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ANALYTICITY OF DIRICHLET–NEUMANN OPERATORS ON HÖLDER AND LIPSCHITZ DOMAINS*

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Abstract. In this paper we take up the question of analyticity properties of Dirichlet–Neumann operators with respect to boundary deformations. In two separate results, we show that if the deformation is sufficiently small and lies either in the class of $C^{1+\alpha}$ (any $\alpha > 0$) or Lipschitz functions, then the Dirichlet–Neumann operator is analytic with respect to this deformation. The proofs of both results utilize the "domain flattening" change of variables recently advocated by Nicholls and Reitich for the stable, high-order numerical simulation of Dirichlet–Neumann operators. We extend their analyticity results through the use of more specialized function spaces, and our new theorems are *optimal* in terms of boundary regularity. In the case of $C^{1+\alpha}$ boundary perturbations the underlying field also lies in the Hölder class $C^{1+\alpha}$ and the theorem follows by appealing to familiar Schauder theory arguments. In contrast, for Lipschitz deformations the field must lie in an L^p-based Sobolev space ($W^{1,p}$), so the relevant elliptic estimates come from Sobolev theory. Additionally, in the case of Lipschitz domains, the Dirichlet–Neumann operator must be reformulated weakly in order to accommodate the lack of regularity at the boundary which these Sobolev-class fields possess.

Key words. Dirichlet–Neumann operators, geometric perturbations, free-boundary problems, boundary value problems, Hölder regularity, Sobolev regularity

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1. Introduction.

Many problems of fundamental importance in engineering and the sciences are posed in terms of partial differential equations formulated on irregular and/or moving boundaries. In many instances the differential equations are quite simple (linear and constant coefficient); however, the nonlinearity of the boundary conditions and/or the geometrical difficulties of the domain usually prevent analytic solution of these problems. Classical examples of such problems are the free-surface evolution of an ideal fluid [15], scattering of electromagnetic radiation from an irregular grating [2], and precipitate growth [13]. For these problems a simplification and reduction in dimension can be achieved by considering surface quantities and, if applicable, the shape of the boundary as fundamental variables. Then, if desired, bulk quantities can be recovered from these boundary measurements via appropriate integral formulas. In general this procedure is complicated by the necessity of normal derivatives of field quantities at the boundary. Therefore, Dirichlet–Neumann operators (DNOs), which deliver normal derivatives ("Neumann data") given boundary measurements ("Dirichlet data"), play a crucial role.

Among the many ways in which the DNO can be simulated numerically (e.g., boundary integrals/elements, finite differences, finite elements, etc.), methods based upon boundary perturbations are particularly appealing. These approaches view the shape of the domain as a (small) deformation of a separable geometry (e.g., disk, torus, infinite strip) and seek solutions as a Taylor series expanded in powers of this small parameter. Aside from being highly accurate within their domain of applicability,

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a particularly appealing property of these methods is that, in contrast with most alternative approaches, the spatial dimension of the problem does not affect their implementation or performance. See [18, 19, 20] for a complete discussion of these issues and presentation of numerical results.

Since perturbation algorithms play such a crucial role in the study of DNOs we take up the mathematical question of their analyticity with respect to boundary perturbations, i.e., with respect to ε, which measures the *size* of the perturbation. The first results along these lines can be derived from the work of Calderón [4] and of Coifman and Meyer [6], who showed that if the upper boundary of a two-dimensional domain is a (one-dimensional) Lipschitz curve, then the DNO maps H^1 to L^2 and is analytic in ε (sufficiently small). Next, Craig, Schanz, and Sulem [10] showed that the DNO maps $W^{k+1,p}$ to $W^{k,p}$ for $k \geq 0$ and is analytic in ε (sufficiently small) for three-dimensional domains provided that the two-dimensional upper boundary is C^1; Craig and Nicholls [8] extended this result to general d dimensions ($(d-1)$–dimensional upper boundary) by the same techniques but, due to the application at hand, also required the boundary deformation to be in the class $W^{k+1,p}$ for $k \geq 0$.

These results are the most general to date but rely heavily on an implicit boundary integral formulation for the DNO which, from a numerical standpoint, undermines the computational advantages of boundary perturbation approaches. With this consideration in mind, Nicholls and Reitich studied analyticity through the transformed field expansion (TFE) approach [18, 19, 20]. While this method did not deliver the sharpest results from a theoretical standpoint (the boundary deformation was required to be in the class $C^{3/2+\delta}$ for any $\delta > 0$), it did produce a new, *stabilized*, high-order numerical procedure for the approximation of DNOs with all the advantages of boundary perturbation methods (e.g., ease of implementation, dimension independent performance) without the shortcomings of classical implementations (e.g., cancellations and high-order instability); please see [18, 19, 20] for a complete discussion and [21, 22] for recent advancements in the setting of acoustic and electromagnetic scattering applications. Finally, we mention the recent work of Buffoni [3] who, in the setting of an existence theory for two-dimensional traveling capillary-gravity waves, utilized the DNO in Zakharov's formulation [23] of surface water wave evolution. However, since other techniques prevailed, the analyticity of the DNO with respect to boundary perturbations was not used.

The goal of this paper is to show that the TFE approach can, in addition to providing a stabilized numerical approach, be used to realize the most general analyticity results possible (in terms of boundary regularity) in *arbitrary* dimension. Concerning smoothness of the boundary, this matches the theorems of Calderón [4] and Coifman and Meyer [6] in two dimensions. However, the underlying function spaces are quite different being based upon L^p-Sobolev spaces rather than L^2-Sobolev spaces. Our results extend those of Craig, Schanz, and Sulem [10] and Craig and Nicholls [8] in higher dimensions. Of course, our method can be extended to spaces with higher regularity if greater smoothness is assumed on the boundary deformation and Dirichlet data. We begin by showing that the TFE method analyzed with Schauder theory in Hölder spaces gives a simple and elegant analyticity theorem for surface deformations in the class $C^{1+\alpha}$ for any $\alpha > 0$. We then follow this analysis with a more involved calculation in $W^{k,p}$ spaces using Sobolev theory and demonstrate that, in fact, the regularity of the surface shape can be reduced to Lipschitz.

The paper is organized as follows: In section 2 we introduce the TFE change of variables and state our main results. In section 3.1 we work in the classical Hölder spaces via Schauder theory and conclude analyticity for boundary deformations of class $C^{1+\alpha}$ for any $\alpha > 0$, and we show that the DNO will map $C^{1+\alpha}$ Dirichlet data to C^{α} Neumann data and is uniformly analytic in ε. In section 3.2 we utilize the Sobolev theory of $W^{k,p}$ spaces and show that, in fact, the regularity of the boundary deformation can be reduced to Lipschitz in *any* spatial dimension; in this case, the DNO is analytic in ε and maps $W^{1-1/p,p}$ Dirichlet data to $W^{-1/p,p}$ Neumann data (see section 2 for the precise definition of $W^{-1/p,p}$). In Appendix A, we review the key elliptic estimates which enable our analysis of the DNO.

2. Problem statement and change of variables. To focus upon a particular problem we consider the classical free-boundary problem of the evolution of a d-dimensional ideal fluid under the effects of gravity. The fluid sits above the bottom of a flat ocean bed at mean depth h and is bounded above by the free surface $\eta(x,t)$, giving the domain

$$S_{h,\eta} = \{(x,y) \in \mathbf{R}^{d-1} \times \mathbf{R} \mid -h < y < \eta\}.$$

The fundamental variables for this problem are the shape of the free surface, η, and the velocity potential $\varphi(x,y,t)$ which gives the velocity of the fluid from $\vec{v} = \nabla\varphi$. The equations of motion are [15]

$$\Delta\varphi = 0 \quad \text{in } S_{h,\eta}, \tag{2.1a}$$

$$\partial_y \varphi(x,-h) = 0, \tag{2.1b}$$

$$\partial_t \eta + \nabla_x \varphi \cdot \nabla_x \eta - \partial_y \varphi = 0 \quad \text{at } y = \eta, \tag{2.1c}$$

$$\partial_t \varphi + \frac{1}{2}\left|\nabla\varphi\right|^2 + g\eta = 0 \quad \text{at } y = \eta. \tag{2.1d}$$

These equations must be supplemented with initial conditions and lateral boundary conditions, which we discuss later.

In a fundamental paper on stability of free-surface ocean waves, Zakharov [23] noted that the Euler equations, (2.1), could be stated as a Hamiltonian system in terms of the canonical variables $(\eta(x,t), \xi(x,t) \equiv \varphi(x,\eta(x,t),t))$. This observation, coupled with the solvability of Laplace's equation on the domain $S_{h,\eta}$ given ξ, leads to the realization that (2.1) can be equivalently stated *at the surface* of the domain $S_{h,\eta}$. The restatement was first made by Craig and Sulem [11] as

$$\partial_t \eta = G(\eta)\, \xi, \tag{2.2a}$$

$$\partial_t \xi = -g\eta - \frac{1}{2(1+\left|\nabla_x \eta\right|^2)} \Big[\left|\nabla_x \xi\right|^2 - (G(\eta)\, \xi)^2$$

$$-2(G(\eta)\, \xi) \nabla_x \xi \cdot \nabla_x \eta + \left|\nabla_x \xi\right|^2 \left|\nabla_x \eta\right|^2 - (\nabla_x \xi \cdot \nabla_x \eta)^2 \Big], \tag{2.2b}$$

where $G(\eta)\, \xi$ is the DNO. This set of equations, (2.2), has been useful in a variety of analytical [8, 7] and numerical [16, 17, 9, 14] treatments of the Euler equations, and clearly a detailed understanding of the DNO is at the heart of these analyses.

Inspired by the geometry of the Euler equations (2.1) and the reduction of Craig and Sulem, we study the DNO, $G(\eta)$, and its associated boundary value problem:

$$\Delta v(x,y) = 0 \quad \text{in } S_{h,\eta}, \tag{2.3a}$$

$$\partial_y v(x,-h) = 0, \tag{2.3b}$$

$$v(x,\eta(x)) = \xi(x). \tag{2.3c}$$

Upon the solution of (2.3) the DNO is defined as

$$G(\eta)\ \xi = \nabla v|_{y=\eta} \cdot N_\eta = \left[-\nabla_x \eta \cdot \nabla_x v + \partial_y v\right]_{y=\eta}, \tag{2.4}$$

where the normal $N = (-\nabla_x \eta, 1)^T$ (not of unit length) is chosen to simplify the restatement of the kinematic condition (2.1c) as (2.2a). Regarding lateral boundary conditions, it is well known that bounded solutions to (2.3) are unique. Thus, $v(x, y)$ is periodic in x if $\eta(x)$ and $\xi(x)$ are periodic in x; similarly, the behavior of $v(x)$ as $x \to \pm\infty$ will be uniquely determined by the behavior of $\xi(x)$ near infinity. In this way we incorporate quite general boundary conditions into the definition of the DNO.

In order to work with more general Lipschitz boundaries, we now derive a weak formulation of the DNO: Take any test function $\psi \in T^1_R(\overline{S_{h,\eta}})$, where

$$T^1_R(\overline{S_{h,\eta}}) = \left\{ f \in C^1(\overline{S_{h,\eta}}) \mid f = 0 \text{ on } \{|x| > R\} \text{ for some large } R \right\}.$$

Then

$$\begin{aligned}
0 &= \int_{S_{h,\eta}} (\Delta v)\psi \,\mathrm{d}V \\
&= \int_{y=\eta(x)} (\partial_\nu v)\psi \,\mathrm{d}S - \int_{S_{h,\eta}} (\nabla_x v \cdot \nabla_x \psi + \partial_y v\ \partial_y \psi)\,\mathrm{d}V \\
&= \int_{\mathbf{R}^{d-1}} \frac{G(\eta)\ \xi}{\sqrt{1+|\nabla_x \eta|^2}}\ \psi(x, \eta(x))\ \sqrt{1+|\nabla_x \eta|^2}\,\mathrm{d}x - \int_{S_{h,\eta}} (\nabla_x v \cdot \nabla_x \psi + \partial_y v\ \partial_y \psi)\,\mathrm{d}V.
\end{aligned}$$

Thus

$$\int_{\mathbf{R}^{d-1}} (G(\eta)\ \xi)\psi(x, \eta(x))\,\mathrm{d}x = \int_{S_{h,\eta}} \nabla_x v \cdot \nabla_x \psi + \partial_y v\ \partial_y \psi \,\mathrm{d}V. \tag{2.5}$$

For any $\psi \in T^1_R(\overline{S_{h,\eta}})$ we can always approximate ψ with $\psi_j \in C^1(\overline{S_{h,\eta}})$ such that

$$\begin{aligned}
\psi_j &\to \psi && \text{strongly in } C(\overline{S_{h,\eta}}), \\
\nabla \psi_j &\to \nabla \psi && \text{weak* in } L^\infty(S_{h,\eta})^d.
\end{aligned}$$

Using this approximation, we find that (2.5) also extends to functions $\psi \in T^{0,1}_R(\overline{S_{h,\eta}})$, where

$$T^{0,1}_R(\overline{S_{h,\eta}}) = \left\{ f \in C^{0,1}(\overline{S_{h,\eta}}) \mid f = 0 \text{ on } \{|x| > R\} \text{ for some large } R \right\}.$$

Using the notation

$$\langle a, b \rangle = \int_{\mathbf{R}^{d-1}} a(x)\ b(x)\,\mathrm{d}x,$$

we restate (2.5) as follows: For any $\psi \in T^{0,1}_R(\overline{S_{h,\eta}})$,

$$\langle G(\eta)\ \xi, \psi(x, \eta(x)) \rangle = \int_{S_{h,\eta}} (\nabla_x v \cdot \nabla_x \psi + \partial_y v\ \partial_y \psi)\,\mathrm{d}V. \tag{2.6}$$

It is clear that the right-hand side of this equality requires v only to be $W^{1,1}_{loc}(\overline{S_{h,\eta}})$.

It has been discovered [18, 19, 20] that an effective technique for establishing the analyticity of DNOs is to make a "domain flattening" change of variables

$$x' = x, \qquad y' = h\frac{y-\eta}{h+\eta}, \tag{2.7}$$

which maps $S_{h,\eta}$ to $S_{h,0}$. Considering the transformed field

$$u(x', y') = v(x', (h+\eta)y'/h+\eta), \tag{2.8}$$

the change of variables induces the formulas

$$(h+\eta)\nabla_x = (h+\eta)\nabla_{x'} - (h+y')(\nabla_{x'}\eta)\partial_{y'}, \tag{2.9a}$$

$$(h+\eta)\ \mathrm{div}_x = (h+\eta)\ \mathrm{div}_{x'} - (h+y')(\nabla_{x'}\eta)\cdot\partial_{y'}, \tag{2.9b}$$

$$(h+\eta)\partial_y = h\partial_{y'}, \tag{2.9c}$$

which include a prefactor of $(h+\eta)$ in order to realize transformed equations with *no quotients involving* η. Upon making this transformation, (2.3) becomes

$$\Delta u(x', y') = F(x', y') \qquad \text{in } S_{h,0}, \tag{2.10a}$$

$$\partial_y u(x', -h) = 0, \tag{2.10b}$$

$$u(x', 0) = \xi(x'), \tag{2.10c}$$

where

$$F(x', y') = \mathrm{div}_{x'}\left[F^{(1)}(x', y')\right] + \partial_{y'}F^{(2)}(x', y') + F^{(3)}(x', y'). \tag{2.11}$$

The form for F can be found most easily from the following calculation:

$$\begin{aligned} 0 &= (h+\eta)^2\{\Delta_x v + \partial_y^2 v\} \\ &= (h+\eta)^2\Delta_x v + (h+\eta)^2\partial_y^2 v \\ &= (h+\eta)\mathrm{div}_x\left[(h+\eta)\nabla_x v\right] - \nabla_x\eta\cdot(h+\eta)\nabla_x v + (h+\eta)\partial_y\left[(h+\eta)\partial_y v\right]. \end{aligned}$$

Using (2.9) it is straightforward to show that

$$\begin{aligned} 0 = {} & h^2\Delta'_x u + h^2\partial_{y'}^2 u \\ & +\eta\ \mathrm{div}_{x'}\left[h\nabla_{x'}u\right] + h\ \mathrm{div}_{x'}\left[\eta\nabla_{x'}u\right] + \eta\ \mathrm{div}_{x'}\left[\eta\nabla_{x'}u\right] - h\ \mathrm{div}_{x'}\left[(h+y)\nabla_{x'}\eta\partial_{y'}u\right] \\ & -\eta\ \mathrm{div}_{x'}\left[(h+y)\nabla_{x'}\eta\partial_{y'}u\right] - (h+y)\nabla_{x'}\eta\cdot\partial_{y'}\left[h\nabla_{x'}u\right] \\ & -(h+y)\nabla_{x'}\eta\cdot\partial_{y'}\left[\eta\nabla_{x'}u\right] + (h+y)\nabla_{x'}\eta\cdot\partial_{y'}\left[(h+y')\nabla_{x'}\eta\partial_{y'}u\right] \\ & -h\nabla_{x'}\eta\cdot\nabla_{x'}u - \eta\nabla_{x'}\eta\cdot\nabla_{x'}u + (h+y')\left|\nabla_{x'}\eta\right|^2\partial_{y'}u. \end{aligned}$$

From this point, several manipulations can be effected to realize the divergence structure of F. Upon dropping primes, this results in

$$F^{(1)} = -\frac{2}{h}\eta\nabla_x u - \frac{1}{h^2}\eta^2\nabla_x u + \frac{h+y}{h}\nabla_x\eta\partial_y u + \frac{(h+y)}{h^2}\eta\nabla_x\eta\partial_y u, \tag{2.12a}$$

$$F^{(2)} = \frac{h+y}{h}\nabla_x\eta\cdot\nabla_x u + \frac{(h+y)}{h^2}\eta\nabla_x\eta\cdot\nabla_x u - \frac{(h+y)^2}{h^2}\left|\nabla_x\eta\right|^2\partial_y u, \tag{2.12b}$$

$$F^{(3)} = \frac{1}{h}\nabla_x\eta\cdot\nabla_x u + \frac{1}{h^2}\eta\nabla_x\eta\cdot\nabla_x u - \frac{(h+y)}{h^2}\left|\nabla_x\eta\right|^2\partial_y u, \tag{2.12c}$$

where $F^{(1)}$, $F^{(2)}$, and $F^{(3)}$ are all $\mathcal{O}(\eta)$. At this point we note that, as claimed above, the right-hand side of (2.10a) contains no quotients involving η.

Of course, we are primarily concerned with the DNO; formula (2.4) transforms as

$$\begin{aligned}(h+\eta)G(\eta)\ \xi &\\ &= \{-\nabla_x\eta\cdot(h+\eta)\nabla_x v+(h+\eta)\partial_y v\}\Big|_{y'=0}\\ &= \{h\partial_{y'}u-h\nabla_{x'}\eta\cdot\nabla_{x'}u-\eta\nabla_{x'}\eta\cdot\nabla_{x'}u+(h+y')\left|\nabla_{x'}\eta\right|^2\partial_{y'}u\}\Big|_{y'=0}.\end{aligned} \tag{2.13}$$

Therefore, again dropping primes,

$$G(\eta)\ \xi(x)=\partial_y u(x,0)+J(x), \tag{2.14}$$

where

$$\begin{aligned}J=&-\frac{\eta}{h}G(\eta)\ \xi-\nabla_x\eta\cdot\nabla_x u(x,0)\\ &-\frac{1}{h}\eta\nabla_x\eta\cdot\nabla_x u(x,0)+\left|\nabla_x\eta\right|^2\partial_y u(x,0),\end{aligned}$$

and clearly $J=\mathcal{O}(\eta)$. The weak statement of the DNO, (2.6), transforms as

$$\begin{aligned}&\langle G(\eta)\ \xi,\psi(x,0)\rangle\\ &=\int_{S_{h,0}}\left\{\left(\nabla_x u-\frac{h+y}{h+\eta}(\nabla_x\eta)\partial_y u\right)\cdot\left(\nabla_x\psi-\frac{h+y}{h+\eta}(\nabla_x\eta)\partial_y\psi\right)\right.\\ &\qquad\left.+\frac{h^2}{(h+\eta)^2}(\partial_y u)\partial_y\psi\right\}\frac{h+\eta}{h}\ \mathrm{d}V\end{aligned} \tag{2.15}$$

for any $\psi\in T_R^{0,1}(\overline{S_{h,0}})$. Finally, we point out that sometimes it is more convenient to write the DNO in the following form:

$$G(\eta)\ \xi(x)=\ -\nabla_x\eta\cdot\nabla_x\xi+\frac{h(1+\left|\nabla_x\eta\right|^2)}{h+\eta}\partial_y u\Bigg|_{y=0}, \tag{2.16}$$

where we have used the fact that $u(x,0)=\xi(x)$.

In the spirit of the boundary perturbation methods we alluded to in the Introduction, we now suppose that we are considering η to be a *small* perturbation of a flat geometry, i.e., $\eta(x)=\varepsilon f(x)$. In this case, for future reference, (2.16) becomes

$$G(\eta)\ \xi(x)=\ -\varepsilon\nabla_x f\cdot\nabla_x\xi+\frac{h(1+\left|\varepsilon\nabla_x f\right|^2)}{h+\varepsilon f}\partial_y u\Bigg|_{y=0}. \tag{2.17}$$

We show the following theorem in section 3.1.

THEOREM 2.1. *Let $f,\xi\in C^{1+\alpha}(\mathbf{R}^{d-1})$, $0<\alpha<1$. Let $v(x,y)$ be the solution of* (2.3) *in the region $S_{h,\eta}$ with $\eta=\varepsilon f$ and define*

$$u(x,y,\varepsilon)=v\left(x,\frac{(h+\varepsilon f)y}{h}+\varepsilon f\right),\quad -\infty<x<\infty,-h<y<0$$

(cf. (2.8)*). Define the DNO $G(\varepsilon f)$ by* (2.14) *with $\eta=\varepsilon f$. Then both the solution $u(x,y,\varepsilon)$ and the DNO $G(\varepsilon f)$ are analytic as functions of ε; i.e., they can be expressed*

as the convergent series

$$u(x,y,\varepsilon)=\sum_{n=0}^{\infty}u_n(x,y)\ \varepsilon^n,\quad G(\varepsilon f)=\sum_{n=0}^{\infty}G_n(f)\ \varepsilon^n, \tag{2.18}$$

for small ε, where u_n and $G_n(f)$ satisfy, for some constants B and C independent of ε,

$$|u_n|_{C^{1+\alpha}(S_{h,0})}\le CB^n\,|\xi|_{C^{1+\alpha}(\mathbf{R}^{d-1})}\,,\quad |G_n(f)|_{\mathcal{L}(C^{1+\alpha}(\mathbf{R}^{d-1}),C^{\alpha}(\mathbf{R}^{d-1}))}\le CB^n.$$

This theorem implies that the DNO maps $C^{1+\alpha}$ Dirichlet data to C^{α} Neumann data.

By working in L^p-based Sobolev spaces, $W^{k,p}$ $(p>d)$, we can refine this result by requiring the boundary to be only Lipschitz continuous. In dealing with these Sobolev spaces, we must appeal to the trace operator and its mapping properties (see [1, Chapter 7 (e.g., Theorem 7.53)] for trace theorems); in particular, if $\partial\Omega\in C^k$, then the trace operator $W^{k,p}(\Omega)\to W^{k-1/p,p}(\partial\Omega)$ is continuous and surjective.

To state the next result with complete accuracy we first define a pair of function spaces. We denote by $B_r(x^*)$ the ball of radius r centered at x^*, and for $p>1$ define

$$X^p=\{\xi\mid \xi\in W^{1-1/p,p}(B_1(x^*))\text{ for any }x^*\in\mathbf{R}^{d-1}\}.$$

For $\xi\in X^p$ we define

$$\|\xi\|_{X^p}=\sup_{x^*\in\mathbf{R}^{d-1}}\|\xi\|_{W^{1-1/p,p}(B_1(x^*))}.$$

Recall that [1, Chapter 7]

$$\|\xi\|_{W^{1-1/p,p}(B_1(x^*))}=\inf\|\zeta\|_{W^{1,p}(B_1(x^*)\times[-h,0])}\,,$$

where the infimum is taken over all functions $\zeta\in W^{1,p}(B_1(x^*)\times[-h,0])$ such that $\zeta(x,0)=\xi(x)$ in the trace sense; i.e., for any C^{∞} function $\gamma(x,y)$ such that $\gamma=0$ on

$$\{\partial B_1(x^*)\times[-h,0]\}\cup\{B_1(x^*)\times\{y=-h\}\}\,,$$

$\gamma(\zeta-\xi)\in W_0^{1,p}((B_1(x^*)\times(-h,0)))$. It is clear that with this definition

$$\|\xi\|_{X^p}\le\sup_{x^*\in\mathbf{R}^{d-1}}\|\xi\|_{W^{1-1/p,p}(B_2(x^*))}\le 2^{(d-1)/p}\,\|\xi\|_{X^p}\,.$$

We also define

$$Y^{k,p}=\{u\mid u\in W^{k,p}(B_1(x^*)\times[-h,0])\text{ for any }x^*\in\mathbf{R}^{d-1}\}$$

and

$$\|u\|_{Y^{k,p}}=\sup_{x^*\in\mathbf{R}^{d-1}}\|u\|_{W^{k,p}(B_1(x^*)\times[-h,0])}\,.$$

In the case of boundary data in X^p, the solution $u(x,y,\varepsilon)$ will only be in the space $W^{1,p}$ in the domain. Therefore, the first order derivative ∇u will only be an L^p function in the domain and the trace operator in (2.14) is not well defined. Thus we shall use the weak formulation (2.15). Since the DNO is local in nature, we shall discuss the DNO only in a neighborhood of an arbitrarily fixed point $\hat{x} \in \mathbf{R}^{d-1}$. We will establish the following in section 3.2.

THEOREM 2.2. *If $f \in C^{0,1}(\mathbf{R}^{d-1})$, $\xi \in X^p$, $p > d$. Let $v(x, y)$ be the solution of* (2.3) *in the region $S_{h,\eta}$ with $\eta = \varepsilon f$ and define*

$$u(x, y, \varepsilon) = v\left(x, \frac{(h+\varepsilon f)y}{h} + \varepsilon f\right), \quad -\infty < x < \infty, -h < y < 0$$

(cf. (2.8)*). Define the DNO $G(\varepsilon f)$ by* (2.15) *with $\eta = \varepsilon f$. Then both the solution $u(x, y, \varepsilon)$ and the DNO $G(\varepsilon f)$ are analytic as functions of ε; i.e., they can be expressed as the convergent series*

$$u(x, y, \varepsilon) = \sum_{n=0}^{\infty} u_n(x, y)\ \varepsilon^n, \quad G(\varepsilon f) = \sum_{n=0}^{\infty} G_n(f)\ \varepsilon^n,$$

for small ε, where u_n and $G_n(f)$ satisfy, for some constants B and C independent of ε,

$$\|u_n\|_{Y^{1,p}} \le CB^n \|\xi\|_{X^p}, \quad \|G_n(f)\|_{\mathcal{L}(X^p,(X_c^q(\hat{x}))^*)} \le CB^n \tag{2.19}$$

for any fixed $\hat{x} \in \mathbf{R}^{d-1}$. In these formulas, q is the conjugate of p, i.e., $q = p/(p-1)$, and $(X_c^q(\hat{x}))^*$ is the dual space of $X_c^q(\hat{x})$:

$$\begin{aligned} X_c^q(\hat{x}) &= \{\varphi \in X^q \mid \varphi = 0 \textit{ for } |x - \hat{x}| > 1\} \\ &\cong W_0^{1-1/q,q}(B_1(\hat{x})). \end{aligned}$$

Remark. Roughly speaking, X^p behaves like $W^{1-1/p,p}$ and X^q behaves like $W^{1-1/q,q}$. Thus, the dual space of $W^{1-1/q,q}$ behaves locally like $W^{-(1-1/q),p} = W^{-1/p,p}$. Therefore, the above theorem states that the DNO "loses one spatial derivative" and is analytic in ε. This is the *optimal* regularity that one can expect for the DNO.

Remark. Theorem 2.2 concerns a field, v, in $W^{1,p}$ with boundary trace, ξ, in $W^{1-1/p,p}$. Such assumptions were made to enable a proof which demands the *weakest* possible regularity on the boundary perturbation. Of course, if the boundary deformation and Dirichlet data are more regular, then the field and DNO will be smoother as well. Results mentioned in the Introduction (e.g., Calderón [4], Coifman and Meyer [6], Craig, Schanz, and Sulem [10], Craig and Nicholls [8], and Nicholls and Reitich [18, 20]) provide such results in a wide array of function spaces.

Remark. We introduced the spaces X^p and $Y^{k,p}$ in order to include quite general behavior at infinity. For instance, we can accommodate periodicity or convergence (at infinity) to a constant. If we specialize to periodic boundary conditions, say on the period cell $Q \subset \mathbf{R}^{d-1}$, we can simplify the statements of the theorem by replacing X^p with $W^{1-1/p,p}(Q)$, $Y^{1,p}$ with $W^{1,p}(Q \times [-h, 0])$, and $B_1(\hat{x})$ with Q in Theorem 2.2.

Remark. Finally, a direct, "method of majorants" approach could be pursued to derive these results; cf. [18, 19, 20]. This would involve (for Theorem 2.1) inserting the expansions (2.18) into (2.10) and (2.14), finding equations satisfied by the u_n and G_n, and then estimating them directly in an appropriate function space. Since our purpose is to simply establish analyticity in ε (rather than *joint* analyticity in x, y, and ε; cf. [20]), we have found that a *complexification* approach greatly simplifies the argument while delivering the most general result possible.

3. Analyticity. In this section we establish analyticity of $u(x, y, \varepsilon)$ in ε via a complexification argument. Of course, in the original system (2.3) we cannot allow ε to be complex-valued as ε measures the magnitude of the (real) deformation of the domain. On the other hand, in the transformed system (2.10) ε has no such interpretation and we are free to allow $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2 \in \mathbf{C}$ and to look for *complex* solutions, u. The advantage of this approach is the availability of the formulas of complex analysis which readily deliver analyticity provided that straightforward estimates are established. Once this is accomplished we may set $\varepsilon_2 = 0$ and obtain the series expansion for u which must be real-valued.

The complexification approach requires us to simply show that $u(x, y, \varepsilon)$ is differentiable in $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2$ for $|\varepsilon|$ sufficiently small. To this end we define the finite difference operator as follows:

$$T_\delta[u](x, y, \varepsilon) = \frac{1}{\delta}[u(x, y, \varepsilon + \delta) - u(x, y, \varepsilon)], \quad \delta = \delta_1 + \mathrm{i}\delta_2.$$

A simple computation shows that

$$T_\delta[u \cdot w](x, y, \varepsilon) = T_\delta[u](x, y, \varepsilon) \cdot w(x, y, \varepsilon) + u(x, y, \varepsilon + \delta) \cdot T_\delta[w](x, y, \varepsilon). \tag{3.1}$$

In the next two subsections we show that $T_\delta[u](x, y, \varepsilon)$ converges as $\delta \to 0$, for ε in a small disk. This is done in Hölder spaces in section 3.1 and in $W^{k,p}$ spaces in section 3.2.

3.1. Hölder estimates. To begin this section we recall the following well-known algebra property of the space C^α.

LEMMA 3.1. *Let* $0 \leq \alpha \leq 1$. *For* $f \in C^\alpha(\mathbf{R}^{d-1})$, $u \in C^\alpha(S_{h,0})$, *the product* $fu \in C^\alpha(S_{h,0})$, *and*

$$|fu|_{C^\alpha} \leq |f|_{C^\alpha} |u|_{C^\alpha}.$$

For convenience, we often use $C^{k+\alpha}$ to denote either $C^{k+\alpha}(\mathbf{R}^{d-1})$ or $C^{k+\alpha}(S_{h,0})$; the meaning should be clear from the context. Now, recalling that since $\varepsilon \in \mathbf{C}$, solutions u of (2.10) will generally be complex-valued (with the real and imaginary parts individually satisfying (2.10)), we establish the following lemma regarding existence and uniqueness of solutions.

LEMMA 3.2. *Given* $f, \xi \in C^{1+\alpha}$ *for any* $\alpha \in (0, 1)$, *there exists* $c_0 > 0$ *such that* (2.10) *(with the right-hand side of* (2.10) *given by* (2.12)*) has a unique solution* $u \in C^{1+\alpha}$ *for all* ε *in the disk* $|\varepsilon| \leq c_0$. *Furthermore,*

$$|u|_{C^{1+\alpha}} \leq C|\xi|_{C^{1+\alpha}}, \tag{3.2}$$

where the constant C *is independent of* ε.

Proof. The contraction mapping principle will be utilized. Consider the space

$$X = \{u \in C^{1+\alpha} \mid u(x, 0) = \xi, \partial_y u(x, -h) = 0\}$$

and the map Φ, defined by the following steps: For $u \in X$, compute $R(x, y) = F(x, y, u(x, y))$ from (2.11) and (2.12), and find the solution of

$$\begin{aligned}
&\Delta w(x, y) = R(x, y) \qquad \text{in } S_{h,0},\\
&w(x, 0) = \xi(x),\\
&\partial_y w(x, -h) = 0,
\end{aligned}$$

guaranteed by Theorem A.2. Setting $\eta = \varepsilon f$, we note that if $u \in C^{1+\alpha}$, then

$$\begin{aligned}\left|F^{(1)}\right|_{C^\alpha} &\le \frac{2}{h}\left|\varepsilon f \nabla_x u\right|_{C^\alpha} + \frac{1}{h^2}\left|\varepsilon^2 f^2 \nabla_x u\right|_{C^\alpha} + \left|\frac{h+y}{h}\varepsilon(\nabla_x f)\partial_y u\right|_{C^\alpha} \\ &\quad + \left|\frac{(h+y)}{h^2}\varepsilon^2 f(\nabla_x f)\partial_y u\right|_{C^\alpha} \\ &\le \frac{2|\varepsilon|}{h}\left|f\right|_{C^\alpha}\left|u\right|_{C^{1+\alpha}} + \frac{|\varepsilon|^2}{h^2}\left|f\right|^2_{C^\alpha}\left|u\right|_{C^{1+\alpha}} + \frac{Y|\varepsilon|}{h}\left|f\right|_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}} \\ &\quad + \frac{Y|\varepsilon|^2}{h^2}\left|f\right|_{C^\alpha}\left|f\right|_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}} \\ &\le |\varepsilon| K_{1,1}\left|f\right|_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}} + |\varepsilon|^2 K_{1,2}\left|f\right|^2_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}},\end{aligned}$$

where we have used Lemma 3.1, and Y is defined by

$$\left|(h+y)u\right|_{C^\alpha} \le Y\left|u\right|_{C^\alpha}.$$

Similarly, it can be shown that

$$\begin{aligned}\left|F^{(2)}\right|_{C^\alpha} &\le |\varepsilon| K_{2,1}\left|f\right|_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}} + |\varepsilon|^2 K_{2,2}\left|f\right|^2_{C^{1+\alpha}}\left|u\right|_{C^{1+\alpha}}, \\ \left|F^{(3)}\right|_{L^\infty} &\le |\varepsilon| K_{3,1}\left|f\right|_{C^1}\left|u\right|_{C^1} + |\varepsilon|^2 K_{3,2}\left|f\right|^2_{C^1}\left|u\right|_{C^1},\end{aligned}$$

so that from (A.1) of Theorem A.2, $w \in C^{1+\alpha}$. Thus, $\Phi : X \to X$ defined by $w = \Phi u$ is well-defined.

Now, if we choose $u, \tilde{u} \in X$, this will generate $w, \tilde{w} \in X$, respectively. Furthermore,

$$\begin{aligned}\left|w - \tilde{w}\right|_{C^{1+\alpha}} &\le C_e\left[\left|R^{(1)} - \tilde{R}^{(1)}\right|_{C^\alpha} + \left|R^{(2)} - \tilde{R}^{(2)}\right|_{C^\alpha} + \left|R^{(3)} - \tilde{R}^{(3)}\right|_{L^\infty}\right] \\ &\le |\varepsilon| K_{4,1}\left|f\right|_{C^{1+\alpha}}\left|u - \widetilde{u}\right|_{C^{1+\alpha}} + |\varepsilon|^2 K_{4,2}\left|f\right|^2_{C^{1+\alpha}}\left|u - \widetilde{u}\right|_{C^{1+\alpha}} \\ &\le \gamma\left|u - \widetilde{u}\right|_{C^{1+\alpha}}\end{aligned}$$

for $\gamma < 1$ if

$$|\varepsilon| \le c_0 \equiv \max\left\{\frac{\gamma}{2K_{4,1}\left|f\right|_{C^{1+\alpha}}}, \frac{\sqrt{\gamma}}{\sqrt{2K_{4,2}}\left|f\right|_{C^{1+\alpha}}}\right\}.$$

Clearly the estimate is uniformly valid for all ε in the disk $|\varepsilon| \le c_0$, and thus the contraction mapping principle gives existence and uniqueness of solutions. Repeating the above estimation procedure we find that (3.2) is valid. ∎

We next establish differentiability of u in ε.

LEMMA 3.3. *By shrinking the constant c_0 in Lemma 3.2 if necessary, we have*

$$\left|T_\delta[u]\right|_{C^{1+\alpha}} \le C \quad \text{for } |\varepsilon| \le c_0, |\delta| \le c_0, \tag{3.3}$$

where the constant C is independent of ε and δ.

Proof. We begin by applying the difference operator T_δ to (2.10) as follows:

$$\Delta T_\delta[u] = \operatorname{div}_x\left[T_\delta[F^{(1)}]\right] + \partial_y T_\delta[F^{(2)}] + T_\delta[F^{(3)}]. \tag{3.4}$$

The product rule (3.1) can be used to derive

$$
\begin{aligned}
&T_\delta[F^{(1)}] \\
&= -\frac{2}{h}\varepsilon f \nabla_x T_\delta[u](\varepsilon) - \frac{2}{h} f \nabla_x u(\varepsilon+\delta) - \frac{\varepsilon^2}{h^2} f^2 \nabla_x T_\delta[u](\varepsilon) - \frac{2\varepsilon}{h^2} f^2 \nabla_x u(\varepsilon+\delta) \\
&\quad -\frac{\delta}{h^2} f^2 \nabla_x u(\varepsilon+\delta) + \frac{\varepsilon(h+y)}{h} \nabla_x f \partial_y T_\delta[u](\varepsilon) + \frac{(h+y)}{h} \nabla_x f \partial_y u(\varepsilon+\delta) \\
&\quad +\frac{\varepsilon^2(h+y)}{h^2} f \nabla_x f \partial_y T_\delta[u](\varepsilon) + \frac{2\varepsilon(h+y)}{h^2} f \nabla_x f \partial_y u(\varepsilon+\delta) \\
&\quad +\frac{\delta(h+y)}{h^2} f \nabla_x f \partial_y u(\varepsilon+\delta).
\end{aligned} \tag{3.5}
$$

Estimating this in C^α we find

$$
\begin{aligned}
\left|T_\delta[F^{(1)}]\right|_{C^\alpha} \le{}& \left\{K_{5,1}|\varepsilon|\,|f|_{C^{1+\alpha}} + K_{5,2}|\varepsilon|^2\,|f|^2_{C^{1+\alpha}}\right\} |T_\delta[u](\varepsilon)|_{C^{1+\alpha}} \\
&+K_6\left\{|f|_{C^{1+\alpha}} + |\varepsilon|\,|f|_{C^{1+\alpha}} + |\delta|\,|f|_{C^{1+\alpha}}\right\} |u(\cdot,\cdot,\varepsilon+\delta)|_{C^{1+\alpha}}\,;
\end{aligned}
$$

similar expressions for $\left|T_\delta[F^{(2)}]\right|_{C^\alpha}$ and $\left|T_\delta[F^{(3)}]\right|_{L^\infty}$ can be found. These results coupled with Theorem A.2 imply that

$$
\begin{aligned}
|T_\delta[u]|_{C^{1+\alpha}} &\le C_e\left[\left|T_\delta[F^{(1)}]\right|_{C^\alpha} + \left|T_\delta[F^{(2)}]\right|_{C^\alpha} + \left|T_\delta[F^{(3)}]\right|_{L^\infty}\right] \\
&\le C_e\left[\left\{K_{7,1}\,|\varepsilon|\,|f|_{C^{1+\alpha}} + K_{7,2}\,|\varepsilon|^2\,|f|^2_{C^{1+\alpha}}\right\} |T_\delta[u]|_{C^{1+\alpha}}\right. \\
&\qquad \left. +K_8\left\{|f|_{C^{1+\alpha}} + |\varepsilon|\,|f|^2_{C^{1+\alpha}} + |\delta|\,|f|^2_{C^{1+\alpha}}\right\} |u|_{C^{1+\alpha}}\right].
\end{aligned}
$$

Clearly, if ε and δ are chosen sufficiently small, then $|T_\delta[u]|_{C^{1+\alpha}}$ is bounded independently of ε and δ. ∎

In the next step we show that this difference quotient converges to the derivative of u with respect to ε.

LEMMA 3.4. *There exists a small positive constant* c_0 *such that, in the disk* $\{|\varepsilon| \le c_0\}$*, the complexified solution* u *of* (2.10) *is differentiable in the complex variable* ε *in the space* $C^{1+\beta}$*, for any* $\beta \in (0,\alpha)$*, i.e.,*

$$
T_\delta[u] \to \partial_\varepsilon u \quad \text{as } |\delta| \to 0.
$$

Proof. For any $\beta \in (0,\alpha)$, we can use the compactness of $C^{1+\alpha}$ to conclude that there exists a subsequence $\delta_n \to 0$ such that

$$
T_{\delta_n}[u] \to w \quad \text{in } C^{1+\beta}(\overline{S_{0,h}} \cap \{|x| \le K\})
$$

for any $K > 1$. By passing δ_n to 0 in the equation, we find that w satisfies

$$
\Delta w = \operatorname{div}_x\left[H^{(1)}\right] + \partial_y H^{(2)} + H^{(3)},
$$

where

$$
\begin{aligned}
H^{(1)} ={}& -\frac{2}{h}\varepsilon f \nabla_x w - \frac{2}{h} f \nabla_x u - \frac{\varepsilon^2}{h^2} f^2 \nabla_x w - \frac{2\varepsilon}{h^2} f^2 \nabla_x u \\
&+\frac{\varepsilon(h+y)}{h} \nabla_x f \partial_y w + \frac{(h+y)}{h} \nabla_x f \partial_y u \\
&+\frac{\varepsilon^2(h+y)}{h^2} f \nabla_x f \partial_y w + \frac{2\varepsilon(h+y)}{h^2} f \nabla_x f \partial_y u,
\end{aligned}
$$

and similar expressions hold for $H^{(2)}$ and $H^{(3)}$. Similar to the proof of Lemma 3.2, the $C^{1+\beta}$ solution w to such a system is unique. This uniqueness implies that the convergence is independent of the subsequence of δ_n. ∎

At this point we can prove Theorem 2.1.

Proof of Theorem 2.1. By Cauchy's formula, for $|\varepsilon| < c_0$,

$$u(x,y,\varepsilon) = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=c_0} \frac{u(x,y,\zeta)}{\zeta - \varepsilon}\, \mathrm{d}\zeta = \sum_{n=0}^{\infty} u_n(x,y)\; \varepsilon^n,$$

where

$$u_n(x,y) = \frac{1}{2\pi i} \int_{|\zeta|=c_0} \frac{u(x,y,\zeta)}{\zeta^{n+1}}\, \mathrm{d}\zeta.$$

From this formula, we obtain the estimates on u_n from the estimates for u as follows:

$$|u_n|_{C^{1+\alpha}} \le \frac{1}{c_0^{n+1}} \max_{|\zeta|=c_0} |u(\cdot,\cdot,\zeta)|_{C^{1+\alpha}} \le CB^n\, |\xi|_{C^{1+\alpha}}\,,$$

where $B = 1/c_0$. Since $G(\varepsilon f)\; \xi(x)$ is expressed in terms of u and its first order derivatives (see (2.17)), we can extend $G(\varepsilon f)\; \xi(x)$ to complex ε. Using the (complex) analyticity of u in ε, we immediately have the differentiability of $G(\varepsilon f)\; \xi$ with respect to ε and

$$|G(\varepsilon f)\; \xi|_{C^\alpha} \le C\, |u|_{C^{1+\alpha}} \le C\, |\xi|_{C^{1+\alpha}}\,.$$

Thus, for $|\varepsilon| < c_0$,

$$G(\varepsilon f)\; \xi = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=c_0} \frac{G(\zeta f)\; \xi}{\zeta - \varepsilon}\, \mathrm{d}\zeta = \sum_{n=0}^{\infty} (G_n(f)\; \xi)\; \varepsilon^n,$$

where

$$G_n(f)\; \xi = \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=c_0} \frac{G(\zeta f)\; \xi}{\zeta^{n+1}}\, \mathrm{d}\zeta.$$

From this, we obtain

$$|G_n(f)\; \xi|_{C^\alpha} \le \frac{1}{c_0^{n+1}} \max_{|\zeta|=c_0} |G(\zeta f)\; \xi|_{C^\alpha} \le \frac{C}{c_0^{n+1}} \max_{|\zeta|=c_0} |u(\cdot,\cdot,\zeta)|_{C^{1+\alpha}} \le CB^n\, |\xi|_{C^{1+\alpha}}\,.$$

This implies

$$|G_n(f)|_{\mathcal{L}(C^{1+\alpha}(\mathbf{R}^{d-1}), C^{\alpha}(\mathbf{R}^{d-1}))} \le CB^n.$$

The theorem is proved. ∎

3.2. $W^{1,p}$ estimates. Using $W^{1,p}(S_{h,\eta})$ ($W^{1-1/p,p}$ on the boundary) estimates, we will extend the result of the previous section to Lipschitz boundaries; i.e., we will assume

$$f \in C^{0,1}, \quad \xi \in X^p \quad (p > d),$$

and approximate f and ξ by smooth functions where necessary. The key result which allows the estimation of Lipschitz boundaries is the following.

Lemma 3.5. *For $f \in C^{0,1}$, $u \in Y^{0,p}$, the product $(\nabla_x f)u \in Y^{0,p}$, and*

$$\|(\nabla_x f)u\|_{Y^{0,p}} \le |f|_{C^{0,1}} \|u\|_{Y^{0,p}} .$$

Given this result we can prove the following lemma.

Lemma 3.6. *Given $f \in C^{0,1}$, $\xi \in X^p$ $(p > d)$, there exists $c_0 > 0$ such that* (2.10) *has a unique solution $u \in Y^{1,p}$ for all ε in the disk $|\varepsilon| \le c_0$. Furthermore,*

$$\|u\|_{Y^{1,p}} \le C \|\xi\|_{X^p} , \tag{3.6}$$

where the constant C is independent of ε.

Proof. The proof is the same as in Lemma 3.2, with the $C^{1+\alpha}$ Hölder estimate (Theorem A.2) replaced by the $W^{1,p}$ estimate (Theorem A.3) given in Appendix A. For instance, the key estimate which guaranteed the contraction property in Lemma 3.2 now reads

$$\begin{aligned}
\left\|F^{(1)}\right\|_{Y^{0,p}} &\le \frac{2}{h} \|\varepsilon f \nabla_x u\|_{Y^{0,p}} + \frac{1}{h^2} \left\|\varepsilon^2 f^2 \nabla_x u\right\|_{Y^{0,p}} + \left\|\frac{h+y}{h} \varepsilon (\nabla_x f) \partial_y u\right\|_{Y^{0,p}} \\
&\quad + \left\|\frac{(h+y)}{h^2} \varepsilon^2 f (\nabla_x f) \partial_y u\right\|_{Y^{0,p}} \\
&\le \frac{2|\varepsilon|}{h} |f|_{C^{0,1}} \|u\|_{Y^{1,p}} + \frac{|\varepsilon|^2}{h^2} |f|^2_{C^{0,1}} \|u\|_{Y^{1,p}} \\
&\quad + \frac{\tilde{Y}|\varepsilon|}{h} |f|_{C^{0,1}} \|u\|_{Y^{1,p}} + \frac{\tilde{Y}|\varepsilon|^2}{h^2} |f|^2_{C^{0,1}} \|u\|_{Y^{1,p}} \\
&\le |\varepsilon| \tilde{K}_{1,1} |f|_{C^{0,1}} \|u\|_{Y^{1,p}} + |\varepsilon|^2 \tilde{K}_{1,2} |f|^2_{C^{0,1}} \|u\|_{Y^{1,p}} ,
\end{aligned}$$

where $\tilde{Y}$ is defined by

$$\|(h+y)u\|_{Y^{0,p}} \le \tilde{Y} \|u\|_{Y^{0,p}} .$$

From this calculation, using Lemma 3.5, we see the *explicit* appearance of the Lipschitz norm on the boundary deformation $f(x)$. ◻

To establish the differentiability in complex ε, we apply the finite difference operator, $T_\delta[\cdot]$.

Lemma 3.7. *By shrinking the positive constant c_0 in Lemma 3.6 if necessary, we have*

$$\|T_\delta[u]\|_{Y^{1,p}} \le C \quad \text{for } |\varepsilon| \le c_0, |\delta| \le c_0, \tag{3.7}$$

where the constant C is independent of ε.

Proof. Again, the proof is essentially the same as for Lemma 3.3 with the $C^{1+\alpha}$ Hölder estimate (Theorem A.2) replaced by the $W^{1,p}$ estimate (Theorem A.3) given in Appendix A. ◻

Now we are ready to establish the differentiability in complex ε.

Lemma 3.8. *If $|\varepsilon| \le c_0$ and u is the solution of* (2.10)*, then u is differentiable in ε as a complex function almost everywhere; i.e.,*

$$T_\delta[u] \to \partial_\varepsilon u \quad \text{as } |\delta| \to 0.$$

Proof. The proof is similar to that of Lemma 3.4. However, since we no longer have compactness for the first order derivatives, the subsequential convergence as

$\delta_n \to 0$ must be replaced by the following:

$$T_{\delta_n}[u] \to w \quad \text{strongly in } C^0(\{|x|<K\}\times[-h,0]) \text{ for any } K>1,$$
$$(\nabla_x,\partial_y)T_{\delta_n}[u] \to (\nabla_x,\partial_y)w \quad \text{weakly in } [L^p(\{|x|<K\}\times[-h,0])]^d \text{ for any } K>1.$$

We note that $T_{\delta_n}[u]$ satisfies equation (3.4) from Lemma 3.3. The terms in (3.5) are all *linear* in $T_{\delta_n}[u]$ and its first order of derivatives; furthermore, all the coefficients are in L^∞ since we assume that f is Lipschitz continuous. These key facts allow us to use weak convergence to take the limit $\delta_n \to 0$. Thus we obtain the equation for w. The rest of the proof remains the same as that of Lemma 3.4. □

Proof of Theorem 2.2. The analyticity of $u(x,y,\varepsilon)$ in ε, and the corresponding estimate for u_n in $Y^{1,p}$, can be obtained in the same manner as in the proof of Theorem 2.1. However, the estimates on $G(\varepsilon f)$ must be modified since we are only permitted the weak formulation of the DNO in this case. It is clear that this weak formulation, (2.15), allows the complexification in ε. To use this definition, however, we have to show that (2.15) defines a DNO in the appropriate space also for complex ε. Namely, we have to show that the value on the right-hand side of (2.15) is *independent* of the way the function $\psi(x,0)$ is extended to $\mathbf{R}^{d-1}\times[-h,0]$.

Since

$$T_R^\infty(\overline{S_{h,0}}) = \big\{ f \in C^\infty(\overline{S_{h,0}}) \mid f = 0 \text{ on } \{|x|>R\} \text{ for some large } R \big\}$$

is dense in $T_R^{0,1}(\overline{S_{h,0}})$, we only need to show that the right-hand side of (2.15) is independent of the extension for such ψ; namely, we need to show

$$\int_{S_{h,0}} \left\{ \left(\nabla_x u - \frac{h+y}{h+\eta}(\nabla_x\eta)\partial_y u\right)\cdot\left(\nabla_x\psi - \frac{h+y}{h+\eta}(\nabla_x\eta)\partial_y\psi\right) + \frac{h^2}{(h+\eta)^2}(\partial_y u)\partial_y\psi \right\} \frac{h+\eta}{h}\,\mathrm{d}V = 0, \tag{3.8}$$

for any $\psi \in C^\infty(\overline{S_{h,0}})$ such that $\psi(x,0)\equiv 0$ for all $x\in\mathbf{R}^{d-1}$ and $\psi(x,y)\equiv 0$ for $|x|>R$ for some $R>1$. Under our assumptions, all boundary terms vanish upon utilization of integration by parts in (3.8), so we can establish (3.8) by using the weak formulation of the complexified equation for u.

We next proceed to establish the estimates for $G(\varepsilon f)$. As in the proof of Theorem 2.1,

$$\langle G(\varepsilon f)\ \xi, \psi(x,0)\rangle = \frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=c_0} \frac{\langle G(\zeta f)\ \xi,\psi(x,0)\rangle}{\zeta-\varepsilon}\,\mathrm{d}\zeta = \sum_{n=0}^{\infty}\langle G_n(f)\ \xi,\psi(x,0)\rangle\ \varepsilon^n,$$

where $\psi(x,0)\in C_c^{0,1}(\mathbf{R}^{d-1})$, and

$$\langle G_n(f)\ \xi,\varphi(x,0)\rangle = \frac{1}{2\pi \mathrm{i}}\int_{|\zeta|=c_0}\frac{\langle G(\zeta f)\ \xi,\varphi(x,0)\rangle}{\zeta^{n+1}}\,\mathrm{d}\zeta.$$

Thus the conclusion of our theorem will follow if we can establish the estimate

$$\|G(\varepsilon f)\|_{\mathcal{L}(X^p,(X_c^q(\hat{x}))^*)} \le C \tag{3.9}$$

for some C, independent of ε, and for all $|\varepsilon|\le c_0$.

For any $\Psi \in X^q_c(\hat{x})$, we use the definition of $X^q_c(\hat{x})$ to extend Ψ to a function $\psi \in W^{1,q}_{loc}(\mathbf{R}^{d-1} \times [-h, 0])$ such that

$$\psi(x,0) = \Psi(x) \quad \text{in the trace sense,} \tag{3.10a}$$

$$\psi(x,y) = 0 \quad \text{for } |x-\hat{x}| > 1, \quad -h < y < 0, \tag{3.10b}$$

$$\|\psi\|_{W^{1,q}(B_1(\hat{x})\times(-h,0))} \le C\,\|\Psi\|_{X^q}\,. \tag{3.10c}$$

Since we have already established a $W^{1,p}(B_1(\hat{x}) \times (-h, 0))$ estimate for u, we can approximate ψ with $C^{0,1}$ functions so that its first order derivatives converge weakly in $L^q(B_1(\hat{x}) \times (-h, 0))$. Thus the test function defined in (3.10) can be used in (2.15). Using (2.15) we find that, for all $\Psi \in X^q_c(\hat{x})$,

$$\begin{aligned} |\langle G(\varepsilon f)\ \xi, \Psi\rangle| &\le C\,\|u\|_{W^{1,p}(B_1(\hat{x})\times(-h,0))}\,\|\nabla\psi\|_{L^q(B_1(\hat{x})\times(-h,0))} \\ &\le C\,\|\xi\|_{X^q}\,\|\Psi\|_{X^q}\,. \end{aligned}$$

This implies that

$$\|G(\varepsilon f)\ \xi\|_{(X^q_c(\hat{x}))^*} \le C\,\|\xi\|_{X^p}\,;$$

i.e., the estimate (3.9) is valid. ∎

Appendix A. Elliptic estimates. In this appendix we present the statements (together with brief proofs) of the elliptic estimates which are at the heart of the analyticity results, Theorems 2.1 and 2.2. Of course, the great simplification of our approach was the use of the "domain flattening" change of variables, (2.7), which maps the domain $S_{h,\eta}$ to the strip $S_{h,0}$. Consequently, it is sufficient to analyze (inhomogeneous) elliptic equations on a much simpler geometry. This, in turn, allows the simple establishment of the following results which, we point out, are true on much more general domains (e.g., see [5]).

We begin with the "comparison principle" on a domain, which implies the uniqueness of bounded solutions.

THEOREM A.1. *If w is bounded and satisfies (in the weak sense)*

$$\begin{aligned} &-\Delta w(x,y) \ge 0 && \textit{in } S_{h,0}, \\ &-\partial_y w(x,-h) \ge 0, && \\ &w(x,0) \ge 0, && \end{aligned}$$

then

$$w(x,y) \ge 0 \qquad \textit{in } S_{h,0}.$$

Proof. Since we can only use weak comparison in the bounded domain, we choose $M = |w|_{L^\infty}$ and let

$$\Phi = \frac{2(d-1)}{R} M \Big[\frac{x_1^2 + \cdots + x_{d-1}^2}{2(d-1)} - \frac{(y+h)^2}{2} + \frac{h^2}{2}\Big] + w.$$

It is clear that

$$\begin{aligned} &-\Delta\Phi \ge 0 && \text{in } S_{h,0}, \\ &-\partial_y \Phi(x,-h) \ge 0, && \\ &\Phi \ge 0 && \text{on } \{y=0\} \cup \{x_1^2 + \cdots + x_{d-1}^2 = R^2\}. \end{aligned}$$

We can now apply the comparison principle on the bounded domain $\{x \mid x_1^2 + \cdots + x_{d-1}^2 < R^2\} \times (-h, 0)$ to conclude that $\Phi \geq 0$ there. If we fix (x, y) and let $M \to \infty$, we obtain $w > 0$ on $S_{h,0}$. □

We next state the Hölder estimate used in section 3.1.

THEOREM A.2. *For any $\alpha \in (0,1)$ there exists a constant C_e such that for any $R^{(1)}, R^{(2)} \in C^\alpha$, $R^{(3)} \in L^\infty$, and $\xi \in C^{1+\alpha}$ there exists a unique solution $w(x,y)$ of*

$$\begin{aligned} &\Delta w = div_x \left[R^{(1)}\right] + \partial_y R^{(2)} + R^{(3)} && \text{in } S_{h,0}, \\ &\partial_y w(x, -h) = 0, \\ &w(x, 0) = \xi(x), \end{aligned}$$

which satisfies

$$|w|_{C^{1+\alpha}} \leq C_e \left\{ \left|R^{(1)}\right|_{C^\alpha} + \left|R^{(2)}\right|_{C^\alpha} + \left|R^{(3)}\right|_{L^\infty} + |\xi|_{C^{1+\alpha}} \right\}. \tag{A.1}$$

Proof. The uniqueness is a corollary of the comparison principle. The existence can be proved using a continuation argument once we obtain the estimate, (A.1), in this theorem. This estimate is a special case of the general $C^{1+\alpha}$ theory for elliptic systems in divergence form which is established using Campanato spaces $\mathcal{L}^{p,\mu}$ (see [5, Theorems 2.6 and 2.7, pp. 152–154]).

Since our system is of constant coefficients and in a special domain, we provide a short proof here. We write

$$w = \sum_{j=1}^{d-1} \partial_{x_j} w_j^{(1)} + \partial_y w^{(2)} + w^{(3)} + w^{(4)} + w^{(5)},$$

where

$$\Delta w_j^{(1)} = R_j^{(1)} \quad \text{in } S_{h,0}, \tag{A.2a}$$
$$\partial_y w_j^{(1)}(x, -h) = 0, \tag{A.2b}$$
$$w_j^{(1)}(x, 0) = 0; \tag{A.2c}$$

$$\Delta w^{(2)} = R^{(2)} \quad \text{in } S_{h,0}, \tag{A.3a}$$
$$\partial_y w^{(2)}(x, -h) = 0, \tag{A.3b}$$
$$w^{(2)}(x, 0) = 0; \tag{A.3c}$$

$$\Delta w^{(3)} = R_j^{(3)} \quad \text{in } S_{h,0}, \tag{A.4a}$$
$$\partial_y w^{(3)}(x, -h) = 0, \tag{A.4b}$$
$$w^{(3)}(x, 0) = 0; \tag{A.4c}$$

$$\Delta w^{(4)} = 0 \quad \text{in } S_{h,0}, \tag{A.5a}$$
$$\partial_y w^{(4)}(x, -h) = 0, \tag{A.5b}$$
$$w^{(4)}(x, 0) = \xi(x); \tag{A.5c}$$

and, finally,

$$\Delta w^{(5)} = 0 \qquad \text{in } S_{h,0}, \tag{A.6a}$$

$$\partial_y w^{(5)}(x,-h) = -\partial_{yy} w^{(2)}(x,-h) = -R^{(2)}(x,-h), \tag{A.6b}$$

$$w^{(5)}(x,0) = 0. \tag{A.6c}$$

We can apply standard Schauder theory [12] to $w^{(1)}$ and $w^{(2)}$ to obtain $C^{2+\alpha}$ estimates for $w^{(1)}$ and $w^{(2)}$. We can apply $W^{2,p}$ estimates to $w^{(3)}$ for $p > d/(1-\alpha)$ and then use an embedding theorem to obtain the $C^{1+\alpha}$ estimate for $w^{(3)}$. Since the Dirichlet boundary data for $w^{(4)}$ is $C^{1+\alpha}$, we obtain $C^{1+\alpha}$ estimates for $w^{(4)}$. Finally, if we let

$$z(x,y) = \int_{-h}^{y} w^{(5)}(x,s)\,\mathrm{d}s,$$

then

$$\Delta z = w_y^{(5)}(x,-h) = -R^{(2)}(x,-h) \qquad \text{in } S_{h,0},$$
$$\partial_y z(x,-h) = 0,$$
$$z(x,0) = 0.$$

Since $R^{(2)}$ is in C^{α}, we can apply the Schauder $C^{2+\alpha}$ estimate for z and obtain an $C^{1+\alpha}$ estimate for $w^{(5)} = \partial_y z$. ☐

Finally, we state the $W^{k,p}$ estimate used in section 3.2.

THEOREM A.3. *For any $p > d$ there exists a constant $\tilde{C}_e$ such that for any $R^{(1)}, R^{(2)}, R^{(3)} \in Y^{0,p}$, and $\xi \in X^p$ there exists a unique solution $w(x,y)$ of*

$$\Delta w = \mathrm{div}_x R^{(1)} + \partial_y\{(h+y)R^{(2)}\} + R^{(3)} \qquad \textit{in } S_{h,0}, \tag{A.7a}$$

$$\partial_y w(x,-h) = 0, \tag{A.7b}$$

$$w(x,0) = \xi(x), \tag{A.7c}$$

which satisfies

$$\|w\|_{Y^{1,p}} \le \tilde{C}_e \left\{ \left\|R^{(1)}\right\|_{Y^{0,p}} + \left\|R^{(2)}\right\|_{Y^{0,p}} + \left\|R^{(3)}\right\|_{Y^{0,p}} + \|\xi\|_{X^p} \right\}.$$

Proof. This estimate is a special case of the general L^p theory for elliptic systems in divergence form which is established in [5] (see page 157, Theorem 2.2 for interior estimates; the boundary estimates can be done in a similar way). In this short proof for our special system, we will assume that the involved functions are smooth since we can always approximate them with smooth functions. The estimate is valid as long as the constants involved are independent of the smoothness. We use the ideas of the earlier proof (Theorem A.2) and divide the proof into two cases.

Case 1: $\xi(x) \equiv 0$. The proof is similar to the proof of Theorem A.2. For any $x^* \in \mathbf{R}^{d-1}$, it suffices to establish estimates on $B_1(x^*) \times (-h,0)$ in terms of norms of $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$ on $B_2(x^*) \times (-h,0)$. We decompose w into $w^{(j)}$ $(j = 1,2,3,4,5)$ as before, and we can then apply the standard $W^{2,p}$ interior-boundary estimates to $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$. Since we have a factor $(h+y)$ on the right-hand side of (A.7) in the $R^{(2)}$ term, $w^{(5)}$ vanishes. Since we have assumed, in this case, that $\xi \equiv 0$, $w^{(4)}$ also vanishes and the estimate is established.

Case 2: General case. We need only estimate $w^{(4)}$; by the maximum principle,

$$\left|w^{(4)}\right|_{L^\infty} \le |\xi|_{L^\infty}.$$

Since $p > d$, we have, by embedding, $|\xi|_{L^\infty} \le C \|\xi\|_{X^p}$. Thus we can use the standard elliptic regularity estimates to derive

$$\left\| w^{(4)} \right\|_{C^2(B_2(x^*)\times[-h,-h/2])} \le C \|\xi\|_{X^p} .$$

For any $x^* \in \mathbf{R}^{d-1}$, we use the definition of X^p to extend the function ξ to a function $\Phi(x,y) \in W^{1,p}_{loc}(B_2(x^*) \times [-h,0]) \cap C^2(\overline{B}_2(x^*) \times [-h,-h/2])$ so that

$$\|\Phi\|_{W^{1,p}(B_2(x^*)\times(-h,0))} \le C \|\xi\|_{X^p} , \tag{A.8}$$

where we understand that $\Phi(\cdot,0) = \xi(\cdot)$ in the trace sense. By using a cut-off function if necessary, we may assume, without loss of generality, that

$$\Phi(x,y) \equiv w^{(4)}(x,y) \quad \text{ for } x \in B_2(x^*), -h \le y \le \frac{-h}{2}.$$

It is clear that $w^{(4)}$ satisfies

$$\begin{aligned}
&\Delta(w^{(4)} - \Phi) = -\mathrm{div}_x \left[\mu_1(y)\nabla_x\Phi\right] - \partial_y\Big((y+h)\frac{\mu_1(y)}{y+h}\partial_y\Phi\Big) \quad \text{ in } S_{h,0},\\
&\partial_y(w^{(4)} - \Phi)(x,-h) = 0,\\
&w^{(4)}(x,0) - \Phi(x,0) = 0,
\end{aligned}$$

where

$$\mu_1(y) = 1 \quad \text{for } \frac{-h}{2} \le y < 0, \qquad \mu_1(y) = 0 \quad \text{for } -h \le y < \frac{-h}{2};$$

we point out that, in fact, $w^{(4)} - \Phi \equiv 0$ for $-h \le y \le -h/2$. Using (A.8), we have

$$\left\| \frac{\mu_1(y)}{(h+y)}\partial_y\Phi \right\|_{L^p(B_2(x^*)\times(-h,0))} \le \frac{2}{h} \|\partial_y\Phi\|_{L^p(B_2(x^*)\times(-h,0))} \le C \|\xi\|_{X^p}$$

and

$$\|\mu_1(y)\nabla_x\Phi\|_{L^p(B_2(x^*)\times(-h,0))} \le C \|\xi\|_{X^p} .$$

We can now apply Case 1 to obtain

$$\left\| w^{(4)} \right\|_{W^{1,p}(B_1(x^*)\times(-h,0))} \le C \|\xi\|_{X^p} .$$

Combining all the estimates for $w^{(j)}$ $(j = 1,2,3,4,5)$ and taking the supremum over all $x^* \in \mathbf{R}^{d-1}$, we conclude the theorem. ∎

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure Appl. Math. 65, Academic Press, New York, 1975.
[2] O. P. Bruno and F. Reitich, *Numerical solution of diffraction problems: A method of variation of boundaries*, J. Opt. Soc. Amer. A, 10 (1993), pp. 1168–1175.
[3] B. Buffoni, *Existence and conditional energetic stability of capillary-gravity solitary water waves by minimisation*, Arch. Ration. Mech. Anal., 173 (2004), pp. 25–68.
[4] A. P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Natl. Acad. Sci. USA, 75 (1977), pp. 1324–1327.
[5] Y.-Z. Chen and L.-C. Wu, *Second Order Elliptic Equations and Elliptic Systems*, Transl. Math. Monogr. 174, AMS, Providence, RI, 1998.
[6] R. Coifman and Y. Meyer, *Nonlinear harmonic analysis and analytic dependence*, in Pseudodifferential Operators and Applications (Notre Dame, IN, 1984), AMS, Providence, RI, 1985, pp. 71–78.
[7] W. Craig, P. Guyenne, D. P. Nicholls, and C. Sulem, *Hamiltonian long wave expansions for water waves over a rough bottom*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 839–873.
[8] W. Craig and D. P. Nicholls, *Traveling two and three dimensional capillary gravity water waves*, SIAM J. Math. Anal., 32 (2000), pp. 323–359.
[9] W. Craig and D. P. Nicholls, *Traveling gravity water waves in two and three dimensions*, Eur. J. Mech. B Fluids, 21 (2002), pp. 615–641.
[10] W. Craig, U. Schanz, and C. Sulem, *The modulation regime of three-dimensional water waves and the Davey-Stewartson system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 14 (1997), pp. 615–667.
[11] W. Craig and C. Sulem, *Numerical simulation of gravity waves*, J. Comput. Phys., 108 (1993), pp. 73–83.
[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin, 1983.
[13] C. Godrèche, ed., *Solids Far from Equilibrium*, Cambridge University Press, Cambridge, UK, 1992.
[14] P. Guyenne and D. P. Nicholls, *Numerical simulation of solitary waves on plane slopes*, Math. Comput. Simulation, to appear.
[15] H. Lamb, *Hydrodynamics*, 6th ed., Cambridge University Press, Cambridge, UK, 1993.
[16] D. P. Nicholls, *Traveling water waves: Spectral continuation methods with parallel implementation*, J. Comput. Phys., 143 (1998), pp. 224–240.
[17] D. P. Nicholls, *On hexagonal gravity water waves*, Math. Comput. Simulation, 55 (2001), pp. 567–575.
[18] D. P. Nicholls and F. Reitich, *A new approach to analyticity of Dirichlet-Neumann operators*, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001), pp. 1411–1433.
[19] D. P. Nicholls and F. Reitich, *Stability of high-order perturbative methods for the computation of Dirichlet-Neumann operators*, J. Comput. Phys., 170 (2001), pp. 276–298.
[20] D. P. Nicholls and F. Reitich, *Analytic continuation of Dirichlet-Neumann operators*, Numer. Math., 94 (2003), pp. 107–146.
[21] D. P. Nicholls and F. Reitich, *Shape deformations in rough surface scattering: Cancellations, conditioning, and convergence*, J. Opt. Soc. Amer. A, 21 (2004), pp. 590–605.
[22] D. P. Nicholls and F. Reitich, *Shape deformations in rough surface scattering: Improved algorithms*, J. Opt. Soc. Amer. A, 21 (2004), pp. 606–621.
[23] V. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys., 9 (1968), pp. 190–194.