}_j$, we can estimate

$$2\int_{S_{h,0}} \phi W_l \nabla\phi\cdot(\mathcal{A}_j\nabla W_m)\,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{C_j}{\varepsilon_j}\int_{S_{h,0}} W_l^2|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y + C_j\varepsilon_j\int_{S_{h,0}}|\nabla W_m|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y,$$

$$\int_{S_{h,0}}\phi^2(W\cdot F)\,\mathrm{d}x\,\mathrm{d}y \leqslant C_0\mu\int_{S_{h,0}}|W|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y + \frac{C_0}{\mu}\int_{S_{h,0}}|F|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y,$$

so that

$$\begin{aligned}(\tau - C_1\varepsilon_1 - C_2\varepsilon_2)\int_{S_{h,0}}\phi^2|\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y &\leqslant \left(\frac{C_1}{\varepsilon_1}+\frac{C_2}{\varepsilon_2}\right)\int_{S_{h,0}}|W|^2|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + C_0\mu\int_{S_{h,0}}|W|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y + \frac{C_0}{\mu}\int_{S_{h,0}}|F|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

'Interpolating' the left-hand side with $0<\alpha<1$, we find that

$$\begin{aligned}\alpha(\tau - C_1\varepsilon_1 - C_2\varepsilon_2)&\int_{S_{h,0}}\phi^2|\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \\ &+ (1-\alpha)(\tau - C_1\varepsilon_1 - C_2\varepsilon_2)\int_{S_{h,0}}\phi^2|\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \left(\frac{C_1}{\varepsilon_1}+\frac{C_2}{\varepsilon_2}\right)\int_{S_{h,0}}|W|^2|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + C_0\mu\int_{S_{h,0}}|W|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y + \frac{C_0}{\mu}\int_{S_{h,0}}|F|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

By using (2.13) in the second term on the left-hand side and bringing the second term on the right-hand side to the left, we obtain

$$\begin{aligned}\alpha(\tau - C_1\varepsilon_1 - C_2\varepsilon_2)&\int_{S_{h,0}}\phi^2|\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \\ &+ \left[\frac{2}{h^2}(1-\alpha)(\tau - C_1\varepsilon_1 - C_2\varepsilon_2) - C_0\mu\right]\int_{S_{h,0}}\phi^2|W|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \left(\frac{C_1}{\varepsilon_1}+\frac{C_2}{\varepsilon_2}\right)\int_{S_{h,0}}|W|^2|\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{C_0}{\mu}\int_{S_{h,0}}|F|^2\phi^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Upon rearranging we are done. To remove assumption (2.12), we replace $\phi(x)$ by $\phi(x)\eta(x/k)$, where $\eta\in C^\infty$ such that

$$0\leqslant\eta(t)\leqslant 1,\quad \eta(t)=1 \text{ for } |t|<1,\quad \eta(t)=0 \text{ for } |t|>2. \tag{2.14}$$

Using (2.10) and letting $k\to\infty$, we conclude the lemma for the general case. □

3. Existence and uniqueness of solutions

With lemma 2.1 in hand, we can establish the existence and uniqueness of solutions of system (2.4). We begin with the uniqueness, which is actually valid in a larger class of functions. We recall that $S_{h,0}$ is the fluid domain in the transformed coordinates, and that H^k corresponds to the Sobolev space of functions with k-many derivatives in L^2.

THEOREM 3.1 (uniqueness). *Let $g_1, g_2 \in W^{1,\infty}(\mathbb{R}^{d-1})$. Suppose that*

$$g(x) = g_1(x) + \mathrm{i}g_2(x)$$

and that assumption (2.7) holds. Then system (2.4) admits at most one weak solution (as usual, a weak solution is defined through integration by parts) $u(x, y, \lambda) = u_1(x, y, \lambda) + \mathrm{i}u_2(x, y, \lambda)$ in the class

$$u \in H^1_{\mathrm{loc}}(\bar{S}_{h,0}, \mathbb{C}), \qquad \int_{S_{h,0}} \mathrm{e}^{-\delta\sqrt{1+|x|^2}}(|u|^2 + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}y < \infty, \tag{3.1}$$

where δ is a small positive number.

Proof. Since the system is linear it suffices to consider the case $\xi(x) \equiv 0$ in (2.4). We consider lemma 2.2 in the case when $F \equiv 0$ with the test function

$$\phi(x) = \mathrm{e}^{-\delta\sqrt{1+|x|^2}}.$$

This function has gradient

$$\nabla\phi = -\frac{\delta x}{\sqrt{1+|x|^2}}\mathrm{e}^{-\delta\sqrt{1+|x|^2}},$$

so that (2.11) implies

$$K_1 \int_{S_{h,0}} \phi^2 |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y + (K_0 - \delta^2\tilde{K}_0) \int_{S_{h,0}} \phi^2 |u|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant 0$$

and, if a sufficiently small δ is chosen, then

$$\int_{S_{h,0}} \mathrm{e}^{-2\delta\sqrt{1+|x|^2}}(|u|^2 + |\nabla u|^2)\,\mathrm{d}x\,\mathrm{d}y = 0,$$

and hence $u \equiv 0$. □

We now turn our attention to the existence of solutions to the complexified system, (2.6).

THEOREM 3.2 (existence). *Let $g_1, g_2, \xi_1, \xi_2 \in C^{1+\alpha}(\mathbb{R}^{d-1})$ with*

$$|g_1|_{C^{1+\alpha}(\mathbb{R}^{d-1})} + |g_2|_{C^{1+\alpha}(\mathbb{R}^{d-1})} + |\xi_1|_{C^{1+\alpha}(\mathbb{R}^{d-1})} + |\xi_2|_{C^{1+\alpha}(\mathbb{R}^{d-1})} < \infty.$$

Suppose that $g(x) = g_1(x) + \mathrm{i}g_2(x)$ and suppose that the assumption (2.7) holds. Then system (2.4) has a solution $u(x, y, \lambda) = u_1(x, y, \lambda) + \mathrm{i}u_2(x, y, \lambda)$ such that $u_1, u_2 \in C^{1+\alpha}(\bar{S}_{h,0})$.

Proof. Using the change of variables

$$U(x,y) := u(x,y) - \xi(x),$$

(2.4) becomes

$$\left.\begin{aligned} \operatorname{div}[\mathcal{A}\nabla U] &= \operatorname{div}_x[F^{(x)}] + F^{(h)} \quad \text{in } S_{h,0}, \\ U(x,0) &= 0, \\ \partial_y U(x,-h) &= 0, \end{aligned}\right\} \tag{3.2}$$

where

$$F^{(x)} = -(h+g)\nabla_x \xi, \qquad F^{(h)} = \nabla_x g \cdot \nabla_x \xi.$$

Clearly,

$$|F^{(x)}|_{C^\alpha} + |F^{(h)}|_{C^\alpha} \leqslant C(|g_1|_{C^{1+\alpha}} + |g_2|_{C^{1+\alpha}})(|\xi_1|_{C^{1+\alpha}} + |\xi_2|_{C^{1+\alpha}}).$$

Akin to our approach earlier, we will set $g = g_1 + \mathrm{i}g_2$, which gives rise to $U = U_1 + \mathrm{i}U_2$, which, upon insertion into (3.2), gives

$$\operatorname{div}[\mathcal{A}_1\nabla U_1] - \operatorname{div}[\mathcal{A}_2\nabla U_2] = \operatorname{div}_x[F_1^{(x)}] + F_1^{(h)}, \tag{3.3 a}$$

$$\operatorname{div}[\mathcal{A}_1\nabla U_2] + \operatorname{div}[\mathcal{A}_2\nabla U_1] = \operatorname{div}_x[F_2^{(x)}] + F_2^{(h)}, \tag{3.3 b}$$

$$U_1(x,0) = U_2(x,0) = 0, \tag{3.3 c}$$

$$\partial_y U_1(x,-h) = \partial_y U_2(x,-h) = 0. \tag{3.3 d}$$

If, in addition to the given assumptions, we further assume that $\xi(x)$ has *compact support*, then the right-hand side of (3.3 *a*) also has compact support. It follows that this right-hand side defines a bounded linear operator on $X \times X$, where X is the closure in $H^1(S_{h,0})$ of $C^\infty(\bar{S}_{h,0})$ functions which vanish on $\{y=0\}$. X is a Banach space with $H^1(S_{h,0})$ norm.

Thus, we can look for solutions $(U_1, U_2) \in X \times X$. The strong Legendre condition implies the coerciveness of this bilinear form in $X \times X$. The Lax–Milgram theorem (cf. [5]) can be applied to obtain the existence of an H^1 solution pair for the corresponding linear problem. A standard procedure can then be applied to obtain a unique H^1 solution for our problem. We then apply Schauder's estimates (cf. [5]) to bound the $C^{1+\alpha}$ norm of our solution.

In order to remove the compactness assumptions for $\xi(x)$, we must derive estimates for this solution. Choosing $\phi(x) = \exp(-\delta\sqrt{1+|x|^2})$ and noting that

$$|\nabla_x \phi|^2 \leqslant \delta^2 |\phi|^2,$$

lemma 2.2 implies

$$K_1 \int_{S_{h,0}} \phi^2 |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y + (K_0 - \delta^2 \tilde{K}_0) \int_{S_{h,0}} \phi^2 |U|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{C_0}{\mu} \int_{S_{h,0}} \phi^2 |F|^2 \,\mathrm{d}x\,\mathrm{d}y$$

so that, for sufficiently small δ,

$$\begin{aligned}\int_{S_{h,0}} \phi^2 |U|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{S_{h,0}} \phi^2 |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y \\ \leqslant C \int_{S_{h,0}} \phi^2 |F^{(x)}|^2 \,\mathrm{d}x\,\mathrm{d}y + C \int_{S_{h,0}} \phi^2 |F^{(h)}|^2 \,\mathrm{d}x\,\mathrm{d}y \\ \leqslant C \|\xi\|^2_{W^{1,\infty}} \|g\|^2_{W^{1,\infty}(\mathbb{R}^{d-1})}.\end{aligned}$$

Notice that the above proof is independent of the choice of origin. In particular, for any $x_0 \in \mathbb{R}^{d-1}$, we have

$$\int_{S_{h,0}} \phi(x-x_0)^2 |U|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_{S_{h,0}} \phi(x-x_0)^2 |\nabla U|^2 \,\mathrm{d}x\,\mathrm{d}y \leqslant C \|\xi\|^2_{W^{1,\infty}} \|g\|^2_{W^{1,\infty}}, \quad (3.4)$$

where the constant C is independent of $x_0 \in \mathbb{R}^{d-1}$.

We can now approximate general ξ with functions of compact support. Using (3.4), we can pass to the limit and obtain a solution in H^1_{loc} satisfying the estimates (3.4). This estimate enables us to apply Schauder's theory [5] to U and to obtain $C^{1+\alpha}$ solutions. This completes the existence proof. □

4. Analyticity

At this point we can prove our analyticity theorem. We proceed by taking

$$g(x) = g_0(x) + \lambda f(x),$$

where g_0, f, λ are all complex valued, and establish the differentiability with respect to λ.

We formally differentiate (2.4) with respect to λ and, upon setting $w(x,y) := \partial_\lambda u(x,y)$, we have

$$\left.\begin{aligned} \operatorname{div}[\mathcal{A}\nabla w] &= -\operatorname{div}[\mathcal{A}_\lambda \nabla u] \quad \text{in } S_{h,0}, \\ w(x,0) &= 0, \\ \partial_y w(x,-h) &= 0, \end{aligned}\right\} \quad (4.1)$$

where

$$\mathcal{A}_\lambda = \begin{pmatrix} fI & -(h+y)\nabla_x f \\ -(h+y)\nabla_x f & Q_\lambda \end{pmatrix},$$

$$Q_\lambda = \frac{\begin{aligned}&2\lambda(h+y)^2 \nabla_x g_0 \cdot \nabla_x f (h + g_0 + \lambda f) \\ &\qquad - \{h^2 + (h+y)^2 (\nabla_x g_0 + \lambda \nabla_x f) \cdot (\nabla_x g_0 + \lambda \nabla_x f)\} f\end{aligned}}{(h + g_0 + \lambda f)^2}.$$

Theorem 4.1. *Under the assumptions of theorem 3.2 (one of the assumptions is that the function $g \equiv g_0 + \lambda f$ satisfies (2.7)), system (4.1) has a unique solution $w(x,y,\lambda) = w_1(x,y,\lambda) + \mathrm{i} w_2(x,y,\lambda)$ such that $w_1, w_2 \in C^{1+\alpha}(\bar{S}_{h,0})$.*

Proof. We can write this system in the same form as (3.3) with the right-hand side replaced by $\operatorname{div}[F]$ for some F that satisfies, by the regularity of u from theorem 3.2,

$$|F|_{C^\alpha} \leqslant C.$$

From this estimate we conclude, as in the proof of theorems 3.1 and 3.2, the existence and uniqueness of the $C^{1+\alpha}$ solution. □

THEOREM 4.2. *The solution to the complexified system $u(x, y, \lambda) = u_1(x, y, \lambda) + \mathrm{i}u_2(x, y, \lambda)$ is an analytic function in λ in the domain in which the assumptions of theorem 3.2 are satisfied.*

Proof. We follow the technique used by Hu and Nicholls [17] and find a system of equations for the finite difference

$$\frac{u(x, y; \lambda + \delta\lambda) - u(x, y; \lambda)}{\delta\lambda}$$

which satisfies an equation much like (4.1), for which $C^{1+\alpha}$ estimates can be derived (cf. theorems 3.2 and 4.1). The compactness of the embedding $C^{1+\alpha} \to C^{1+\beta}$ for $0 < \beta < \alpha$ implies that a subsequence of solutions of the system satisfied by the finite difference converges in $C^{1+\beta}$ to the solution of (4.1). By the uniqueness theorem (theorem 3.1), the limit is independent of the choice of subsequence. Thus, the finite difference converges to the unique limit w, thereby proving that $u(x, y, \lambda)$ is analytic in λ. □

THEOREM 4.3. *The Dirichlet–Neumann operator G is a holomorphic (analytic) map from $C^{1+\alpha}(\mathbb{R}^{d-1})$ into $\mathcal{L}(C^{1+\alpha}(\mathbb{R}^{d-1}), C^{\alpha}(\mathbb{R}^{d-1}))$ in the domain where (2.7) is satisfied.*

Proof. Let g_0 be any (complex) function in $C^{1+\alpha}(\mathbb{R}^{d-1})$ satisfying (2.7). For any (complex) $f \in C^{1+\alpha}(\mathbb{R}^{d-1})$ and any $|\lambda| \ll 1$, $g_0 + \lambda f$ will clearly satisfy (2.7) (with a different δ). Theorem 4.2 shows that, for the above g_0 and f, the corresponding $u(x, y, \lambda)$ is analytic in λ, and hence G is (complex) Gâteaux differentiable at g_0 in the direction f. The estimates derived also clearly show that G is a locally bounded operator. Thus, the Grave–Taylor–Hille–Zorn theorem implies that G is holomorphic [4, ch. 14, p. 198]. □

We now go back to our original problem in the real spaces and establish the radius of convergence. All that we have to do now is find the conditions under which (2.7) will hold when $f(x)$ is real. We note that, in the case when $g(x) = (\lambda_1 + \mathrm{i}\lambda_2)f(x)$, $f(x)$ real, (2.7) is satisfied (for some different $\delta > 0$) if, for some $\delta > 0$,

$$|\lambda_1|\frac{|f|_{L^\infty}}{h} + |\lambda_2||\nabla_x f|_{L^\infty} \leqslant 1 - \delta. \tag{4.2}$$

REMARK 4.4. If

$$|\lambda| \leqslant \frac{1}{\sqrt{[|f|_{L^\infty}/h]^2 + |\nabla_x f|^2}_{L^\infty}} - \delta,$$

then clearly (4.2) is satisfied. In other words, the analytic extension of u depends on the oscillation of f: more specifically, the L^∞ norm of the derivative; smaller oscillation gives a larger domain of analyticity.

THEOREM 4.5. *Consider the special case in which $g(x) = \lambda f(x)$, where λ is a real number and f is a real function. If $f, \xi \in C^{1+\alpha}(\mathbb{R}^{d-1})$ then both the solution $u(x, y, \lambda)$ of (2.1) defined by (2.4) and the DNO $G(\lambda f)$ defined in (2.2) are analytic as functions of λ, i.e. there exists $\delta > 0$ such that they can be expressed as the convergent series*

$$u(x, y, \lambda) = \sum_{n=0}^{\infty} u_n(x, y)\lambda^n, \qquad G(\lambda f) = \sum_{n=0}^{\infty} G_n(f)\lambda^n \tag{4.3}$$

for all

$$|\lambda| \leqslant \frac{1}{\sqrt{[|f|_{L^\infty}/h]^2 + |\nabla_x f|_{L^\infty}^2}} - \delta.$$

Furthermore, u_n and $G_n(f)$ satisfy

$$|u_n|_{C^{1+\alpha}} \leqslant CB^n |\xi|_{C^{1+\alpha}}, \qquad \|G_n(f)\|_{\mathcal{L}(C^{1+\alpha}, C^\alpha)} \leqslant CB^n,$$

where C is a constant independent of λ and B is a positive constant such that

$$B < \sqrt{[|f|_{L^\infty}/h]^2 + |\nabla_x f|_{L^\infty}^2} + \delta.$$

Proof of theorem 4.5. We assume that $\nabla_x f \not\equiv 0$, otherwise the change of variables is trivial and there is nothing to prove. Take

$$c_0 = \frac{1}{\sqrt{[|f|_{L^\infty}/h]^2 + |\nabla_x f|_{L^\infty}^2}} - \frac{\delta}{2}, \quad 0 < \delta \ll 1.$$

Then, by theorem 4.2, the function u is analytic in λ for $|\lambda| \leqslant c_0$. Furthermore, uniform $C^{1+\alpha}$ estimates hold for $u(\cdot, \cdot, \lambda)$ for $|\lambda| = c_0$. By Cauchy's formula, if $|\lambda| < c_0$,

$$u(x, y, \lambda) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta| = c_0} \frac{u(x, y, \zeta)}{\zeta - \lambda} \, \mathrm{d}\zeta = \sum_{n=0}^{\infty} u_n(x, y)\lambda^n,$$

where

$$u_n(x, y) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta| = c_0} \frac{u(x, y, \zeta)}{\zeta^{n+1}} \, \mathrm{d}\zeta.$$

Using this formula, we obtain estimates on u_n from estimates for u:

$$|u_n|_{C^{1+\alpha}} \leqslant \frac{1}{c_0^{n+1}} \max_{|\zeta| = c_0} |u(\cdot, \cdot, \zeta)|_{C^{1+\alpha}} \leqslant CB^n |\xi|_{C^{1+\alpha}},$$

where $B = 1/c_0$. Restricting λ to being real and restricting $|\lambda|$ to less than c_0, we obtain the expansion for $u(x, y, \lambda)$. Similarly, we can extend $G(\lambda f)[\xi]$ to complex λ. Using the (complex) analyticity of u in λ, we immediately have the differentiability of $G(\lambda f)[\xi]$ with respect to λ and

$$|G(\lambda f)\xi|_{C^\alpha} \leqslant C|u|_{C^{1+\alpha}} \leqslant C|\xi|_{C^{1+\alpha}}.$$

Thus, for $|\lambda| < c_0$,

$$G(\lambda f)\xi = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta|=c_0} \frac{G(\zeta f)\xi}{\zeta - \lambda}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty} (G_n(f)\xi)\lambda^n,$$

where

$$G_n(f)\xi = \frac{1}{2\pi\mathrm{i}} \int_{|\zeta|=c_0} \frac{G(\zeta f)\xi}{\zeta^{n+1}}\,\mathrm{d}\zeta.$$

From this, we obtain

$$|G_n(f)\xi|_{C^\alpha} \leqslant \frac{1}{c_0^{n+1}} \max_{|\zeta|=c_0} |G(\zeta f)\xi|_{C^\alpha} \leqslant \frac{C}{c_0^{n+1}} \max_{|\zeta|=c_0} |u(\cdot,\cdot,\zeta)|_{C^{1+\alpha}} \leqslant CB^n |\xi|_{C^{1+\alpha}}.$$

This implies

$$\|G_n(f)\|_{\mathcal{L}(C^{1+\alpha},C^\alpha)} \leqslant CB^n$$

and the theorem is proved. □

5. Numerical examples

In this section we illustrate the theoretical results of the previous sections by investigating, numerically, the location of the singularities of the expansion

$$G(\lambda f) = \sum_{n=0}^{\infty} G_n(f)\lambda^n$$

(cf. (4.3)). In [27] we described a stable, high-order, accurate numerical scheme for the simulation of a DNO in the context of water waves for the Laplace equation on an ocean of depth h. The method is, in fact, based upon the simulation of the u_n and G_n, which appear in the expansions (4.3) where, at every order n,

$$u_n(x,y) \approx u_n^{N_x,N_y}(x,y) = \sum_{k=-N_x/2}^{N_x/2-1} \sum_{l=0}^{N_y} \hat{u}_n^{k,l} T_l\left(\frac{2y+h}{h}\right) \mathrm{e}^{\mathrm{i}kx},$$

$$G_n(x) \approx G_n^{N_x}(x) = \sum_{k=-N_x/2}^{N_x/2-1} \hat{G}_n^k \mathrm{e}^{\mathrm{i}kx},$$

where T_l is the Chebyshev polynomial of degree l. We refer the interested reader to [27] for the details of how the $\{\hat{u}_n^{k,l},\ \hat{G}_n^k\}$ are approximated.

Regarding a numerical approximation of the singularities of the expansion (4.3) for $G(\lambda f)$, we have chosen the method of Padé approximants [2, § 2.2]. To summarize our procedure, consider a function $c(\lambda)$ with Taylor series

$$c(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n,$$

which converges for $|\lambda| < \rho$. The method of Padé approximation seeks to approximate the Nth partial sum

$$c^N(\lambda) := \sum_{n=0}^{N} c_n \lambda^n$$

by the rational function

$$\left[\frac{L}{M}\right](x) = \frac{a^L(\lambda)}{b^M(\lambda)} = \frac{\sum_{l=0}^{L} a_l \lambda^l}{\sum_{m=0}^{M} b_m \lambda^m},$$

where $L + M = N$ and

$$\left[\frac{L}{M}\right](x) = c^N(\lambda) + \mathcal{O}(\lambda^{L+M+1})$$

(see [2] for details and a numerical scheme for obtaining the $\{a_l, b_m\}$).

Given this rational function approximation, it is reasonable to consider the set of denominator zeros

$$D_M := \{\lambda \in \mathbb{C} \mid b^M(\lambda) = 0\}$$

as an approximation of the set of singularities of $c(\lambda)$. However, we quickly discover that a 'false singularity' may be detected if we do not account for the possibility that a numerator zero may cancel a member of D_M. Thus, we define

$$N_L := \{\lambda \in \mathbb{C} \mid a^L(\lambda) = 0\}$$

and set

$$P_{L,M} := D_M \setminus N_L.$$

Of course, in a numerical simulation the members of D_M and N_L can only be identified up to machine precision and thus we define the approximate sets $\tilde{D}_M$ and $\tilde{N}_L$. Furthermore, we have found a 'cancellation tolerance', τ, to be necessary to achieve meaningful results, so, in what follows, we compute

$$\tilde{P}^{\tau}_{L,M} := \{\lambda \in \tilde{D}_M \mid |\lambda - \mu| > \tau \text{ for all } \mu \in \tilde{N}_L\}.$$

For a complete discussion of the capabilities of this approach to approximating the singularities of a function we refer the reader to [2, § 2.2] and the insightful calculations in [3, § 8.3].

For a given profile $f(x)$, our numerical method [27] furnishes us with the data

$$\{\hat{G}^k_n\}, \quad 0 \leqslant n \leqslant N, \qquad -\tfrac{1}{2}N_x \leqslant k \leqslant \tfrac{1}{2}N_x - 1,$$

and, thus, we form

$$c^{k,N}(\lambda) = \sum_{n=0}^{N} \hat{G}^k_n \lambda^n, \qquad -\tfrac{1}{2}N_x \leqslant k \leqslant \tfrac{1}{2}N_x - 1$$

and compute $\tilde{P}^{\tau}_{L,M,k}$. In the figures that follow we plot

$$\boldsymbol{P} = \bigcup_{k=-N_x/2}^{N_x/2-1} \tilde{P}^{\tau}_{N/2,N/2,k}$$

so that we consider the singularities of *all* possible data associated to the DNO with diagonal Padé approximants.

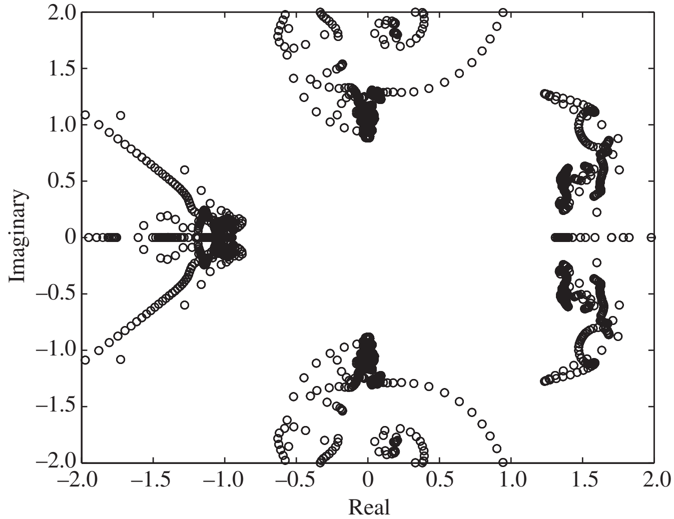

Figure 1. Uncancelled singularities in the DNO for the sinusoidal profile with $h = 1$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$, $\tau = 10^{-4}$.

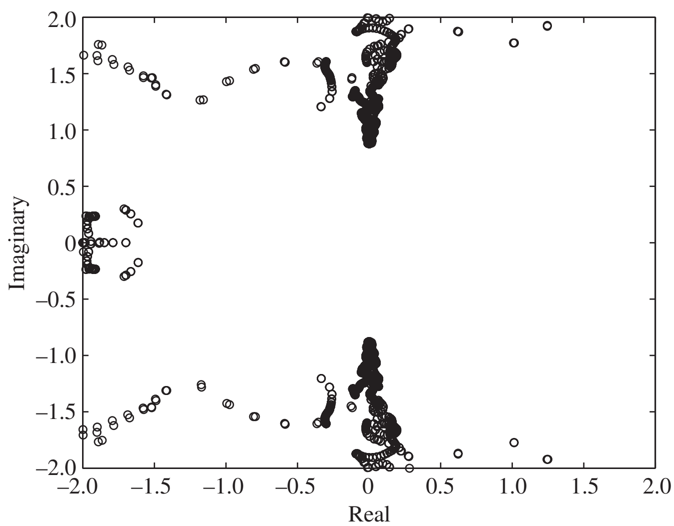

Figure 2. Uncancelled singularities in the DNO for the C^4 profile with $h = 1$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$, $\tau = 10^{-4}$.

Of course, the structure of the singular values of $G(\lambda f)$ changes as f is varied, particularly as it varies from smooth to rough. To investigate this property, we have chosen three profiles (all 2π-periodic or otherwise periodically extended with period 2π) which possess very different smoothness properties: a sinusoid

$$f_s(x) = \cos(x), \tag{5.1 a}$$

a C^4 profile,

$$f_4(x) = (2 \times 10^{-4})x^4(2\pi - x)^4 - c_0, \tag{5.1 b}$$

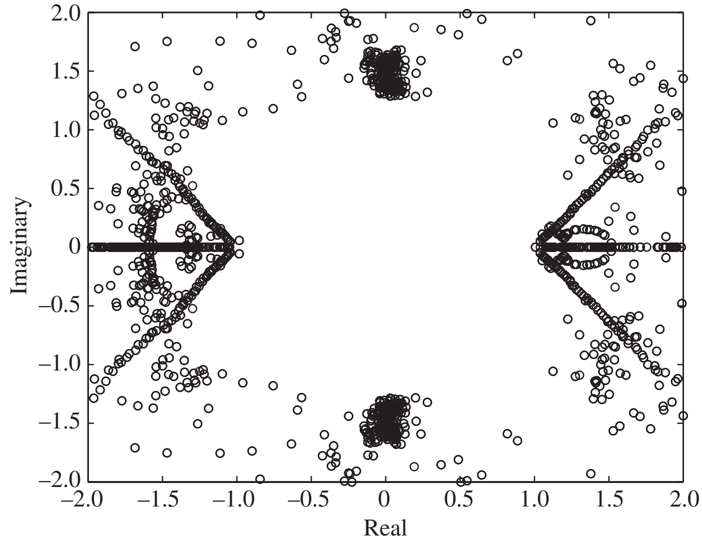

Figure 3. Uncancelled singularities in the DNO for the Lipschitz profile with $h = 1$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$, $\tau = 10^{-4}$.

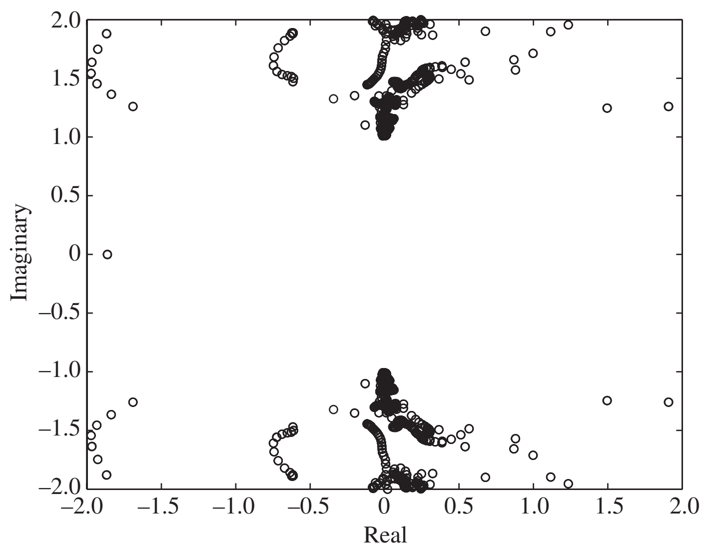

Figure 4. Uncancelled singularities in the DNO for the sinusoidal profile, with $h = 2$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$, $\tau = 10^{-4}$.

where c_0 is chosen so that f_4 has zero mean, and a Lipschitz curve

$$f_L(x) = \begin{cases} -\dfrac{2}{\pi}x + 1, & 0 \leqslant x \leqslant \pi, \\ \dfrac{2}{\pi}x - 3, & \pi \leqslant x \leqslant 2\pi, \end{cases} \tag{5.1 c}$$

(cf. [29]). As pointed out in [29], the latter two equations admit the Fourier series

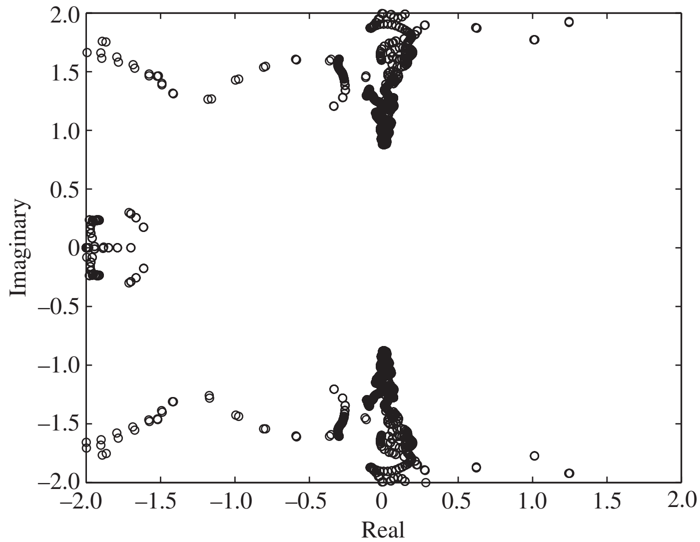

Figure 5. Uncancelled singularities in the DNO for the C^4 profile, with $h = 2$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$ and $\tau = 10^{-4}$.

representations

$$f_4(x) = \sum_{k=1}^{\infty} \frac{96(2k^2\pi^2 - 21)}{125k^8} \cos(kx),$$

$$f_L(x) = \sum_{k=1}^{\infty} \frac{8}{\pi^2(2k-1)^2} \cos((2k-1)x),$$

and we have found it useful in numerical simulations to approximate these by the truncations

$$f_{4,P}(x) = \sum_{k=1}^{P} \frac{96(2k^2\pi^2 - 21)}{125k^8} \cos(kx), \tag{5.2 a}$$

$$f_{L,P}(x) = \sum_{k=1}^{P/2} \frac{8}{\pi^2(2k-1)^2} \cos((2k-1)x), \tag{5.2 b}$$

where we set $P = 40$ in the simulations presented here.

In figures 1–3 we plot $\boldsymbol{P}$ for the profiles f_s, $f_{4,40}$ and $f_{L,40}$, respectively (cf. (5.1 *a*), (5.2 *a*) and (5.2 *b*)), in the case when $h = 1$. In these simulations we chose the numerical parameters $N_x = 256$, $N_y = 64$ and $N = 40$ [27]. Here we see, in each case, a clear region of analyticity near the origin. However, in each case, we also see a clustering of singularities on the *real* axis, seemingly contradicting our results. However, closer inspection reveals that the singularities occur at $\lambda = \pm 1$, corresponding to the *physical* obstruction presented by our domain of depth $h = 1$. Recall that (2.7) implies

$$|\lambda_1| \frac{|f|_{L^\infty}}{h} \leqslant 1 - \delta.$$

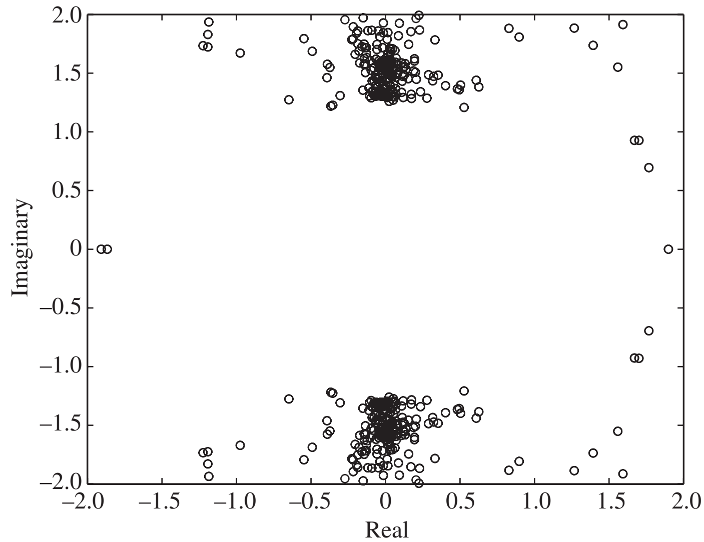

Figure 6. Uncancelled singularities in the DNO for the Lipschitz profile, with $h = 2$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$ and $\tau = 10^{-4}$.

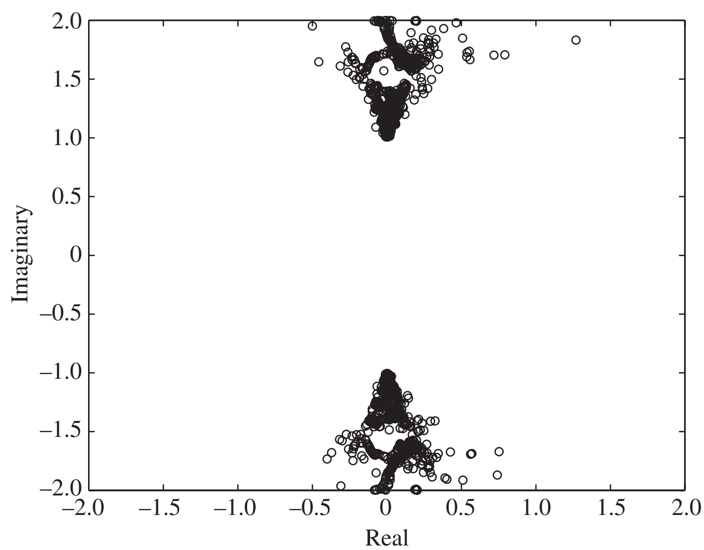

Figure 7. Uncancelled singularities in the DNO for the sinusoidal profile, with $h = \infty$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$ and $\tau = 10^{-4}$.

To further illustrate, we simulate the profiles f_s, $f_{4,40}$ and $f_{L,40}$ in the case when $h = 2$ (the numerical parameters are once again $N_x = 256$, $N_y = 64$ and $N = 40$). The results for $\boldsymbol{P}$ are presented in figures 4, 5 and 6, respectively. We again notice a well-defined region of analyticity and a clustering of singularities on the *real* axis for all three profiles. However, the clustering occurs around $\lambda = \pm 2$ which is, again, the location of the physical obstruction presented by the finite extent of the problem domain.

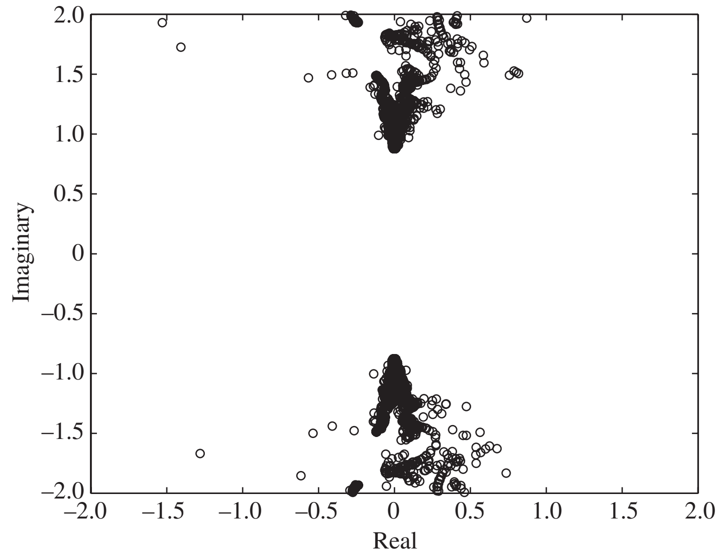

Figure 8. Uncancelled singularities in the DNO for the C^4 profile with $h = \infty$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$, $\tau = 10^{-4}$.

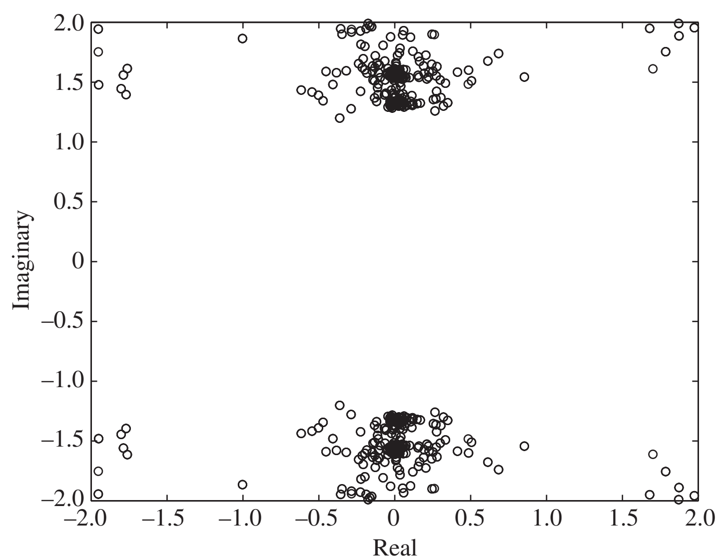

Figure 9. Uncancelled singularities in the DNO for the Lipschitz profile, with $h = \infty$ and $L = 2\pi$. Here $N_x = 256$, $N = 40$, $N_y = 64$ and $\tau = 10^{-4}$.

Finally, we consider the case when $h = \infty$ (in fact, we chose $h = 10^6$) for all three profiles. The results are given in figures 7–9. Here, $|\lambda_1||f|_{L^\infty}/h \leqslant 1 - \delta$ is always satisfied and the singularities for all three profiles have disappeared *entirely* from the real axis, showing that the DNO can be continued analytically for any real λ (cf. [28]).

Remark 5.1. Before closing, we observe that our results give information not only about the permitted location of poles on the real axis, but also about their appearance on the imaginary axis. While (1.1) indicates that the latter poles are essentially governed by h (the depth of the fluid which parametrizes the physical obstruction at the bottom of the fluid domain), the same equation states that poles on the imaginary axis are restricted only by the L^∞ norm of the *gradient* of f and are *independent* of h. A quick inspection of figures 1–9 reveals that the smallest singularity on the imaginary axis is always in a neighbourhood of $\pm\mathrm{i}$, both independent of h and reasonably close to $|\partial_x f|_{L^\infty}^{-1}$.

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References

1 D. J. Acheson. *Elementary fluid dynamics* (New York: Clarendon Press, 1990).
2 G. A. Baker Jr and P. Graves-Morris. *Padé approximants*, 2nd edn (Cambridge University Press, 1996).
3 C. M. Bender and S. A. Orszag. *Advanced mathematical methods for scientists and engineers* (New York: McGraw-Hill, 1978).
4 S. B. Chae. *Holomorphy and calculus in normed spaces*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 92 (New York: Marcel Dekker, 1985).
5 Y.-Z. Chen and L.-C. Wu. *Second order elliptic equations and elliptic systems*, Translations of Mathematical Monographs, vol. 174 (Providence, RI: American Mathematical Society, 1998).
6 R. Coifman and Y. Meyer. Nonlinear harmonic analysis and analytic dependence. In *Pseudodifferential operators and applications*, Proceedings of Symposia on Pure Mathematics, vol. 43, pp. 71–78 (Providence, RI: American Mathematical Society, 1985).
7 D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, 2nd edn (Springer, 1998).
8 W. Craig and D. P. Nicholls. Traveling two and three dimensional capillary gravity water waves. *SIAM J. Math. Analysis* **32** (2000), 323–359.
9 W. Craig and C. Sulem. Numerical simulation of gravity waves. *J. Computat. Phys.* **108** (1993), 73–83.
10 W. Craig, U. Schanz and C. Sulem. The modulation regime of three-dimensional water waves and the Davey–Stewartson system. *Annales Inst. Henri Poincaré* **14** (1997), 615–667.
11 H. De Han and X. N. Wu. Approximation of infinite boundary condition and its application to finite element methods. *J. Computat. Math.* **3** (1985), 179–192.
12 C. Fazioli and D. P. Nicholls. Parametric analyticity of functional variations of Dirichlet–Neumann operators. *Diff. Int. Eqns* **21** (2008), 541–574.
13 D. Givoli. Nonreflecting boundary conditions. *J. Computat. Phys.* **94** (1991), 1–29.
14 D. Givoli. *Numerical methods for problems in infinite domains*, Studies in Applied Mechanics, vol. 33 (Elsevier, 1992).
15 D. Givoli and J. B. Keller. Special finite elements for use with high-order boundary conditions. *Computat. Meth. Appl. Mech. Engng* **119** (1994), 199–213.
16 C. Godrèche. *Solids far from equilibrium* (Cambridge University Press, 1992).
17 B. Hu and D. P. Nicholls. Analyticity of Dirichlet–Neumann operators on Hölder and Lipschitz domains. *SIAM J. Math. Analysis* **37** (2005), 302–320.
18 F. Ihlenburg. *Finite element analysis of acoustic scattering* (Springer, 1998).

19 J. B. Keller and D. Givoli. Exact nonreflecting boundary conditions. *J. Computat. Phys.* **82** (1989), 172–192.

20 H. Lamb. *Hydrodynamics*, 6th edn (Cambridge University Press, 1993).

21 D. M. Milder. An improved formalism for rough-surface scattering of acoustic and electromagnetic waves. In *Proceedings of SPIE: The International Society for Optical Engineering (San Diego, 1991)*, vol. 1558, pp. 213–221 (Bellingham, WA: International Society for Optical Engineering, 1991).

22 D. M. Milder. An improved formalism for wave scattering from rough surfaces. *J. Acoust. Soc. Am.* **89** (1991), 529–541.

23 D. M. Milder and H. T. Sharp. Efficient computation of rough surface scattering. In *Proc. 1991 Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation Phenomena (Strasbourg)*, pp. 314–322 (Philadelphia, PA: SIAM, 1991).

24 D. M. Milder and H. T. Sharp. An improved formalism for rough surface scattering. II. Numerical trials in three dimensions. *J. Acoust. Soc. Am.* **91** (1992), 2620–2626.

25 D. P. Nicholls. Traveling water waves: spectral continuation methods with parallel implementation. *J. Computat. Phys.* **143** (1998), 224–240.

26 D. P. Nicholls and F. Reitich. A new approach to analyticity of Dirichlet–Neumann operators. *Proc. R. Soc. Edinb.* A **131** (2001), 1411–1433.

27 D. P. Nicholls and F. Reitich. Stability of high-order perturbative methods for the computation of Dirichlet–Neumann operators. *J. Computat. Phys.* **170** (2001), 276–298.

28 D. P. Nicholls and F. Reitich. Analytic continuation of Dirichlet–Neumann operators. *Numer. Math.* **94** (2003), 107–146.

29 D. P. Nicholls and F. Reitich. Shape deformations in rough surface scattering: cancellations, conditioning, and convergence. *J. Opt. Soc. Am.* A **21** (2004), 590–605.

30 D. P. Nicholls and M. Taber. Joint analyticity and analytic continuation for Dirichlet–Neumann operators on doubly perturbed domains. *J. Math. Fluid Mech.* **10** (2008), 238–271.

31 V. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9** (1968), 190–194.

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