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Joint Analyticity and Analytic Continuation of Dirichlet–Neumann Operators on Doubly Perturbed Domains

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Abstract. In this paper we take up the question of analyticity properties of Dirichlet–Neumann operators (DNO) which arise in boundary value and free boundary problems from a wide variety of applications (e.g., fluid and solid mechanics, electromagnetic and acoustic scattering). More specifically, we consider DNO defined on domains inspired by the simulation of ocean waves over bathymetry, i.e. domains perturbed independently at both the top and bottom. Our analysis shows that the DNO, when perturbed from an arbitrary smooth domain, is parametrically analytic (as a function of deformation height/slope) for profiles of finite smoothness. Additionally, we extend these results to joint spatial and parametric analyticity when the perturbations are real analytic. This analysis is novel not only in that it accounts for the doubly perturbed nature of the geometry, but also in that the technique of proof establishes the full joint analyticity from an arbitrary smooth profile simultaneously.

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1. Introduction

Boundary value and free boundary problems arise in a wide variety of applications in the physical and engineering sciences. From electromagnetics and acoustics [3] to fluid [12] and solid mechanics [9], boundary value and free boundary models are indispensable as a source of quantitative information for real-world phenomena. As important tools for scientists and engineers alike, the analysis (both theoretical and numerical) of these problems is clearly of crucial importance in understanding basic physical processes. In this paper we present a novel analysis of analyticity properties of a boundary operator (the "Dirichlet–Neumann operator"), as a function of boundary deformation, which appears in the analysis of many boundary value and free boundary problems.

For a large sub-class of boundary value and free boundary problems, a simplification and reduction in dimension can be achieved by considering boundary quantities as fundamental variables. This is usually possible when the unknown functions satisfy particularly simple differential equations on the *interior* of the problem domain. This is the case, e.g., for potential fluid flow [12] (the velocity potential satisfies Laplace's equation) and linear time-harmonic acoustics [3] (the reduced pressure satisfies Helmholtz's equation). In such cases the field quantity at the boundary and (in the case of a free boundary problem) the boundary shape typically suffice as fundamental variables. From these the value of the field at any point in the domain can be recovered from a suitable integral formula.

Of course, derivatives of the field at the boundary may be of physical interest and/or necessary to correctly pose the physical problem. In this case a challenge arises in producing *normal* boundary derivatives as these involve, in a fundamental way, the solution of the differential equation *inside* the problem domain. For this reason, normal derivative operators such as the Dirichlet–Neumann operator (DNO), also known as the Steklov–Poincaré operator [3], which produce a first normal derivative (Neumann data) from boundary measurements (Dirichlet data) play a large role. Clearly, a detailed understanding of the analytical properties of these DNO is crucial to not only the theoretical study of boundary value and free boundary problems, but also their reliable and accurate numerical simulation.

In this paper we take up such questions in the setting of ideal, free-boundary fluid mechanics (the water wave problem) in d dimensions ($(d-1)$-many horizontal dimensions and one vertical dimension). In particular, we focus upon *analyticity* properties of DNO with respect to boundary variations. These results are important for numerical simulation as they justify boundary perturbation methods for the approximation of DNO [22, 14, 7, 17]. In the case of infinite depth or trivial (i.e. flat) bathymetry, analyticity of DNO with respect to surface variation, say $\eta = \varepsilon f$, has been investigated by several authors. Coifman & Meyer [2] (based upon the work of Calderón [1]) were the first to show that the DNO varies analytically as a function of ε for f Lipschitz when $d = 2$. Craig, Schanz, and Sulem [6] extended this method (based upon an integral equation formulation) to $d = 3$ for f in the class of C^1 functions, while Craig & Nicholls [5] produced the corresponding result for any d.

In [16] Nicholls & Reitich devised a new, direct strategy for establishing analyticity of DNO in arbitrary dimensions using a non-conformal change of variables and the classic existence and regularity theory of elliptic partial differential equations. Subsequently this method of "Transformed Field Expansions" (TFE) has been expanded in many new directions and applied to several different problems. Of this work, the most closely related to the current research is that of Nicholls & Reitich [18] in which the joint parametric and spatial analyticity of the DNO is established, and a theorem is proven justifying methods of analytic continuation for these operators.

In the current research we apply and extend the TFE method in several important new directions. First of all, we consider the DNO in the setting of water waves over non-trivial bathymetry which gives a more realistic description of ocean waves, particularly in the shallow-water regime. To our knowledge the only previous work on DNO in this geometry is that of Smith [21], who derived forms (via "Operator Expansions," cf. [16]) for the n-th term in the Taylor expansion of the DNO; Guyenne & Nicholls [10], who performed numerical simulations based upon these formulas; and Craig, Guyenne, Nicholls, & Sulem [4] who derived long-wave approximations to the water wave equations over bathymetry. However, none of these rigorously justifies the expansion of the DNO, and the current research provides this justification. Of course, this result was long anticipated, however, the double perturbation technique used is novel and worthy of note.

Perhaps more importantly, in this work we generalize the technique of proof used in [18] to establish "analytic continuation" results. The paper of Nicholls & Reitich [18] established two results: First, that the DNO is jointly analytic as a function of both spatial ($x \in \mathbf{R}^{d-1}$ and $y \in \mathbf{R}$) and parametric (ε) variables for ε sufficiently small and f real analytic. Second, it was shown that the DNO depends analytically on variations from *arbitrary* smooth domains, say $\eta(x) = f_0(x)$. More precisely, it was shown that if the top perturbation $\eta(x)$ is shaped by $f_0(x)+\varepsilon f(x)$ then the DNO is analytic as a function of ε (sufficiently small) for *any* f_0 and f sufficiently smooth. This implies that the domain of (parametric) analyticity of the DNO includes a neighborhood of the *entire* real axis, an "analytic continuation" result. However, the two results were not proven simultaneously, i.e. that the DNO is *jointly* analytic in spatial and parametric variables as a variation of an arbitrary smooth domain. As we demonstrate, this extension is highly non-trivial and requires (see § 4) the proof of a generalized elliptic regularity theorem (see § A) based upon subtle commutator estimates (see § B).

Our new result is the following: If the top of our problem domain (the water surface) is shaped by $y = \eta(x) = \tilde{\varepsilon} f(x)$ and the bottom (the ocean bottom with mean depth h) is given by $y = -h + \zeta(x) = -h + \tilde{\delta} b(x)$, then the DNO is jointly analytic as a function of the parameters $\tilde{\varepsilon}$ and $\tilde{\delta}$, and the spatial variables x and y. Furthermore, this disk of analyticity can be centered at *any* (f_0, b_0) thereby including a neighborhood of the full, real two-plane in (ε, δ) space. More precisely, let us set $\eta(x) = f_0(x) + \varepsilon f(x)$ and $\zeta(x) = b_0(x) + \delta b(x)$, then if $\xi(x)$ gives the Dirichlet data at the surface, u is the field (satisfying Laplace's equation), and G is the DNO, then we can make the Taylor expansions

$$u(x, y; \varepsilon, \delta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{n,m}(x, y) \varepsilon^n \delta^m, \quad G(x; \varepsilon, \delta)[\xi] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} G_{n,m}[\xi] \varepsilon^n \delta^m,$$

which converge strongly in the sense of the following two theorems, cf. [18].

Theorem 1. *If f, b, f_0, b_0, ξ are real analytic functions then*

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{n,m} \right\|_{H^2} \le K_0 B^n E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2},$$

for constants $K_0, B, E, A, D > 0$.

Theorem 2. *If* f, b, f_0, b_0, ξ *are real analytic functions then*

$$\left\| \frac{\partial_x^k}{k!} G_{n,m}[\xi] \right\|_{H^{s+1/2}} \leq \tilde{K}_0 B^n E^m \frac{A^k}{(k+1)^2},$$

for constants $\tilde{K}_0, B, E, A > 0$.

The outline of the paper is as follows: In § 2 we review the relevant governing equations, while in § 3 we establish the parametric analyticity of the DNO for variations of arbitrary smooth functions. In § 4 we extend this result to joint parametric and spatial analyticity for analytic deformations (Theorems 1 and 2). Finally, in § A and § B we prove two crucial results on elliptic regularity and smoothness of commutators, respectively.

2. Governing equations

While Dirichlet–Neumann operators (DNO) arise in a wide array of physical contexts, we choose as motivation the physics and geometry of free surface ideal fluid flows (the water wave problem). Consider a d-dimensional ($d = 2, 3$) ideal (inviscid, irrotational, incompressible) fluid occupying the domain

$$S_{h,\zeta,\eta} := \left\{ (x, y) \in \mathbf{R}^{d-1} \times \mathbf{R} \mid -h + \zeta(x) < y < \eta(x,t) \right\},$$

meant to represent a fluid of mean depth h, with bottom topography ζ, and time dependent free surface η. The irrotational and incompressible nature of the flow dictates that the fluid velocity inside $S_{h,\zeta,\eta}$ can be expressed as the gradient of a potential, $u = \nabla\varphi$. The Euler equations [12] govern the evolution of the potential and the surface shape under the effects of gravity and surface tension by:

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in } S_{h,\zeta,\eta} \\ \partial_y\varphi - \nabla_x\zeta \cdot \nabla_x\varphi &= 0 \quad \text{at } y = -h + \zeta \\ \partial_t\varphi + \frac{1}{2}\left|\nabla\varphi\right|^2 + g\eta - \sigma\kappa(\eta) &= 0 \quad \text{at } y = \eta \\ \partial_t\eta - \partial_y\varphi + \nabla_x\eta \cdot \nabla_x\varphi &= 0 \quad \text{at } y = \eta, \end{aligned}$$

where g and σ are the constants of gravity and capillarity, respectively, and κ is the curvature:

$$\kappa(\eta) := \operatorname{div}_x \left[\frac{\nabla_x\eta}{\sqrt{1 + \left|\nabla_x\eta\right|^2}} \right].$$

For simplicity we consider periodic boundary conditions with respect to the lattice $\Gamma \subset \mathbf{R}^{d-1}$ giving period cell $P(\Gamma)$ and wavenumbers in the conjugate lattice Γ'.

A simplification and reduction in dimension can be achieved for the water wave problem upon the realization that, given the surface deformation $\eta(x,t)$ and the Dirichlet trace of the potential at the surface $\xi(x,t)$, the full potential, $\varphi(x,y,t)$, can be recovered anywhere inside the domain $S_{h,\zeta,\eta}$ via an appropriate integral formula [8]. Of course other surface quantities could be used, however, the Dirichlet data is distinguished by the discovery of Zakharov [23] that the pair (η,ξ) are, in fact, canonical variables in a Hamiltonian formulation of the water wave problem. The Hamiltonian presented by Zakharov is somewhat implicit in nature as the quantity ξ does not make an explicit appearance, however, this was rectified by Craig & Sulem [7] with the introduction of the DNO to the formulation.

The problem which defines the DNO for surface water waves is:

$$\Delta v = 0 \qquad \text{in } S_{h,\zeta,\eta} \tag{1a}$$

$$v(x,\eta) = \xi(x) \tag{1b}$$

$$\partial_y v - \nabla_x \zeta \cdot \nabla_x v = 0 \qquad y = -h + \zeta, \tag{1c}$$

coupled with periodic boundary conditions. From this, the DNO, which maps Dirichlet data ξ to an (unnormalized) normal derivative of v at η, is defined by

$$G(\eta,\zeta)[\xi] := [\nabla v \cdot N_\eta]_{y=\eta} = [\partial_y v - \nabla_x \eta \cdot \nabla_x v]_{y=\eta}, \tag{2}$$

where $N_\eta := (-\nabla_x \eta, 1)^T$. The choice of this particular normal is two-fold: First, it accommodates a particularly simple restatement of the water wave problem [7]. Second, and more importantly, this DNO (with normal N_η) is self-adjoint which permits the implementation of rapid Boundary Perturbation schemes for its numerical simulation [15, 17].

2.1. Change of variables

To facilitate our analysis we effect a change of variables which we have found quite useful in establishing analyticity properties of boundary operators such as the DNO [16, 18]. Consider the mapping,

$$x' = x, \quad y' = h\left(\frac{y-\eta}{h-\zeta+\eta}\right), \tag{3}$$

which takes the fluid domain $S_{h,\zeta,\eta}$ to the simpler geometry $S_{h,0,0}$. To clarify our presentation we introduce the notation

$$M(x) := h - \zeta(x) + \eta(x)$$

$$\tilde{M}(x) := M(x) - h = -\zeta(x) + \eta(x)$$

$$N(x,y) := -(y+h)\nabla_x \eta(x) + y \nabla_x \zeta(x),$$

and point out the following useful formulas

$$M(x)\nabla_x = M(x')\nabla_{x'} + N(x',y')\partial_{y'} \tag{4a}$$

$$M(x)\mathrm{div}_x\,[\cdot] = M(x')\mathrm{div}_{x'}\,[\cdot] + N(x',y')\cdot\partial_{y'}[\cdot] \tag{4b}$$

$$M(x)\partial_y = h\partial_{y'}. \tag{4c}$$

The field v transforms to

$$u(x',y') := v(x',(h-\zeta(x')+\eta(x'))y'/h+\eta(x')),$$

and (1) transforms (upon dropping primes) to

$$\Delta u = F \qquad -h<y<0 \tag{5a}$$

$$u(x,0) = \xi(x) \tag{5b}$$

$$\partial_y u(x,-h) = J(x), \tag{5c}$$

where

$$F := \mathrm{div}_x\left[F^{(x)}\right] + \partial_y F^{(y)} + F^{(h)},$$

and

$$h^2F^{(x)} = -(2h\tilde{M}+\tilde{M}^2)\nabla_x u - NM\partial_y u \tag{5d}$$

$$h^2F^{(y)} = -MN\cdot\nabla_x u - |N|^2\,\partial_y u \tag{5e}$$

$$h^2F^{(h)} = M\nabla_x M\cdot\nabla_x u + N\cdot\nabla_x M\partial_y u \tag{5f}$$

$$hJ = M\nabla_x\zeta\cdot\nabla_x u(x,-h) - h\,|\nabla_x\zeta|^2\,\partial_y u(x,-h). \tag{5g}$$

The DNO, (2), transforms to

$$G = \partial_y u(x,0) + I, \tag{6a}$$

where

$$hI = \zeta G - \eta G - \nabla_x\eta\cdot M\nabla_x u(x,0) + h\,|\nabla_x\eta|^2\,\partial_y u(x,0). \tag{6b}$$

Remark 1. We remark at this point that the form (5) is not the only one which can be realized with the change of variables (3). For instance, to derive (5a) we premultiplied Laplace's equation (1a) by a factor of M^2, rearranged terms so that (4) could be used, and then moved all terms involving powers of η and/or ζ to the right-hand side. This last step is taken since, as we shall see in § 2.3, we wish to expand the field u in (essentially) powers of the perturbation functions η and ζ. This formulation has the advantage that the "base operator" on the left-hand side remains the Laplacian while the right-hand side contains no quotients.

However, as noted in Lannes [13], one can also attain a purely second-order divergence form in (5a) using different manipulations. In particular, if one premultiplies Laplace's equation (1a) with *one* power of M we can replace (5a) with

$$\mathrm{div}\,[P\nabla u] := \mathrm{div}\left[\begin{pmatrix} M & -N \\ -N & \frac{h^2+N^2}{M}\end{pmatrix}\nabla u\right] = 0. \tag{7}$$

This form has the aesthetic advantage of being in purely second-order divergence form, however, we have *not* separated out factors which depend upon η or ζ so that, to truly compare (7) to (5), we must separate $P(\eta,\zeta) = hI + \tilde{P}(\eta,\zeta)$ and rewrite (7) as

$$h\Delta u = -\text{div}\left[\tilde{P}(\eta,\zeta)\nabla u\right] =: \tilde{F}(\eta,\zeta). \tag{8}$$

Furthermore, this representation includes terms which are quotients in η and ζ.

At this point one can wonder whether one formulation is to be preferred over the other. A brief comparison of (5) to (8) shows the difference to be rather small and, indeed, the proofs presented later in this paper would proceed with little alteration. However, as we typically have a numerical implementation in mind, the first formulation, (5) has a *significant* advantage in terms of computational complexity. This can be realized with an inspection of, e.g., (11d) which, as a result of the lack of quotients in (5), features a *fixed* number of terms *regardless* of the perturbation order (n,m). By contrast, a similar expansion using (8) will result, as a consequence of the quotients appearing in $\tilde{F}$, in right-hand sides with a number of terms *proportional* to (n,m). Clearly the former approach will be greatly advantaged in terms of execution time in a numerical simulation and it is for this reason that we utilize (5) rather than (8).

2.2. Analytic continuation

Following the work of Nicholls & Reitich [18] we shall demonstrate that the analyticity of the field, u, and the DNO, G, extends beyond the disk in (ε,δ) centered at the origin to include disks centered at any *real* value of the parameters ε and δ. In fact, our theory will allow us to conclude that the field and DNO depend analytically (both parametrically and spatially) on variations of *arbitrary* smooth domains. In this sense our results are ones of analytic continuation, and provide justification for schemes such as Padé approximation which have been applied to the computation of DNO and related quantities (see, e.g., [17, 19, 20]).

To begin, we consider a fixed profile pair $(f(x), b(x))$ and we show that the field and DNO depend analytically upon $(\tilde{\varepsilon}f, \tilde{\delta}b)$ for any $(\tilde{\varepsilon},\tilde{\delta}) \in U \subset \mathbf{R}^2$. Here U is the set of allowable parameters, i.e. the set of $(\tilde{\varepsilon},\tilde{\delta})$ such that the top and bottom deformations do not intersect. Consider a fixed pair $(\tilde{\varepsilon}_0,\tilde{\delta}_0) \in U$, if we now write

$$f_0(x) = \tilde{\varepsilon}_0 f(x), \qquad \varepsilon = \tilde{\varepsilon} - \tilde{\varepsilon}_0,$$
$$b_0(x) = \tilde{\delta}_0 f(x), \qquad \delta = \tilde{\delta} - \tilde{\delta}_0,$$

then we must prove joint analyticity of the field and DNO in (ε,δ) about $(\varepsilon = 0, \delta = 0)$. In light of this we now make the change of variables (3) with

$$\eta(x) = f_0(x) + \varepsilon f(x), \qquad \zeta(x) = -h + b_0(x) + \delta b(x),$$

so that

$$\begin{aligned} M &= \{h - b_0 + f_0\} - \delta b + \varepsilon f \\ &=: M_0 - \delta b + \varepsilon f \end{aligned}$$

$$\begin{aligned}
\tilde{M} &= \{-b_0 + f_0\} - \delta b + \varepsilon f \\
&=: \tilde{M}_0 - \delta b + \varepsilon f \\
N &= \{-(y+h)\nabla_x f_0 + y\nabla_x b_0\} - \varepsilon(y+h)\nabla_x f + \delta y \nabla_x b \\
&=: N_0 - \varepsilon(y+h)\nabla_x f + \delta y \nabla_x b.
\end{aligned}$$

In writing (5) & (6) we separated, to the right-hand side, all terms of order $\mathcal{O}(\eta+\zeta)$. For our proof of analytic continuation we can utilize a double induction provided that terms of order $\mathcal{O}(\varepsilon+\delta)$ are isolated on the right-hand side of the differential equation and boundary conditions. To this end we notice that

$$\begin{aligned}
h^2 F^{(x)} &= -(2h\tilde{M}_0 + \tilde{M}_0^2)\nabla_x u - N_0 M_0 \partial_y u + h^2 R^{(x)} \\
h^2 F^{(y)} &= -M_0 N_0 \cdot \nabla_x u - |N_0|^2 \partial_y u + h^2 R^{(y)} \\
h^2 F^{(h)} &= M_0 \nabla_x M_0 \cdot \nabla_x u + N_0 \cdot \nabla_x M_0 \partial_y u + h^2 R^{(h)} \\
hJ &= M_0 \nabla_x b_0 \cdot \nabla_x u(x,-h) - h |\nabla_x b_0|^2 \partial_y u(x,-h) + hQ,
\end{aligned}$$

where

$$\begin{aligned}
h^2 R^{(x)} = {} & \delta\big\{2hb\nabla_x u + 2b\tilde{M}_0 \nabla_x u + bN_0 \partial_y u - y\nabla_x b M_0 \partial_y u\big\} \\
& + \varepsilon\big\{-2hf\nabla_x u + 2f\tilde{M}_0 \nabla_x u - fN_0\partial_y u + (y+h)\nabla_x f M_0 \partial_y u\big\} \\
& + \delta^2\big\{-b^2\nabla_x u + yb\nabla_x b \partial_y u\big\} \\
& + \varepsilon^2\big\{-f^2\nabla_x u + (y+h) f \nabla_x f \partial_y u\big\} \\
& + \delta\varepsilon\big\{2fb\nabla_x u - yf\nabla_x b\partial_y u - (y+h)\nabla_x f b \partial_y u\big\},
\end{aligned}$$

$$\begin{aligned}
h^2 R^{(y)} = {} & \delta\big\{bN_0\nabla_x u - yM_0\nabla_x b \cdot \nabla_x u - 2y\nabla_x b \cdot N_0 \partial_y u\big\} \\
& + \varepsilon\big\{-fN_0 \cdot \nabla_x u + (y+h)M_0\nabla_x f \cdot \nabla_x u + 2(y+h)\nabla_x f \cdot N_0 \partial_y u\big\} \\
& + \delta^2\big\{yb\nabla_x b \cdot \nabla_x u - y^2 |\nabla_x b|^2 \partial_y u\big\} \\
& + \varepsilon^2\big\{(y+h)f\nabla_x f \cdot \nabla_x u - (y+h)^2 |\nabla_x f|^2 \partial_y u\big\} \\
& + \delta\varepsilon\big\{-(y+h)b\nabla_x f \cdot \nabla_x u - yf\nabla_x b \cdot \nabla_x u + 2y(y+h)\nabla_x f \cdot \nabla_x b \partial_y u\big\},
\end{aligned}$$

$$\begin{aligned}
h^2 R^{(h)} = {} & \delta\big\{-M_0\nabla_x b \cdot \nabla_x u - b\nabla_x M_0 \cdot \nabla_x u - \nabla_x b \cdot N_0 \partial_y u + y\nabla_x b \cdot \nabla_x M_0 \partial_y u\big\} \\
& + \varepsilon\big\{M_0\nabla_x f \cdot \nabla_x u + f\nabla_x M_0 \cdot \nabla_x u + \nabla_x f \cdot N_0 \partial_y u - (y+h)\nabla_x f \cdot \nabla_x M_0 \partial_y u\big\} \\
& + \delta^2\big\{b\nabla_x b \cdot \nabla_x u - y |\nabla_x b|^2 \partial_y u\big\} \\
& + \varepsilon^2\big\{f\nabla_x f \cdot \nabla_x u - (y+h) |\nabla_x f|^2 \partial_y u\big\} \\
& + \delta\varepsilon\big\{-b\nabla_x f \cdot \nabla_x u - f\nabla_x b \cdot \nabla_x u + (2y+h)\nabla_x f \cdot \nabla_x b \partial_y u\big\},
\end{aligned}$$

and

$$
\begin{aligned}
hQ &= \delta\big\{-b\nabla_x b_0 \cdot \nabla_x u + M_0 \nabla_x b \cdot \nabla_x u - 2h \nabla_x b_0 \cdot \nabla_x b \partial_y u\big\} \\
&\quad + \varepsilon\big\{f \nabla_x b_0 \cdot \nabla_x u\big\} + \delta^2\big\{-b\nabla_x b \cdot \nabla_x u - h\left|\nabla_x b\right|^2 \partial_y u\big\} \\
&\quad + \delta\varepsilon\big\{f \nabla_x b \cdot \nabla_x u\big\}.
\end{aligned}
$$

We can now restate (5) as

$$\mathcal{L}\left\{u\right\} = R \qquad -h < y < 0 \tag{9a}$$

$$u(x,0) = \xi(x) \tag{9b}$$

$$\mathcal{B}\left\{u(x,-h)\right\} = Q(x), \tag{9c}$$

where

$$R := \operatorname{div}_x \left[R^{(x)}\right] + \partial_y R^{(y)} + R^{(h)},$$

and

$$\mathcal{L}\left\{w\right\} := \operatorname{div}\left[A\nabla w\right] + B \cdot \nabla w, \tag{9d}$$

$$h^2 A := \begin{pmatrix} M_0^2 & M_0 N_0 \\ M_0 N_0^T & h^2 + \left|N_0\right|^2 \end{pmatrix}, \quad h^2 B := \begin{pmatrix} M_0 \nabla_x M_0 \\ N_0 \cdot \nabla_x M_0 \end{pmatrix}, \tag{9e}$$

$$\mathcal{B}\left\{w\right\} := E \cdot \nabla w(x,-h), \quad hE := \begin{pmatrix} -M_0 \nabla_x b_0 \\ h + h\left|\nabla_x b_0\right|^2 \end{pmatrix}. \tag{9f}$$

Additionally, we write (6) as

$$G = \mathcal{G}\left\{u(x,0)\right\} + H(x), \tag{10a}$$

where

$$\mathcal{G}\left\{w(x,0)\right\} := L \cdot \nabla w(x,0), \quad L := \frac{1}{M_0}\begin{pmatrix} -M_0 \nabla_x f_0 \\ h + h\left|\nabla_x f_0\right|^2 \end{pmatrix}, \tag{10b}$$

and

$$
\begin{aligned}
M_0 H(x) &= \delta\big\{bG + b\nabla_x f_0 \cdot \nabla_x u(x,0)\big\} \\
&\quad + \varepsilon\big\{-fG - M_0 \nabla_x f \cdot \nabla_x u(x,0) - f\nabla_x f_0 \cdot \nabla_x u(x,0) \\
&\qquad\qquad + 2h \nabla_x f \cdot \nabla_x f_0 \partial_y u(x,0)\big\} \\
&\quad + \varepsilon^2\big\{-f\nabla_x f \cdot \nabla_x u(x,0) + h\left|\nabla_x f\right|^2 \partial_y u(x,0)\big\} \\
&\quad + \delta\varepsilon\big\{b\nabla_x f \cdot \nabla_x u(x,0)\big\}.
\end{aligned} \tag{10c}
$$

2.3. Transformed field expansions

Having made the change of variables (3) about the arbitrary profile pair (f_0, b_0) we now follow the Transformed Field Expansions approach [16, 18] to establishing analyticity by expanding the transformed field:

$$u(x,y;\varepsilon,\delta)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}u_{n,m}(x,y)\varepsilon^n\delta^m.$$

Upon inserting this into (9) we find that we must solve

$$\mathcal{L}\{u_{n,m}\}=R_{n,m} \qquad -h<y<0 \tag{11a}$$

$$u_{n,m}(x,0)=\delta_{n,0}\delta_{m,0}\xi(x) \tag{11b}$$

$$\mathcal{B}\{u_{n,m}(x,-h)\}=Q_{n,m}(x), \tag{11c}$$

where $\delta_{n,p}$ is the Kronecker delta,

$$R_{n,m}:=\operatorname{div}_x\left[R_{n,m}^{(x)}\right]+\partial_y R_{n,m}^{(y)}+R_{n,m}^{(h)},$$

and

$$\begin{aligned}h^2R_{n,m}^{(x)}=&\left\{2hb\nabla_x u_{n,m-1}+2b\tilde{M}_0\nabla_x u_{n,m-1}+bN_0\partial_y u_{n,m-1}-y\nabla_x bM_0\partial_y u_{n,m-1}\right\}\\&+\left\{-2hf\nabla_x u_{n-1,m}+2f\tilde{M}_0\nabla_x u_{n-1,m}-fN_0\partial_y u_{n-1,m}\right.\\&\qquad\left.+(y+h)\nabla_x fM_0\partial_y u_{n-1,m}\right\}\\&+\left\{-b^2\nabla_x u_{n,m-2}+yb\nabla_x b\partial_y u_{n,m-2}\right\}\\&+\left\{-f^2\nabla_x u_{n-2,m}+(y+h)f\nabla_x f\partial_y u_{n-2,m}\right\}\\&+\left\{2fb\nabla_x u_{n-1,m-1}-yf\nabla_x b\partial_y u_{n-1,m-1}-(y+h)\nabla_x fb\partial_y u_{n-1,m-1}\right\},\end{aligned} \tag{11d}$$

$$\begin{aligned}h^2R_{n,m}^{(y)}=&\left\{bN_0\nabla_x u_{n,m-1}-yM_0\nabla_x b\cdot\nabla_x u_{n,m-1}-2y\nabla_x b\cdot N_0\partial_y u_{n,m-1}\right\}\\&+\left\{-fN_0\cdot\nabla_x u_{n-1,m}+(y+h)M_0\nabla_x f\cdot\nabla_x u_{n-1,m}\right.\\&\qquad\left.+2(y+h)\nabla_x f\cdot N_0\partial_y u_{n-1,m}\right\}\\&+\left\{yb\nabla_x b\cdot\nabla_x u_{n,m-2}-y^2\left|\nabla_x b\right|^2\partial_y u_{n,m-2}\right\}\\&+\left\{(y+h)f\nabla_x f\cdot\nabla_x u_{n-2,m}-(y+h)^2\left|\nabla_x f\right|^2\partial_y u_{n-2,m}\right\}\\&+\left\{-(y+h)b\nabla_x f\cdot\nabla_x u_{n-1,m-1}-yf\nabla_x b\cdot\nabla_x u_{n-1,m-1}\right.\\&\qquad\left.+2y(y+h)\nabla_x f\cdot\nabla_x b\partial_y u_{n-1,m-1}\right\},\end{aligned} \tag{11e}$$

$$\begin{aligned}h^2R_{n,m}^{(h)}=&\left\{-M_0\nabla_x b\cdot\nabla_x u_{n,m-1}-b\nabla_x M_0\cdot\nabla_x u_{n,m-1}\right.\\&\qquad\left.-\nabla_x b\cdot N_0\partial_y u_{n,m-1}+y\nabla_x b\cdot\nabla_x M_0\partial_y u_{n,m-1}\right\}\\&+\left\{M_0\nabla_x f\cdot\nabla_x u_{n-1,m}+f\nabla_x M_0\cdot\nabla_x u_{n-1,m}+\nabla_x f\cdot N_0\partial_y u_{n-1,m}\right.\\&\qquad\left.-(y+h)\nabla_x f\cdot\nabla_x M_0\partial_y u_{n-1,m}\right\}\\&+\left\{b\nabla_x b\cdot\nabla_x u_{n,m-2}-y\left|\nabla_x b\right|^2\partial_y u_{n,m-2}\right\}\end{aligned}$$

$$+\left\{f\nabla_x f\cdot\nabla_x u_{n-2,m}-(y+h)\left|\nabla_x f\right|^2\partial_y u_{n-2,m}\right\}$$
$$+\left\{-b\nabla_x f\cdot\nabla_x u_{n-1,m-1}-f\nabla_x b\cdot\nabla_x u_{n-1,m-1}\right.$$
$$\left.+(2y+h)\nabla_x f\cdot\nabla_x b\partial_y u_{n-1,m-1}\right\}, \tag{11f}$$

and

$$\begin{aligned}hQ_{n,m}=&\left\{-b\nabla_x b_0\cdot\nabla_x u_{n,m-1}+M_0\nabla_x b\cdot\nabla_x u_{n,m-1}\right.\\&\left.-2h\nabla_x b_0\cdot\nabla_x b\partial_y u_{n,m-1}\right\}+\left\{f\nabla_x b_0\cdot\nabla_x u_{n,m-1}\right\}\\&+\left\{-b\nabla_x b\cdot\nabla_x u_{n,m-2}-h\left|\nabla_x b\right|^2\partial_y u_{n,m-2}\right\}\\&+\left\{f\nabla_x b\cdot\nabla_x u_{n-1,m-1}\right\}.\end{aligned} \tag{11g}$$

Furthermore, if we expand the DNO in a series

$$G(\eta,\zeta)[\xi]=G(f_0+\varepsilon f,b_0+\delta b)[\xi]=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}G_{n,m}(f,b)[\xi]\varepsilon^n\delta^m,$$

then the terms $G_{n,m}$ are given by

$$G_{n,m}=\mathcal{G}\left\{u_{n,m}(x,0)\right\}+H_{n,m}(x), \tag{12a}$$

where

$$\begin{aligned}M_0H_{n,m}=&\left\{bG_{n,m-1}+b\nabla_x f_0\cdot\nabla_x u_{n,m-1}(x,0)\right\}\\&+\left\{-fG_{n-1,m}-M_0\nabla_x f\cdot\nabla_x u_{n-1,m}(x,0)\right.\\&\left.-f\nabla_x f_0\cdot\nabla_x u_{n-1,m}(x,0)+2h\nabla_x f\cdot\nabla_x f_0\partial_y u_{n-1,m}(x,0)\right\}\\&+\left\{-f\nabla_x f\cdot\nabla_x u_{n-2,m}(x,0)+h\left|\nabla_x f\right|^2\partial_y u_{n-2,m}(x,0)\right\}\\&+\left\{b\nabla_x f\cdot\nabla_x u_{n-1,m-1}(x,0)\right\}.\end{aligned} \tag{12b}$$

3. Finite smoothness boundaries: parametric analyticity

To begin, we establish the joint *parametric* analyticity of the field, u, and the DNO, G, i.e., analytic dependence with respect to ε and δ. For this we can make a double inductive estimation of the recursions (11) and (12). To accomplish this we recall two tools of classical analysis: An "algebra property" for Sobolev spaces [16, 18], Lemma 1, and an elliptic estimate, Theorem 3, for divergence-form elliptic partial differential equations [11, 8].

Lemma 1. *Given an integer $s\geq 0$ and any $\sigma>0$, there exists a constant $\mathcal{M}=\mathcal{M}(d,s)$ such that if $f\in C^s(P(\Gamma))$, $w\in H^s(S_{h,0,0})$ then*

$$\left\|fw\right\|_{H^s}\leq\mathcal{M}(d,s)\left|f\right|_{C^s}\left\|w\right\|_{H^s},$$

and if $\tilde{f} \in C^{s+1/2+\sigma}(P(\Gamma))$, $\tilde{w} \in H^{s+1/2}(P(\Gamma))$ *then*

$$\left\| \tilde{f}\tilde{w} \right\|_{H^{s+1/2}} \leq \mathcal{M}(d,s) \left| \tilde{f} \right|_{C^{s+1/2+\sigma}} \left\| \tilde{w} \right\|_{H^{s+1/2}} .$$

Theorem 3. *Given any integer* $s \geq 0$, *if* $b_0, f_0 \in C^{s+2}(P(\Gamma))$; $R \in H^s(S_{h,0,0})$; $\xi \in H^{s+3/2}(P(\Gamma))$; *and* $Q \in H^{s+1/2}(P(\Gamma))$, *then the unique solution* $w \in H^{s+2}(S_{h,0,0})$ *of*

$$\mathcal{L}\{w\} = R \qquad -h < y < 0 \tag{13a}$$

$$w(x,0) = \xi(x) \tag{13b}$$

$$\mathcal{B}\{w\}(x,-h) = Q(x), \tag{13c}$$

satisfies

$$\|w\|_{H^{s+2}} \leq C_e \left[\|R\|_{H^s} + \|\xi\|_{H^{s+3/2}} + \|Q\|_{H^{s+1/2}} \right], \tag{14}$$

for some constant $C_e = C_e(|b_0|_{C^{s+2}}, |f_0|_{C^{s+2}}, s, d)$.

Our goal in this section is to show the following joint *parametric* analyticity result.

Theorem 4. *Given any integer* $s \geq 0$, *if* $f, b, f_0, b_0 \in C^{s+2}(P(\Gamma))$ *and* $\xi \in H^{s+3/2}(P(\Gamma))$ *then* $u_{n,m} \in H^{s+2}(S_{h,0,0})$ *and*

$$\|u_{n,m}\|_{H^{s+2}} \leq K_1 B^n E^m$$

for constants $K_1, B, E > 0$.

Once we have this we can quickly obtain the analogous result for the DNO.

Theorem 5. *Given any integer* $s \geq 0$, *if* $f, b, f_0, b_0 \in C^{s+2}(P(\Gamma))$ *and* $\xi \in H^{s+3/2}(P(\Gamma))$ *then* $G_{n,m}[\xi] \in H^{s+1/2}(P(\Gamma))$ *and*

$$\|G_{n,m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^n E^m$$

for constants $\tilde{K}_1, B, E > 0$.

We establish these results via a double induction and, as such, we require individual analyticity theorems for u and G as functions of ε and δ separately. This analyticity in ε (for a "flat bottomed ocean," i.e. $\delta = 0$) was established in [18] and we simply restate the results here for completeness.

Theorem 6. *Given any integer* $s \geq 0$, *if* $f, f_0, b_0 \in C^{s+2}(P(\Gamma))$ *and* $\xi \in H^{s+3/2}(P(\Gamma))$ *then* $u_{n,0} \in H^{s+2}(S_{h,0,0})$ *and*

$$\|u_{n,0}\|_{H^{s+2}} \leq K_1 B^n$$

for constants $K_1, B > 0$.

Theorem 7. *Given any integer $s \geq 0$, if $f, f_0, b_0 \in C^{s+2}(P(\Gamma))$ and $\xi \in H^{s+3/2}(P(\Gamma))$ then $G_{n,0}[\xi] \in H^{s+1/2}(P(\Gamma))$ and*

$$\|G_{n,0}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^n$$

for constants $\tilde{K}_1, B > 0$.

However, in the case of a "flat surface," i.e. $\varepsilon = 0$, these are new results and we present them here with their complete proofs.

Theorem 8. *Given any integer $s \geq 0$, if $b, b_0, f_0 \in C^{s+2}(P(\Gamma))$ and $\xi \in H^{s+3/2}(P(\Gamma))$ then $u_{0,m} \in H^{s+2}(S_{h,0,0})$ and*

$$\|u_{0,m}\|_{H^{s+2}} \leq K_1 E^m$$

for constants $K_1, E > 0$.

Theorem 9. *Given any integer $s \geq 0$, if $b, b_0, f_0 \in C^{s+2}(P(\Gamma))$ and $\xi \in H^{s+3/2}(P(\Gamma))$ then $G_{0,m}[\xi] \in H^{s+1/2}(P(\Gamma))$ and*

$$\|G_{0,m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 E^m$$

for constants $\tilde{K}_1, E > 0$.

For this proof we need the following inductive lemma.

Lemma 2. *Given any integer $s \geq 0$, if $b, b_0, f_0 \in C^{s+2}(P(\Gamma))$ and*

$$\|u_{0,m}\|_{H^s} \leq K_1 E^m, \qquad \forall m < \bar{m},$$

for constants $K_1, E > 0$, then there exists a constant $\bar{C}_1 > 0$ such that

$$\|R_{0,\bar{m}}\|_{H^s} \leq K_1 \bar{C}_1 \left[|b|_{C^{s+2}} E^{\bar{m}-1} + |b|_{C^{s+2}}^2 E^{\bar{m}-2} \right]$$
$$\|Q_{0,\bar{m}}\|_{H^{s+1/2}} \leq K_1 \bar{C}_1 \left[|b|_{C^{s+2}} E^{\bar{m}-1} + |b|_{C^{s+2}}^2 E^{\bar{m}-2} \right].$$

Proof. For brevity we consider only a portion of $R_{0,\bar{m}}$, $\operatorname{div}_x \left[R_{0,\bar{m}}^{(x)} \right]$:

$$\begin{aligned}
\left\| R_{0,\bar{m}}^{(x)} \right\|_{H^{s+1}} \leq\ & 2h\mathcal{M} \, |b|_{C^{s+1}} \, \|\nabla_x u_{0,\bar{m}-1}\|_{H^{s+1}} \\
& + 2\mathcal{M} \, |b|_{C^{s+1}} \left\| \tilde{M}_0 \nabla_x u_{0,\bar{m}-1} \right\|_{H^{s+1}} \\
& + \mathcal{M} \, |b|_{C^{s+1}} \, \|N_0 \partial_y u_{0,\bar{m}-1}\|_{H^{s+1}} \\
& + Y\mathcal{M} \, |b|_{C^{s+2}} \, \|M_0 \partial_y u_{0,\bar{m}-1}\|_{H^{s+1}} \\
& + \mathcal{M}^2 \, |b|_{C^{s+1}}^2 \, \|\nabla_x u_{0,\bar{m}-2}\|_{H^{s+1}} \\
& + Y\mathcal{M}^2 \, |b|_{C^{s+1}} \, |b|_{C^{s+2}} \, \|\partial_y u_{0,\bar{m}-2}\|_{H^{s+1}}
\end{aligned}$$

$$\leq \left[2h\mathcal{M} + 2\mathcal{M}^2 \left|\tilde{M}_0\right|_{C^{s+1}} + \mathcal{M}^2 \left|N_0\right|_{C^{s+1}} + Y\mathcal{M}^2 \left|M_0\right|_{C^{s+1}}\right]$$
$$\times \left|b\right|_{C^{s+2}} K_1 E^{\bar{m}-1}$$
$$+ (1+Y)\mathcal{M}^2 \left|b\right|^2_{C^{s+2}} K_1 E^{\bar{m}-2},$$

where $Y = Y(d,s,h)$ is the largest constant such that both

$$\|yw\|_{H^s} \leq Y \|w\|_{H^s}, \quad \|(y+h)w\|_{H^s} \leq Y \|w\|_{H^s}.$$

We are done if

$$\bar{C}_1 > \max\left\{2h\mathcal{M} + 2\mathcal{M}^2 \left|\tilde{M}_0\right|_{C^{s+1}} + \mathcal{M}^2 \left|N_0\right|_{C^{s+1}} + Y\mathcal{M}^2 \left|M_0\right|_{C^{s+1}}, (1+Y)\mathcal{M}^2\right\},$$

where, for instance, we can bound

$$\left|N_0\right|_{C^{s+1}} \leq Y(\left|f_0\right|_{C^{s+2}} + \left|b_0\right|_{C^{s+2}}), \quad \left|\tilde{M}_0\right|_{C^{s+1}} \leq \left|f_0\right|_{C^{s+2}} + \left|b_0\right|_{C^{s+2}},$$
$$\left|M_0\right|_{C^{s+1}} \leq \left|f_0\right|_{C^{s+2}} + \left|b_0\right|_{C^{s+2}}.$$

We are now in a position to establish Theorem 8.

Proof of Theorem 8. We work by induction in m; at order $m = 0$ we use Theorem 3 to see that

$$\|u_{0,0}\|_{H^{s+2}} \leq C_e \|\xi\|_{H^{s+3/2}},$$

and we set $K_1 := C_e \|\xi\|_{H^{s+3/2}}$. Now we suppose that (8) holds for all $m < \bar{m}$ and examine $u_{0,\bar{m}}$. From Theorem 3 we have

$$\|u_{0,\bar{m}}\|_{H^{s+2}} \leq C_e \left[\|R_{0,\bar{m}}\|_{H^s} + \|Q_{0,\bar{m}}\|_{H^{s+1/2}}\right],$$

and from Lemma 2 we deduce that

$$\|u_{0,\bar{m}}\|_{H^{s+2}} \leq C_e 2\bar{C}_1 K_1 \left[\left|b\right|_{C^{s+2}} E^{\bar{m}-1} + \left|b\right|^2_{C^{s+2}} E^{\bar{m}-2}\right].$$

We are done provided that

$$E > \max\left\{4C_e\bar{C}_1 \left|b\right|_{C^{s+2}}, 2\sqrt{C_e\bar{C}_1} \left|b\right|_{C^{s+2}}\right\}.$$

Given this result we can show the analyticity of the DNO with respect to δ (provided $\varepsilon = 0$).

Proof of Theorem 9. Again, we work by induction, and at $m = 0$ we recall that

$$G_{0,0}[\xi] = \mathcal{G}\left\{u_{0,0}(x,0)\right\},$$

so that

$$\begin{aligned}
\|G_{0,0}\|_{H^{s+1/2}} &= \|\mathcal{G}\{u_{0,0}\}\|_{H^{s+1/2}} \\
&\leq \left\| \frac{1}{M_0}\Big(h\partial_y u_{0,0} - M_0 \nabla_x f_0 \cdot \nabla_x u_{0,0} + h\left|\nabla_x f_0\right|^2 \partial_y u_{0,0}\Big)\right\|_{H^{s+1/2}} \\
&\leq \mathcal{M}\left|\frac{1}{M_0}\right|_{C^{s+1/2+\sigma}} \Big(h\,\|\partial_y u_{0,0}\|_{H^{s+1/2}} \\
&\quad + \mathcal{M}^2 \left|M_0\right|_{C^{s+1/2+\sigma}} \left|\nabla_x f_0\right|_{C^{s+1/2+\sigma}} \|\nabla_x u_{0,0}\|_{H^{s+1/2}} \\
&\quad + h\mathcal{M}^2 \left|\nabla_x f_0\right|^2_{C^{s+1/2+\sigma}} \|\partial_y u_{0,0}\|_{H^{s+1/2}}\Big) \\
&\leq \mathcal{M}\left|\frac{1}{M_0}\right|_{C^{s+2}} \Big(h + \mathcal{M}^2 \left|M_0\right|_{C^{s+2}} \left|f_0\right|_{C^{s+2}} + h\mathcal{M}^2 \left|f_0\right|^2_{C^{s+2}}\Big) \\
&\quad \times \|u_{0,0}\|_{H^{s+3/2}}.
\end{aligned}$$

We choose $\tilde{K}_1$ by

$$\frac{\tilde{K}_1}{2K_1} := \mathcal{M}\left|\frac{1}{M_0}\right|_{C^{s+2}} \Big(h + \mathcal{M}^2 \left|M_0\right|_{C^{s+2}} \left|f_0\right|_{C^{s+2}} + h\mathcal{M}^2 \left|f_0\right|^2_{C^{s+2}}\Big),$$

(which is finite by the smoothness of b_0, f_0) and observe that

$$\|\mathcal{G}\{w\}\|_{H^{s+1/2}} \leq \frac{\tilde{K}_1}{2K_1} \|w\|_{H^{s+3/2}}$$

for any $w \in H^{s+3/2}$. In particular, since $\|u_{0,0}\|_{H^{s+3/2}} \leq K_1$ then

$$\|G_{0,0}[\xi]\|_{H^{s+1/2}} \leq \frac{\tilde{K}_1}{2} < \tilde{K}_1.$$

We now assume that

$$\|G_{0,m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 E^m, \quad \forall m < \bar{m},$$

and estimate

$$\begin{aligned}
\|G_{0,\bar{m}}\|_{H^{s+1/2}} &\leq \|\mathcal{G}\{u_{0,\bar{m}}\}\|_{H^{s+1/2}} + \|H_{0,\bar{m}}\|_{H^{s+1/2}} \\
&\leq \frac{\tilde{K}_1}{2K_1} \|u_{0,\bar{m}}\|_{H^{s+3/2}} + \mathcal{M}\left|\frac{1}{M_0}\right|_{C^{s+1/2+\sigma}} \big(\mathcal{M}\left|b\right|_{C^{s+1/2+\sigma}} \|G_{0,\bar{m}-1}\|_{H^{s+1/2}} \\
&\quad + \mathcal{M}^2 \left|b\right|_{C^{s+1/2+\sigma}} \left|\nabla_x f_0\right|_{C^{s+1/2+\sigma}} \|\nabla_x u_{0,\bar{m}-1}\|_{H^{s+1/2}}\big) \\
&\leq \frac{\tilde{K}_1}{2K_1} K_1 E^{\bar{m}} + \mathcal{M}\left|\frac{1}{M_0}\right|_{C^{s+2}} \Big(\mathcal{M}\left|b\right|_{C^{s+2}} \tilde{K}_1 E^{\bar{m}-1} \\
&\quad + \mathcal{M}^2 \left|b\right|_{C^{s+2}} \left|f_0\right|_{C^{s+2}} K_1 E^{\bar{m}-1}\Big) \\
&\leq \tilde{K}_1 E^{\bar{m}},
\end{aligned}$$

provided that

$$E > 4\mathcal{M} \left| \frac{1}{M_0} \right|_{C^{s+2}} \max \left\{ \mathcal{M} \left| b \right|_{C^{s+2}}, \mathcal{M}^2 \left| b \right|_{C^{s+2}} \left| f_0 \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} \right\}.$$

To establish the full joint parametric analyticity of u (Theorem 4) we will need another inductive lemma.

Lemma 3. *Given any integer $s \geq 0$, if $f, b, f_0, b_0 \in C^{s+2}(P(\Gamma))$ and*

$$\|u_{n,m}\|_{H^s} \leq K_1 B^n E^m, \qquad \forall n < \bar{n}, \quad \forall m$$
$$\|u_{\bar{n},m}\|_{H^s} \leq K_1 B^{\bar{n}} E^m, \qquad \forall m < \bar{m},$$

for constants $K_1, B, E > 0$, then there exists a constant $\bar{C}_2 > 0$ such that

$$\begin{aligned}\|R_{\bar{n},\bar{m}}\|_{H^s} \leq K_1 \bar{C}_2 \Big[&|b|_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-1} + |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}} + |b|^2_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-2} \\ &+ |f|^2_{C^{s+2}} B^{\bar{n}-2} E^{\bar{m}} + |b|_{C^{s+2}} |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}-1} \Big] \\ \|Q_{\bar{n},\bar{m}}\|_{H^s} \leq K_1 \bar{C}_2 \Big[&|b|_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-1} + |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}} + |b|^2_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-2} \\ &+ |f|^2_{C^{s+2}} B^{\bar{n}-2} E^{\bar{m}} + |b|_{C^{s+2}} |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}-1} \Big].\end{aligned}$$

Proof. For brevity we consider only $R^{(x)}_{\bar{n},\bar{m}}$:

$$\begin{aligned}\left\| R^{(x)}_{\bar{n},\bar{m}} \right\|_{H^{s+1}} \leq\ & 2h\mathcal{M} |b|_{C^{s+1}} \|\nabla_x u_{\bar{n},\bar{m}-1}\|_{H^{s+1}} \\ &+ 2\mathcal{M} |b|_{C^{s+1}} \left\| \tilde{M}_0 \nabla_x u_{\bar{n},\bar{m}-1} \right\|_{H^{s+1}} \\ &+ \mathcal{M} |b|_{C^{s+1}} \|N_0 \partial_y u_{\bar{n},\bar{m}-1}\|_{H^{s+1}} \\ &+ Y\mathcal{M} |b|_{C^{s+2}} \|M_0 \partial_y u_{\bar{n},\bar{m}-1}\|_{H^{s+1}} \\ &+ 2h\mathcal{M} |f|_{C^{s+1}} \|\nabla_x u_{\bar{n}-1,\bar{m}}\|_{H^{s+1}} \\ &+ 2\mathcal{M} |f|_{C^{s+1}} \left\| \tilde{M}_0 \nabla_x u_{\bar{n}-1,\bar{m}} \right\|_{H^{s+1}} \\ &+ \mathcal{M} |f|_{C^{s+1}} \|N_0 \partial_y u_{\bar{n}-1,\bar{m}}\|_{H^{s+1}} \\ &+ Y\mathcal{M} |f|_{C^{s+2}} \|M_0 \partial_y u_{\bar{n}-1,\bar{m}}\|_{H^{s+1}} \\ &+ \mathcal{M}^2 |b|^2_{C^{s+2}} \|\nabla_x u_{\bar{n},\bar{m}-2}\|_{H^{s+1}} \\ &+ Y\mathcal{M}^2 |b|_{C^{s+1}} |b|_{C^{s+2}} \|\partial_y u_{\bar{n},\bar{m}-2}\|_{H^{s+1}} \\ &+ \mathcal{M}^2 |f|^2_{C^{s+1}} \|\nabla_x u_{\bar{n}-2,\bar{m}}\|_{H^{s+1}} \\ &+ Y\mathcal{M}^2 |f|_{C^{s+1}} |f|_{C^{s+2}} \|\partial_y u_{\bar{n}-2,\bar{m}}\|_{H^{s+1}}\end{aligned}$$

$$+ 2\mathcal{M}^2 |f|_{C^{s+1}} |b|_{C^{s+1}} \|\nabla_x u_{\bar{n}-1,\bar{m}-1}\|_{H^{s+1}}$$
$$+ Y\mathcal{M}^2 |f|_{C^{s+1}} |b|_{C^{s+2}} \|\partial_y u_{\bar{n}-1,\bar{m}-1}\|_{H^{s+1}}$$
$$+ Y\mathcal{M}^2 |f|_{C^{s+2}} |b|_{C^{s+1}} \|\partial_y u_{\bar{n}-1,\bar{m}-1}\|_{H^{s+1}} .$$

Using the inductive hypothesis,

$$\begin{aligned}\left\|R^{(x)}_{\bar{n},\bar{m}}\right\|_{H^{s+1}} \leq & \Big\{2h\mathcal{M} |b|_{C^{s+1}} + \mathcal{M}^2 |b|_{C^{s+1}} \left(2\left|\tilde{M}_0\right|_{C^{s+1}} + |N_0|_{C^{s+1}}\right) \\ & \qquad + Y\mathcal{M}^2 |b|_{C^{s+2}} |M_0|_{C^{s+1}}\Big\} K_1 B^{\bar{n}} E^{\bar{m}-1} \\ & + \Big\{2h\mathcal{M} |f|_{C^{s+1}} + \mathcal{M}^2 |f|_{C^{s+1}} \left(2\left|\tilde{M}_0\right|_{C^{s+1}} + |N_0|_{C^{s+1}}\right) \\ & \qquad + Y\mathcal{M}^2 |f|_{C^{s+2}} |M_0|_{C^{s+1}}\Big\} K_1 B^{\bar{n}-1} E^{\bar{m}} \\ & + \Big\{\mathcal{M}^2 |b|^2_{C^{s+2}} + Y\mathcal{M}^2 |b|_{C^{s+1}} |b|_{C^{s+2}}\Big\} K_1 B^{\bar{n}} E^{\bar{m}-2} \\ & + \Big\{\mathcal{M}^2 |f|^2_{C^{s+1}} + Y\mathcal{M}^2 |f|_{C^{s+1}} |f|_{C^{s+2}}\Big\} K_1 B^{\bar{n}-2} E^{\bar{m}} \\ & + \Big\{2\mathcal{M}^2 |f|_{C^{s+1}} |b|_{C^{s+1}} + Y\mathcal{M}^2 |f|_{C^{s+1}} |b|_{C^{s+2}} \\ & \qquad + Y\mathcal{M}^2 |f|_{C^{s+2}} |b|_{C^{s+1}}\Big\} K_1 B^{\bar{n}-1} E^{\bar{m}-1}.\end{aligned}$$

We are done if

$$\begin{aligned}\bar{C}_2 > \max\Big\{ & 2h\mathcal{M} + \mathcal{M}^2 \left(2\left|\tilde{M}_0\right|_{C^{s+1}} + |N_0|_{C^{s+1}}\right) + Y\mathcal{M}^2 |M_0|_{C^{s+1}} , \\ & \mathcal{M}^2(Y+1), 2\mathcal{M}^2(Y+1)\Big\}.\end{aligned}$$

Finally, we can prove Theorem 4.

Proof of Theorem 4. We work using an induction in n. At order $n = 0$ we must prove

$$\|u_{0,m}\|_{H^{s+2}} \leq K_1 E^m, \quad \forall m,$$

but this is simply Theorem 8. We now assume

$$\|u_{n,m}\|_{H^{s+2}} \leq K_1 B^n E^m, \quad \forall m,$$

for all $n < \bar{n}$ and seek to prove

$$\|u_{\bar{n},m}\|_{H^{s+2}} \leq K_1 B^{\bar{n}} E^m, \quad \forall m.$$

For this we consider an induction in m. At order $m = 0$ we need

$$\|u_{\bar{n},0}\|_{H^{s+2}} \leq K_1 B^{\bar{n}},$$

but this follows from Theorem 6. Finally, we assume

$$\|u_{\bar{n},m}\|_{H^{s+2}} \leq K_1 B^{\bar{n}} E^m,$$

for all $m < \bar{m}$ and seek

$$\|u_{\bar{n},\bar{m}}\|_{H^{s+2}} \leq K_1 B^{\bar{n}} E^{\bar{m}}.$$

From Theorem 3 we estimate

$$\|u_{\bar{n},\bar{m}}\|_{H^{s+2}} \leq C_e \left[\|R_{\bar{n},\bar{m}}\|_{H^s} + \|Q_{\bar{n},\bar{m}}\|_{H^{s+1/2}} \right],$$

which we estimate, from Lemma 3, by

$$\begin{aligned} \|u_{\bar{n},\bar{m}}\|_{H^{s+2}} \leq C_e 2\bar{C}_2 K_1 \Big[& |b|_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-1} + |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}} \\ & + |b|^2_{C^{s+2}} B^{\bar{n}} E^{\bar{m}-2} + |f|^2_{C^{s+2}} B^{\bar{n}-2} E^{\bar{m}} + |b|_{C^{s+2}} |f|_{C^{s+2}} B^{\bar{n}-1} E^{\bar{m}-1} \Big]. \end{aligned}$$

We are done provided that

$$\begin{aligned} B &> \max \left\{ 10 C_e \bar{C}_2 |f|_{C^{s+2}}, \sqrt{10 C_e \bar{C}_2}\, |f|_{C^{s+2}} \right\} \\ E &> \max \left\{ 10 C_e \bar{C}_2 |b|_{C^{s+2}}, \sqrt{10 C_e \bar{C}_2}\, |b|_{C^{s+2}} \right\}. \end{aligned}$$

Now, the joint parametric analyticity of the DNO (Theorem 5) can be demonstrated.

Proof of Theorem 5. We work by induction in n; at $n = 0$ we seek

$$\|G_{0,m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 E^m, \quad \forall m$$

which is simply Theorem 9. Now we assume

$$\|G_{n,m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^n E^m, \quad \forall m,$$

for all $n < \bar{n}$, and require

$$\|G_{\bar{n},m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^{\bar{n}} E^m, \quad \forall m.$$

For this we work using induction on m: For $m = 0$ we have, from Theorem 7,

$$\|G_{\bar{n},0}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^{\bar{n}}.$$

Now we assume

$$\|G_{\bar{n},m}[\xi]\|_{H^{s+1/2}} \leq \tilde{K}_1 B^{\bar{n}} E^m, \quad \forall m < \bar{m}$$

and estimate

$$\begin{aligned} \|G_{\bar{n},\bar{m}}\|_{H^{s+1/2}} &\leq \|\mathcal{G}\{u_{\bar{n},\bar{m}}\}\|_{H^{s+1/2}} + \|H_{\bar{n},\bar{m}}\|_{H^{s+1/2}} \\ &\leq \frac{\tilde{K}_1}{2K_1} \|u_{\bar{n},\bar{m}}\|_{H^{s+3/2}} \end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{M} \left| \frac{1}{M_0} \right|_{C^{s+1/2+\sigma}} \Big(\mathcal{M} \left| b \right|_{C^{s+1/2+\sigma}} \left\| G_{\bar{n},\bar{m}-1} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M}^2 \left| b \right|_{C^{s+1/2+\sigma}} \left| f_0 \right|_{C^{s+3/2+\sigma}} \left\| \nabla_x u_{\bar{n},\bar{m}-1} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M} \left| f \right|_{C^{s+1/2+\sigma}} \left\| G_{\bar{n}-1,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M}^2 \left| M_0 \right|_{C^{s+1/2+\sigma}} \left| f \right|_{C^{s+3/2+\sigma}} \left\| \nabla_x u_{\bar{n}-1,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M}^2 \left| f \right|_{C^{s+1/2+\sigma}} \left| f_0 \right|_{C^{s+3/2+\sigma}} \left\| \nabla_x u_{\bar{n}-1,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ 2h\mathcal{M}^2 \left| f \right|_{C^{s+3/2+\sigma}} \left| f_0 \right|_{C^{s+3/2+\sigma}} \left\| \partial_y u_{\bar{n}-1,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M}^2 \left| f \right|_{C^{s+1/2+\sigma}} \left| f \right|_{C^{s+3/2+\sigma}} \left\| \nabla_x u_{\bar{n}-2,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ h\mathcal{M}^2 \left| f \right|^2_{C^{s+3/2+\sigma}} \left\| \partial_y u_{\bar{n}-2,\bar{m}} \right\|_{H^{s+1/2}} \\
&+ \mathcal{M}^2 \left| b \right|_{C^{s+1/2+\sigma}} \left| f \right|_{C^{s+3/2+\sigma}} \left\| \nabla_x u_{\bar{n}-1,\bar{m}-1} \right\|_{H^{s+1/2}} \Big) .
\end{aligned}
$$

Using the inductive hypotheses:

$$
\begin{aligned}
\left\| G_{\bar{n},\bar{m}} \right\|_{H^{s+1/2}} \leq \frac{\tilde{K}_1}{2} B^{\bar{n}} E^{\bar{m}} + \mathcal{M} \left| \frac{1}{M_0} \right|_{C^{s+2}} &\\
\times \Bigg(\left[\mathcal{M} \left| b \right|_{C^{s+2}} + \mathcal{M}^2 \left| b \right|_{C^{s+2}} \left| f_0 \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} \right] \tilde{K}_1 B^{\bar{n}} E^{\bar{m}-1} &\\
+ \Bigg[\mathcal{M} \left| f \right|_{C^{s+2}} + \mathcal{M}^2 \left| M_0 \right|_{C^{s+2}} \left| f \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} &\\
+ \mathcal{M}^2 \left| f \right|_{C^{s+2}} \left| f_0 \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} &\\
+ 2h\mathcal{M}^2 \left| f \right|_{C^{s+2}} \left| f_0 \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} \Bigg] \tilde{K}_1 B^{\bar{n}-1} E^{\bar{m}} &\\
+ \left[\mathcal{M}^2 \left| f \right|^2_{C^{s+2}} \frac{K_1}{\tilde{K}_1} + h\mathcal{M}^2 \left| f \right|^2_{C^{s+2}} \frac{K_1}{\tilde{K}_1} \right] \tilde{K}_1 B^{\bar{n}-2} E^{\bar{m}} &\\
+ \left[\mathcal{M}^2 \left| b \right|_{C^{s+2}} \left| f \right|_{C^{s+2}} \frac{K_1}{\tilde{K}_1} \right] \tilde{K}_1 B^{\bar{n}-1} E^{\bar{m}-1} \Bigg) . &
\end{aligned}
$$

The theorem is complete provided that

$$
\begin{aligned}
B > \mathcal{M} \max \Bigg\{ & 8 \left[\mathcal{M} + \mathcal{M}^2 \left| M_0 \right|_{C^{s+2}} (K_1/\tilde{K}_1) + \mathcal{M}^2 (1+2h) \left| f_0 \right|_{C^{s+2}} (K_1/\tilde{K}_1) \right], \\
& \mathcal{M} \sqrt{8(1+h)(K_1/\tilde{K}_1)}, \mathcal{M} \sqrt{8(K_1/\tilde{K}_1)} \Bigg\} \left| \frac{1}{M_0} \right|_{C^{s+2}} \left| f \right|_{C^{s+2}} \\
E > \mathcal{M} \max \Bigg\{ & 8 \left[\mathcal{M} + \mathcal{M}^2 \left| f_0 \right|_{C^{s+2}} (K_1/\tilde{K}_1) \right], \mathcal{M} \sqrt{8(K_1/\tilde{K}_1)} \Bigg\} \left| \frac{1}{M_0} \right|_{C^{s+2}} \left| b \right|_{C^{s+2}} .
\end{aligned}
$$

4. Analytic boundaries: joint analyticity

At this point we take up the proof of the analyticity of the field, u, and DNO, G, jointly in parameter and spatial variable on variations of arbitrary smooth domains (Theorems 1 & 2). Of course, in this setting we can no longer expect finite smoothness in the profiles f_0, b_0, f, and b to suffice; all of these must be real analytic. We characterize this analyticity (more precisely its domain of analyticity) in the following definition which is most convenient for our proof (see § B and the remark therein). It is possible that this estimate could be further optimized so that a weaker norm could be used, however, this would only affect our estimate of the *size* of the domain of analyticity which is not, in any case, specified with great precision by our method.

Definition 1. A function f is a member of the space $C_3^\omega(P(\Gamma))$ if it is real analytic and satisfies the estimate

$$\left| \frac{\partial_x^k}{k!} f \right|_{C^3} \leq C_f \frac{A^k}{(k+1)^2}, \qquad \forall k.$$

The notation C_3^ω is meant to indicate the space of real analytic functions, C^ω, with radius of analyticity (characterized by A) measured in the C^3 norm.

The key to the estimates of this section is the following generalization of Theorem 3 to the case of analytic coefficients, f_0 and b_0, and inhomogeneities ξ, Q, and R. This result depends on subtle commutator estimates (established in § B) and is proven in § A, however, once verified, it renders the proof of the joint analyticity results quite straightforward.

Theorem 10. *Suppose $f_0, b_0 \in C_3^\omega(P(\Gamma))$; $\xi, Q \in C^\omega(P(\Gamma))$; and $R \in C^\omega(S_{h,0,0})$ satisfying*

$$\left\| \frac{\partial_x^k}{k!} \xi \right\|_{H^{3/2}} \leq C_\xi \frac{A^k}{(k+1)^2}, \qquad \left\| \frac{\partial_x^k}{k!} Q \right\|_{H^{1/2}} \leq C_Q \frac{A^k}{(k+1)^2}, \quad \forall k$$

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} R \right\|_{H^0} \leq C_R \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l.$$

Then the unique solution $w \in C^\omega(S_{h,0,0})$ of

$$\mathcal{L}\{w\} = R \qquad -h < y < 0 \tag{15a}$$

$$w(x,0) = \xi(x) \tag{15b}$$

$$\mathcal{B}\{w\}(x,-h) = Q(x) \tag{15c}$$

satisfies

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} w \right\|_{H^2} \leq \bar{C}_e \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \qquad \forall k, l, \tag{16}$$

where $\bar{C}_e = \alpha(C_R + C_\xi + C_Q)$ *and* $\alpha = \alpha(d, h)$.

Again, we establish our results via induction and so we require individual analyticity theorems for u and G as functions of ε and δ separately. Analyticity in ε (for $\delta = 0$) is simply stated here for completeness.

Theorem 11. *If* $f, f_0, b_0 \in C_3^\omega(P(\Gamma))$ *and* $\xi \in C^\omega(P(\Gamma))$ *then* $u_{n,0} \in C^\omega(S_{h,0,0})$ *and*

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{n,0} \right\|_{H^2} \le K_0 B^n \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2},$$

for constants $K_0, B, A, D > 0$.

Theorem 12. *If* $f, f_0, b_0 \in C_3^\omega(P(\Gamma))$ *and* $\xi \in C^\omega(P(\Gamma))$ *then* $G_{n,0}[\xi] \in C^\omega(P(\Gamma))$ *and*

$$\left\| \frac{\partial_x^k}{k!} G_{n,0}[\xi] \right\|_{H^{1/2}} \le \tilde{K}_0 B^n \frac{A^k}{(k+1)^2},$$

for constants $\tilde{K}_0, B, A > 0$.

We present the complete joint analyticity proof in the case $\varepsilon = 0$ for the field (Theorem 13); the analyticity of the DNO is straightforward given this estimate and follows quite closely the method of § 3.

Theorem 13. *If* $b, f_0, b_0 \in C_3^\omega(P(\Gamma))$ *and* $\xi \in C^\omega(P(\Gamma))$ *then* $u_{0,m} \in C^\omega(S_{h,0,0})$ *and*

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{0,m} \right\|_{H^2} \le K_0 E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2},$$

for constants $K_0, E, A, D > 0$.

Theorem 14. *If* $b, f_0, b_0 \in C_3^\omega(P(\Gamma))$ *and* $\xi \in C^\omega(P(\Gamma))$ *then* $G_{0,m}[\xi] \in C^\omega(P(\Gamma))$ *and*

$$\left\| \frac{\partial_x^k}{k!} G_{0,m}[\xi] \right\|_{H^{1/2}} \le \tilde{K}_0 E^m \frac{A^k}{(k+1)^2},$$

for constants $\tilde{K}_0, E, A > 0$.

Again, we require an inductive lemma.

Lemma 4. *If* $b, b_0, f_0 \in C_3^\omega(P(\Gamma))$ *and*

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{0,m} \right\|_{H^2} \le K_0 E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \qquad \forall m < \bar{m}, \qquad \forall k, l$$

for constants $K_0, E, A, D > 0$, then there exists a constant $\bar{C}_3 > 0$ such that

$$\|R_{0,\bar{m}}\|_{H^0} \le K_0 \bar{C}_3 \left[C_b E^{\bar{m}-1} + C_b^2 E^{\bar{m}-2} \right] \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l$$

$$\|Q_{0,\bar{m}}\|_{H^{1/2}} \le K_0 \bar{C}_3 \left[C_b E^{\bar{m}-1} + C_b^2 E^{\bar{m}-2} \right] \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l.$$

Proof. For brevity we consider only a portion of $R^{(x)}_{0,\bar{m}}$, which is representative of all terms:

$$Z := b(x) N_0(x,y) \partial_y u_{0,\bar{m}-1}(x,y).$$

We begin with

$$\begin{aligned}
\frac{\partial_x^k \partial_y^l}{(k+l)!} Z &= \frac{k!l!}{(k+l)!} \frac{\partial_x^k}{k!} \frac{\partial_y^l}{l!} \left(b N_0 \partial_y u_{0,\bar{m}-1} \right) \\
&= \frac{k!l!}{(k+l)!} \sum_{p=0}^{k} \sum_{r=0}^{p} \left(\frac{\partial_x^{k-p}}{(k-p)!} b \right) \left(\frac{\partial_x^{p-r}}{(p-r)!} N_0 \right) \left(\frac{\partial_x^r}{r!} \frac{\partial_y^l}{l!} \partial_y u_{0,\bar{m}-1} \right) \\
&\quad + \left(\frac{\partial_x^{k-p}}{(k-p)!} b \right) \left(\frac{\partial_x^{p-r}}{(p-r)!} \partial_y N_0 \right) \left(\frac{\partial_x^r}{r!} \frac{\partial_y^{l-1}}{(l-1)!} \partial_y u_{0,\bar{m}-1} \right) \\
&= \sum_{p=0}^{k} \sum_{r=0}^{p} \left(\frac{\partial_x^{k-p}}{(k-p)!} b \right) \left(\frac{\partial_x^{p-r}}{(p-r)!} N_0 \right) \left(\frac{\partial_x^r \partial_y^l}{(r+l)!} \partial_y u_{0,\bar{m}-1} \right) \Lambda_{k,l,r} \\
&\quad + \left(\frac{\partial_x^{k-p}}{(k-p)!} b \right) \left(\frac{\partial_x^{p-r}}{(p-r)!} \partial_y N_0 \right) \left(\frac{\partial_x^r \partial_y^{l-1}}{(r+l-1)!} \partial_y u_{0,\bar{m}-1} \right) \tilde{\Lambda}_{k,l,r},
\end{aligned}$$

where

$$\Lambda_{k,l,r} := \frac{k!l!(r+l)!}{(k+l)!r!l!} \le 1, \quad \tilde{\Lambda}_{k,l,r} := \frac{k!l!(r+l-1)!}{(k+l)!r!(l-1)!} \le 1,$$

since $r \le k$. Of course we must estimate $\operatorname{div}_x \left[R^{(x)}_{0,\bar{m}} \right]$ in H^0, and, in light of the calculation above,

$$\begin{aligned}
\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} \operatorname{div}_x [Z] \right\|_{H^0} &\le \left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} Z \right\|_{H^1} \\
&\le \sum_{p=0}^{k} \sum_{r=0}^{p} \mathcal{M} \left| \frac{\partial_x^{k-p}}{(k-p)!} b \right|_{C^1} \\
&\quad \times \left[\left\| \left(\frac{\partial_x^{p-r}}{(p-r)!} N_0 \right) \left(\frac{\partial_x^r \partial_y^l}{(r+l)!} \partial_y u_{0,\bar{m}-1} \right) \right\|_{H^1} \right.
\end{aligned}$$

$$+ \left\| \left(\frac{\partial_x^{p-r}}{(p-r)!} (\partial_y N_0) \right) \left(\frac{\partial_x^r \partial_y^{l-1}}{(r+l-1)!} \partial_y u_{0,\bar{m}-1} \right) \right\|_{H^1} \Bigg] .$$

Recalling that

$$N_0(x,y) = -(y+h)\nabla_x f_0 + y \nabla_x b_0,$$

it is not difficult to see that, for any $F \in H^1$,

$$\left\| \left(\frac{\partial_x^{p-r}}{(p-r)!} N_0 \right) F \right\|_{H^1} \leq Y(C_{f_0} + C_{b_0}) \frac{A^{p-r}}{(p-r+1)^2} \|F\|_{H^1}$$
$$\left\| \left(\frac{\partial_x^{p-r}}{(p-r)!} \partial_y N_0 \right) F \right\|_{H^1} \leq (C_{f_0} + C_{b_0}) \frac{A^{p-r}}{(p-r+1)^2} \|F\|_{H^1} ,$$

so that

$$\begin{aligned} \left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} Z \right\|_{H^1} &\leq \sum_{p=0}^{k} \sum_{r=0}^{p} \mathcal{M} C_b \frac{A^{k-p}}{(k-p+1)^2} (C_{f_0} + C_{b_0}) \frac{A^{p-r}}{(p-r+1)^2} \\ &\quad \times \left[Y \left\| \frac{\partial_x^r \partial_y^l}{(r+l)!} u_{0,\bar{m}-1} \right\|_{H^2} + \left\| \frac{\partial_x^r \partial_y^{l-1}}{(r+l-1)!} u_{0,\bar{m}-1} \right\|_{H^2} \right] \\ &\leq \sum_{p=0}^{k} \sum_{r=0}^{p} \mathcal{M} C_b \frac{A^{k-p}}{(k-p+1)^2} (C_{f_0} + C_{b_0}) \frac{A^{p-r}}{(p-r+1)^2} \\ &\quad \times \left[Y \frac{A^r}{(r+1)^2} \frac{D^l}{(l+1)^2} + \frac{A^r}{(r+1)^2} \frac{D^{l-1}}{(l-1+1)^2} \right] K_0 E^{\bar{m}-1} \\ &\leq \frac{A^k}{(k+1)^2} K_0 \frac{D^l}{(l+1)^2} E^{\bar{m}-1} \mathcal{M} C_b (C_{f_0} + C_{b_0})(Y+4) \\ &\quad \times \sum_{p=0}^{k} \sum_{r=0}^{p} \frac{(k+1)^2}{(r+1)^2 (p-r+1)^2 (k-p+1)^2}, \end{aligned}$$

since

$$\frac{1}{(l-1+1)^2} = \frac{(l+1)^2}{(l-1+1)^2} \frac{1}{(l+1)^2} = \frac{(1+1/l)^2}{1} \frac{1}{(l+1)^2} \leq 4 \frac{1}{(l+1)^2}.$$

The final double-sum can be bounded by a constant S^2 (cf. the proof of Lemma 11 in [18]) in the following way:

$$\begin{aligned} &\sum_{p=0}^{k} \sum_{r=0}^{p} \frac{(k+1)^2}{(r+1)^2 (p-r+1)^2 (k-p+1)^2} \\ &\qquad \leq \sum_{p=0}^{k} \frac{(k+1)^2}{(p+1)^2 (k-p+1)^2} \left[\sum_{r=0}^{p} \frac{(p+1)^2}{(r+1)^2 (p-r+1)^2} \right] \end{aligned}$$

$$\leq \sum_{p=0}^{k} \frac{(k+1)^2}{(p+1)^2(k-p+1)^2} \; [S] \leq S^2.$$

So

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} Z \right\|_{H^1} \leq K_0 \mathcal{M}(C_{f_0} + C_{b_0})(Y+4) C_b E^{\bar{m}-1} \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2},$$

and we are done provided that $E > C_b$ and $\bar{C}_3 > \mathcal{M}(C_{f_0} + C_{b_0})(Y+4)$.

We are now in a position to establish Theorem 13.

Proof of Theorem 13. We work by induction in m; at order $m = 0$, since $\xi \in C^\omega$, we use Theorem 10 to see that

$$\|u_{0,0}\|_{H^2} \leq \alpha C_\xi \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \qquad \forall k, l,$$

and we set $K_0 := \alpha C_\xi$. Now we suppose that

$$\|u_{0,m}\|_{H^2} \leq K_0 E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \qquad \forall k, l,$$

for all $m < \bar{m}$, and examine $u_{0,\bar{m}}$. By Lemma 4 we have that the hypotheses of Theorem 10 hold with

$$C_R = C_Q = K_0 \bar{C}_3 \left[C_b E^{\bar{m}-1} + C_b^2 E^{\bar{m}-2} \right].$$

Now,

$$\begin{aligned} \|u_{0,\bar{m}}\|_{H^2} &\leq 2\bar{C}_e \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \\ &\leq 2\alpha K_0 \bar{C}_3 \left[C_b E^{\bar{m}-1} + C_b^2 E^{\bar{m}-2} \right] \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \end{aligned}$$

and we are done provided that

$$E > \max\left\{ 4\alpha \bar{C}_3, 2\sqrt{\alpha \bar{C}_3} \right\} C_b.$$

To establish the full joint analyticity we will need a final inductive lemma.

Lemma 5. *If $f, b, f_0, b_0 \in C_3^\omega(P(\Gamma))$ and*

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{n,m} \right\|_{H^2} \leq K_0 B^n E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \qquad \forall n < \bar{n}, \quad \forall m, \quad \forall k, l$$

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{\bar{n},m} \right\|_{H^2} \leq K_0 B^{\bar{n}} E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \qquad \forall m < \bar{m}, \quad \forall k, l$$

for constants $K_0, B, E, A, D > 0$, then there exists a constant $\bar{C}_0 > 0$ such that

$$\begin{aligned} \|R_{\bar{n},\bar{m}}\|_{H^0} &\le K_0\bar{C}_0\left[C_bB^{\bar{n}}E^{\bar{m}-1} + C_fB^{\bar{n}-1}E^{\bar{m}}\right. \\ &\quad \left. + C_b^2B^{\bar{n}}E^{\bar{m}-2} + C_f^2B^{\bar{n}-2}E^{\bar{m}} + C_bC_fB^{\bar{n}-1}E^{\bar{m}-1}\right] \\ &\quad \times \frac{A^k}{(k+1)^2}\frac{D^l}{(l+1)^2} \\ \|Q_{\bar{n},\bar{m}}\|_{H^{1/2}} &\le K_0\bar{C}_0\left[C_bB^{\bar{n}}E^{\bar{m}-1} + C_fB^{\bar{n}-1}E^{\bar{m}}\right. \\ &\quad \left. + C_b^2B^{\bar{n}}E^{\bar{m}-2} + C_f^2B^{\bar{n}-2}E^{\bar{m}} + C_bC_fB^{\bar{n}-1}E^{\bar{m}-1}\right] \\ &\quad \times \frac{A^k}{(k+1)^2}\frac{D^l}{(l+1)^2}. \end{aligned}$$

Proof. For brevity we again consider only a portion of $R^{(x)}_{\bar{n},\bar{m}}$:

$$\tilde{Z} := -yf(x)\nabla_x b(x)\partial_y u_{\bar{n}-1,\bar{m}-1}(x,y).$$

Using the same techniques as in the proof of Lemma 4, we estimate $\tilde{Z}$ in H^1:

$$\begin{aligned} \left\|\frac{\partial_x^k\partial_y^l}{(k+l)!}\tilde{Z}\right\|_{H^1} &\le \left\|\frac{\partial_x^k\partial_y^l}{(k+l)!}\left[yf\nabla_x b\partial_y u_{\bar{n}-1,\bar{m}-1}\right]\right\|_{H^1} \\ &\le \sum_{p=0}^{k}\sum_{r=0}^{p}\left\|y\left(\frac{\partial_x^{k-p}}{(k-p)!}f\right)\left(\frac{\partial_x^{p-r}}{(p-r)!}\nabla_x b\right)\left(\frac{\partial_x^r\partial_y^l}{(r+l)!}\partial_y u_{\bar{n}-1,\bar{m}-1}\right)\right\|_{H^1} \\ &\quad + \left\|\left(\frac{\partial_x^{k-p}}{(k-p)!}f\right)\left(\frac{\partial_x^{p-r}}{(p-r)!}\nabla_x b\right)\left(\frac{\partial_x^r\partial_y^{l-1}}{(r+l-1)!}\partial_y u_{\bar{n}-1,\bar{m}-1}\right)\right\|_{H^1} \\ &\le \sum_{p=0}^{k}\sum_{r=0}^{p}\mathcal{M}^2C_f\frac{A^{k-p}}{(k-p+1)^2}C_b\frac{A^{p-r}}{(p-r+1)^2} \\ &\quad \times\left(Y\left\|\frac{\partial_x^r\partial_y^l}{(r+l)!}u_{\bar{n}-1,\bar{m}-1}\right\|_{H^2} + \left\|\frac{\partial_x^r\partial_y^{l-1}}{(r+l-1)!}u_{\bar{n}-1,\bar{m}-1}\right\|_{H^2}\right) \\ &\le \mathcal{M}^2C_fC_b\frac{A^k}{(k+1)^2}\frac{D^l}{(l+1)^2}(Y+4)K_0B^{\bar{n}-1}E^{\bar{m}-1} \\ &\quad \times\sum_{p=0}^{k}\sum_{r=0}^{p}\frac{(k+1)^2}{(k-p+1)^2(p-r+1)^2(r+1)^2} \\ &\le K_0\mathcal{M}^2C_fC_bS^2(Y+4)\frac{A^k}{(k+1)^2}\frac{D^l}{(l+1)^2}B^{\bar{n}-1}E^{\bar{m}-1}. \end{aligned}$$

Again, we are done if $B > C_f$, $E > C_b$, and $\bar{C}_0 > \mathcal{M}^2S^2(Y+4)$.

Proof of Theorem 1. We work using an induction in n. At order $n = 0$ we must prove

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{0,m} \right\|_{H^2} \leq K_0 E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall m, k, l,$$

but this is simply Theorem 13. We now assume

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{n,m} \right\|_{H^2} \leq K_0 B^n E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall m, k, l,$$

for all $n < \bar{n}$ and seek to prove

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{\bar{n},m} \right\|_{H^2} \leq K_0 B^{\bar{n}} E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall m, k, l.$$

For this we consider an induction in m. At order $m = 0$ we need

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{\bar{n},0} \right\|_{H^2} \leq K_0 B^{\bar{n}} \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l,$$

but this follows from Theorem 11. Finally, we assume

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{\bar{n},m} \right\|_{H^2} \leq K_0 B^{\bar{n}} E^m \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l,$$

for all $m < \bar{m}$ and seek

$$\left\| \frac{\partial_x^k \partial_y^l}{(k+l)!} u_{\bar{n},\bar{m}} \right\|_{H^2} \leq K_0 B^{\bar{n}} E^{\bar{m}} \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2}, \quad \forall k, l.$$

By Lemma 5 we have that the hypotheses of Theorem 10 hold with

$$C_R = C_Q = K_0 \bar{C}_0 \left[\frac{C_f}{B} + \frac{C_b}{E} + \frac{C_f^2}{B^2} + \frac{C_b^2}{E^2} + \frac{C_f C_b}{BE} \right] B^{\bar{n}} E^{\bar{m}}.$$

Now,

$$\begin{aligned} \|u_{\bar{n},\bar{m}}\|_{H^2} &\leq \bar{C}_e \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \\ &\leq 2\alpha K_0 \bar{C}_0 \left[\frac{C_f}{B} + \frac{C_b}{E} + \frac{C_f^2}{B^2} + \frac{C_b^2}{E^2} + \frac{C_f C_b}{BE} \right] B^{\bar{n}} E^{\bar{m}} \\ &\quad \times \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \end{aligned}$$

and we are done provided that

$$B > \max\left\{ 10\alpha\bar{C}_0, \sqrt{10\alpha\bar{C}_0} \right\} C_f, \quad E > \max\left\{ 10\alpha\bar{C}_0, \sqrt{10\alpha\bar{C}_0} \right\} C_b.$$

A. Generalized elliptic estimates

In this appendix we establish the generalized elliptic estimate (Theorem 10) which is crucial to the inductive estimates for the spatial analyticity results of § 4. The key to the argument is to apply the classical elliptic theorem (Theorem 3) to arbitrary spatial derivatives of the system (13). In the work of Nicholls & Reitich [18] this was straightforward: They worked in the setting where $f_0 \equiv 0$ so that $\mathcal{L}$ is the Laplacian, Δ. In this case the operators $\frac{\partial_x^k}{k!}$ and Δ commute and a direct application of Theorem 3 is immediate. For us the estimation is more subtle as $\frac{\partial_x^k}{k!}$ and $\mathcal{L}$ do not commute and one must account for the remainders, i.e. the commutator.

To begin, apply $\frac{\partial_x^k}{k!}$ to (13):

$$\begin{aligned}\frac{\partial_x^k}{k!}\mathcal{L}\{w\} &= \frac{\partial_x^k}{k!}R && -h < y < 0\\ \frac{\partial_x^k}{k!}w(x,0) &= \frac{\partial_x^k}{k!}\xi(x) \\ \frac{\partial_x^k}{k!}\mathcal{B}\{w(x,-h)\} &= \frac{\partial_x^k}{k!}Q(x),\end{aligned}$$

which simplifies to

$$\mathcal{L}\left\{\frac{\partial_x^k}{k!}w\right\} = \frac{\partial_x^k}{k!}R + \left[\mathcal{L}, \frac{\partial_x^k}{k!}\right] w \qquad -h < y < 0 \tag{17a}$$

$$\frac{\partial_x^k}{k!}w(x,0) = \frac{\partial_x^k}{k!}\xi(x) \tag{17b}$$

$$\mathcal{B}\left\{\frac{\partial_x^k}{k!}w(x,-h)\right\} = \frac{\partial_x^k}{k!}Q(x) + \left[\mathcal{B}, \frac{\partial_x^k}{k!}\right] w(x,-h), \tag{17c}$$

where $[\cdot,\cdot]$ denotes the commutator,

$$[A,B] = AB - BA.$$

It is now clear that the following estimate (proven in § B) will be crucial to our analysis.

Lemma 6. *If* $f_0, b_0 \in C_3^\omega(P(\Gamma))$ *and*

$$\left\| \frac{\partial_x^k}{k!} w \right\|_{H^2} \leq \bar{K} \frac{A^k}{(k+1)^2}, \qquad \forall k < \bar{k}, \tag{18}$$

for constants $\bar{K}, A > 0$ *then*

$$\left\| \left[\mathcal{L}, \frac{\partial_x^{\bar{k}}}{\bar{k}!}\right] w \right\|_{H^0} \leq \bar{K}\tilde{K} \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2} \tag{19a}$$

$$\left\| \left[\mathcal{B}, \frac{\partial_x^{\bar{k}}}{\bar{k}!} \right] w \right\|_{H^{1/2}} \leq \bar{K} \tilde{K} \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2}, \tag{19b}$$

for a constant $\tilde{K} > 0$.

Now, given Lemma 6, to prove Theorem 10 we need the following preliminary theorem.

Theorem 15. *Suppose* $f_0, b_0 \in C_3^\omega(P(\Gamma))$*;* $\xi, Q \in C^\omega(P(\Gamma))$*; and* $R \in C^\omega(S_{h,0,0})$ *satisfying*

$$\left\| \frac{\partial_x^k}{k!} \xi \right\|_{H^{3/2}} \leq C_\xi \frac{A^k}{(k+1)^2}, \qquad \left\| \frac{\partial_x^k}{k!} Q \right\|_{H^{1/2}} \leq C_Q \frac{A^k}{(k+1)^2}, \quad \forall k \tag{20}$$

$$\left\| \frac{\partial_x^k}{k!} R \right\|_{H^0} \leq C_R \frac{A^k}{(k+1)^2}, \quad \forall k. \tag{21}$$

Then the unique solution $w \in C^\omega(S_{h,0,0})$ *of*

$$\mathcal{L}\{w\} = R \qquad -h < y < 0 \tag{22a}$$

$$w(x,0) = \xi(x) \tag{22b}$$

$$\mathcal{B}\{w\}(x,-h) = Q(x) \tag{22c}$$

satisfies

$$\left\| \frac{\partial_x^k}{k!} w \right\|_{H^2} \leq \bar{C}_e \frac{A^k}{(k+1)^2} \qquad \forall k, \tag{23}$$

where $\bar{C}_e = \alpha(C_R + C_\xi + C_Q)$ *and* $\alpha = \alpha(d,h)$.

Proof. We specialize to $d = 1$ (the higher dimensional case simply requires $(d-1)$-many more inductions), and work by induction in k. The case $k = 0$ is Theorem 3. We now assume that (23) holds for all $k < \bar{k}$. Theorem 3 states that solutions of (17) satisfy

$$\left\| \frac{\partial_x^{\bar{k}}}{\bar{k}!} w \right\|_{H^2} \leq C_e \left[\left\| \frac{\partial_x^{\bar{k}}}{\bar{k}!} R \right\|_{H^0} + \left\| \frac{\partial_x^{\bar{k}}}{\bar{k}!} \xi \right\|_{H^{3/2}} + \left\| \frac{\partial_x^{\bar{k}}}{\bar{k}!} Q \right\|_{H^{1/2}} + \left\| \left[\mathcal{L}, \frac{\partial_x^{\bar{k}}}{\bar{k}!} \right] w \right\|_{H^0} + \left\| \left[\mathcal{B}, \frac{\partial_x^{\bar{k}}}{\bar{k}!} \right] w \right\|_{H^{1/2}} \right].$$

Using (20) and Lemma 6, we find that

$$\left\| \frac{\partial_x^{\bar{k}}}{\bar{k}!} w \right\|_{H^2} \leq C_e \left[(C_R + C_\xi + C_Q) \frac{A^{\bar{k}}}{(\bar{k}+1)^2} \right] + 2\bar{C}_e \tilde{K} \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2}.$$

Our proof is complete provided

$$\bar{C}_e \geq 2C_e(C_R + C_\xi + C_Q), \quad A \geq 4\tilde{K},$$

so that, in this case, $\alpha = 2C_e$.

We can now prove Theorem 10.

Proof of Theorem 10. We work by induction in l and notice that $l = 0$ is Theorem 15. We now assume (16) for all $l < \bar{l}$ and all k, and examine

$$\begin{aligned}\left\| \frac{\partial_x^k \partial_y^{\bar{l}}}{(k+\bar{l})!} w \right\|_{H^2} &= \left\| \frac{\partial_x^k \partial_y^{\bar{l}}}{(k+\bar{l})!} w \right\|_{H^1} + \left\| \frac{\partial_x^k \partial_y^{\bar{l}}}{(k+\bar{l})!} \partial_x w \right\|_{H^1} + \left\| \frac{\partial_x^k \partial_y^{\bar{l}}}{(k+\bar{l})!} \partial_y w \right\|_{H^1} \\ &\leq \left\| \frac{\partial_x^k \partial_y^{\bar{l}-1}}{(k+\bar{l})!} w \right\|_{H^2} + \left\| \frac{\partial_x^k \partial_y^{\bar{l}-1}}{(k+\bar{l})!} \partial_x w \right\|_{H^2} + \left\| \frac{\partial_x^k \partial_y^{\bar{l}-1}}{(k+\bar{l})!} \partial_y^2 w \right\|_{H^1}.\end{aligned}$$

The first two of these terms can be handled by the inductive hypothesis as they involve y-derivatives of order $\bar{l} - 1$. The third, which we denote T_3, requires the use of (22a):

$$\begin{aligned}\partial_y^2 w &= \partial_y \left[\frac{h^2 + |N_0|^2}{h^2 + |N_0|^2} \partial_y w \right] \\ &= \partial_y \left[S(x,y)(h^2 + |N_0|^2) \partial_y w \right] \\ &= (\partial_y S)(h^2 + |N_0|^2)(\partial_y w) + S \partial_y \left[(h^2 + |N_0|^2) \partial_y w \right] \\ &= \bar{S}(\partial_y w) + S \left(R - \tilde{\mathcal{L}} \{w\} \right),\end{aligned}$$

where

$$\begin{aligned}S &:= \frac{1}{h^2 + |N_0|^2} \\ \bar{S} &:= (\partial_y S)(h^2 + |N_0|^2) \\ \tilde{\mathcal{L}} \{w\} &:= \mathcal{L} \{w\} - \partial_y \left[(h^2 + |N_0|^2) \partial_y w \right].\end{aligned}$$

We point out that $\tilde{\mathcal{L}}$ involves only first order derivatives in y which will prove important in our proof. Since $f_0, b_0 \in C_3^\omega$ and S is the reciprocal of a quadratic in y, clearly, for any $F \in H^2$, the analyticity estimates

$$\begin{aligned}\left\| \left(\frac{\partial_x^k \partial_y^l}{(k+l)!} S \right) F \right\|_{H^2} &\leq C_S \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \|F\|_{H^2} \qquad \forall k, l \\ \left\| \left(\frac{\partial_x^k \partial_y^l}{(k+l)!} \bar{S} \right) F \right\|_{H^2} &\leq C_{\bar{S}} \frac{A^k}{(k+1)^2} \frac{D^l}{(l+1)^2} \|F\|_{H^2} \qquad \forall k, l,\end{aligned}$$

hold for some constants C_S and $C_{\bar{S}}$. From this we can now estimate

$$
\begin{aligned}
T_3 &= \left\| \frac{\partial_x^k \partial_y^{\bar{l}-1}}{(k+\bar{l})!} \partial_y^2 w \right\|_{H^1} = \left\| \frac{\partial_x^k \partial_y^{\bar{l}-1}}{(k+\bar{l})!} \left[\bar{S}(\partial_y w) + S\left(R - \tilde{\mathcal{L}}\{w\}\right) \right] \right\|_{H^1} \\
&\leq \left\| \frac{\partial_x^k \partial_y^{\bar{l}-2}}{(k+\bar{l})!} \left[\bar{S}(\partial_y w) + S\left(R - \tilde{\mathcal{L}}\{w\}\right) \right] \right\|_{H^2} \\
&\leq \sum_{p=0}^{k} \sum_{q=0}^{\bar{l}-2} \left\| \frac{\partial_x^{k-p} \partial_y^{\bar{l}-2-q}}{((k-p)+(\bar{l}-q))!} [\bar{S}] \frac{\partial_x^p \partial_y^q}{(p+q)!} [\partial_y w] \right\|_{H^2} \\
&\qquad + \left\| \frac{\partial_x^{k-p} \partial_y^{\bar{l}-2-q}}{((k-p)+(\bar{l}-q))!} [S] \frac{\partial_x^p \partial_y^q}{(p+q)!} [R] \right\|_{H^2} \\
&\qquad + \left\| \frac{\partial_x^{k-p} \partial_y^{\bar{l}-2-q}}{((k-p)+(\bar{l}-q))!} [S] \frac{\partial_x^p \partial_y^q}{(p+q)!} \left[\tilde{\mathcal{L}}\{w\} \right] \right\|_{H^2} \\
&\leq \sum_{p=0}^{k} \sum_{q=0}^{\bar{l}-2} C_{\bar{S}} \frac{A^{k-p}}{(k-p+1)^2} \frac{D^{\bar{l}-2-q}}{(\bar{l}-2-q+1)^2} \bar{C}_e \frac{A^p}{(p+1)^2} \frac{D^{q+1}}{(q+1+1)^2} \\
&\qquad + C_S \frac{A^{k-p}}{(k-p+1)^2} \frac{D^{\bar{l}-2-q}}{(\bar{l}-2-q+1)^2} C_R \frac{A^p}{(p+1)^2} \frac{D^q}{(q+1)^2} \\
&\qquad + C_S \frac{A^{k-p}}{(k-p+1)^2} \frac{D^{\bar{l}-2-q}}{(\bar{l}-2-q+1)^2} \left\| \frac{\partial_x^p \partial_y^q}{(p+q)!} \left[\tilde{\mathcal{L}}\{w\} \right] \right\|_{H^2} .
\end{aligned}
$$

Since $\tilde{\mathcal{L}}\{w\}$ involves only single y-derivatives it is not difficult to show that

$$
\left\| \frac{\partial_x^p \partial_y^q}{(p+q)!} \left[\tilde{\mathcal{L}}\{w\} \right] \right\|_{H^2} \leq \bar{C}_e C_{\tilde{\mathcal{L}}} \frac{A^{p+2}}{(p+2+1)^2} \frac{D^{q+1}}{(q+1+1)^2},
$$

where $C_{\tilde{\mathcal{L}}}$, of course, depends up C_{b_0} and C_{f_0}. Thus, since $0 \leq q \leq \bar{l}-2$, we have that

$$
\begin{aligned}
T_3 &\leq \frac{A^k}{(k+1)^2} \frac{D^{\bar{l}-1}}{(\bar{l}+1)^2} \left[\bar{C}_e (C_{\bar{S}} + C_S C_{\tilde{\mathcal{L}}}) + C_S C_R \right] \\
&\qquad \times \sum_{p=0}^{k} \sum_{q=0}^{\bar{l}-2} \frac{(k+1)^2 (\bar{l}+1)^2}{(k-p+1)^2 (\bar{l}-2-q+1)^2 (p+1)^2 (q+1)^2} \\
&\leq S^2 \left[\bar{C}_e (C_{\bar{S}} + C_S C_{\tilde{\mathcal{L}}}) + C_S C_R \right] \frac{A^k}{(k+1)^2} \frac{D^{\bar{l}-1}}{(\bar{l}+1)^2} \\
&\leq \bar{C}_e \frac{A^k}{(k+1)^2} \frac{D^{\bar{l}}}{(\bar{l}+1)^2},
\end{aligned}
$$

provided that $D > 2S^2(C_{\bar{S}} + C_S C_{\tilde{\mathcal{L}}})$ and $\bar{C}_e > 2DS^2 C_S C_R$.

B. Proof of a commutator estimate

In this appendix we provide the proof of the commutator estimate, Lemma 6, which is the key to establishing Theorem 10.

Proof of Lemma 6. Recall that

$$h^2\mathcal{L}\{w\} = \mathrm{div}_x\left[M_0^2\nabla_x w\right] + \mathrm{div}_x\left[M_0N_0\partial_y w\right] + \partial_y\left[M_0N_0\cdot\nabla_x w\right] + \partial_y\left[(h^2+|N_0|^2)\partial_y w\right] + M_0\nabla_x M_0\cdot\nabla_x w + N_0\cdot\nabla_x M_0\partial_y w.$$

Using Leibniz's rule

$$\begin{aligned} h^2\frac{\partial_x^{\bar{k}}}{\bar{k}!}\mathcal{L}\{w\} = \sum_{p=0}^{\bar{k}} &\mathrm{div}_x\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0^2\right)\nabla_x\frac{\partial_x^p}{p!}w\right] \\ &+ \mathrm{div}_x\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\partial_y\frac{\partial_x^p}{p!}w\right] \\ &+ \partial_y\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\cdot\nabla_x\frac{\partial_x^p}{p!}w\right] \\ &+ \partial_y\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(h^2+|N_0|^2\right)\partial_y\frac{\partial_x^p}{p!}w\right] \\ &+ \frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x M_0\right)\cdot\nabla_x\frac{\partial_x^p}{p!}w \\ &+ \frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(N_0\cdot\nabla_x M_0\right)\partial_y\frac{\partial_x^p}{p!}w, \end{aligned}$$

so that

$$\begin{aligned} h^2\left[\mathcal{L},\frac{\partial_x^{\bar{k}}}{\bar{k}!}\right]w &= h^2\mathcal{L}\left\{\frac{\partial_x^{\bar{k}}}{\bar{k}!}w\right\} - h^2\frac{\partial_x^{\bar{k}}}{\bar{k}!}\mathcal{L}\{w\} \\ &= -\left\{\sum_{p=0}^{\bar{k}-1}\mathrm{div}_x\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0^2\right)\nabla_x\frac{\partial_x^p}{p!}w\right]\right. \\ &\quad + \mathrm{div}_x\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\partial_y\frac{\partial_x^p}{p!}w\right] \\ &\quad + \partial_y\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\cdot\nabla_x\frac{\partial_x^p}{p!}w\right] \end{aligned}$$

$$
\begin{aligned}
&+\partial_y\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(h^2+|N_0|^2\right)\partial_y\frac{\partial_x^p}{p!}w\right]\\
&+\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x M_0\right)\cdot\nabla_x\frac{\partial_x^p}{p!}w\\
&\left.+\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(N_0\cdot\nabla_x M_0\right)\partial_y\frac{\partial_x^p}{p!}w\right\}.
\end{aligned}
$$

To illustrate the essential difficulties we analyze the second term more closely and note that the other terms can be estimated in a similar way. For $0\le p\le\bar{k}-1$,

$$
\begin{aligned}
&\left\|\operatorname{div}_x\left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\partial_y\frac{\partial_x^p}{p!}w\right]\right\|_{H^0}\\
&\le\left\|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0N_0\right)\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}\\
&\le\left\|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(-M_0(y+h)\nabla_x f_0+M_0y\nabla_x b_0\right)\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}\\
&\le Y\left\|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x f_0\right)\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}+Y\left\|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x b_0\right)\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}\\
&\le Y\mathcal{M}\left|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x f_0\right)\right|_{C^1}\left\|\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}\\
&\quad+Y\mathcal{M}\left|\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!}\left(M_0\nabla_x b_0\right)\right|_{C^1}\left\|\partial_y\frac{\partial_x^p}{p!}w\right\|_{H^1}.
\end{aligned}
$$

Consider the first of these terms and set $q=\bar{k}-p$; we note that $q\ge 1$ which we use explicitly in the estimate below.

$$
\begin{aligned}
\left|\frac{\partial_x^q}{q!}\left(M_0\nabla_x f_0\right)\right|_{C^1}&=\left|\frac{\partial_x}{q}\frac{\partial_x^{q-1}}{(q-1)!}\left(M_0\nabla_x f_0\right)\right|_{C^1}\le\frac{1}{q}\left|\frac{\partial_x^{q-1}}{(q-1)!}\left(M_0\nabla_x f_0\right)\right|_{C^2}\\
&\le\left|\sum_{m=0}^{q-1}\frac{\partial_x^{q-m-1}}{(q-m-1)!}(M_0)\frac{\partial_x^m}{m!}(\nabla_x f_0)\right|_{C^2}\\
&\le\sum_{m=0}^{q-1}\left|\frac{\partial_x^{q-m-1}}{(q-m-1)!}M_0\right|_{C^2}\left|\frac{\partial_x^m}{m!}f_0\right|_{C^3}\\
&\le\sum_{m=0}^{q-1}C_{M_0}\frac{A^{q-m-1}}{(q-m)^2}C_{f_0}\frac{A^m}{(m+1)^2}
\end{aligned}
$$

$$\le C_{M_0} C_{f_0} \frac{A^{q-1}}{(q+1)^2} \left(\sum_{m=0}^{q-1} \frac{(q+1)^2}{(q-m)^2 (m+1)^2} \right)$$
$$\le S C_{M_0} C_{f_0} \frac{A^{q-1}}{(q+1)^2}, \tag{24}$$

where the sum in parentheses is bounded by the universal constant S (cf. Lemma 11 of [18]). Using (18) we deduce that

$$\left\| \operatorname{div}_x \left[\frac{\partial_x^{\bar{k}-p}}{(\bar{k}-p)!} (M_0 N_0) \partial_y \frac{\partial_x^p}{p!} w \right] \right\|_{H^0} \le Y \mathcal{M} S C_{M_0} (C_{f_0} + C_{b_0}) \bar{K} \times \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2} \frac{(\bar{k}+1)^2}{(\bar{k}-p)^2 (p+1)^2},$$

so that

$$\left\| \left[\mathcal{L}, \frac{\partial_x^{\bar{k}}}{\bar{k}!} \right] w \right\|_{H^0} \le \frac{Y \mathcal{M} S C_{M_0} (C_{f_0} + C_{b_0})}{h^2} \bar{K} \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2} \sum_{p=0}^{\bar{k}-1} \frac{(\bar{k}+1)^2}{(\bar{k}-p)^2 (p+1)^2} + \dots \le \tilde{K} \bar{K} \frac{A^{\bar{k}-1}}{(\bar{k}+1)^2}$$

if $\tilde{K} > (Y \mathcal{M} S^2 C_{M_0} (C_{f_0} + C_{b_0}))/h^2$. From this (19a) follows easily; (19b) is established using the same techniques which are omitted here for brevity.

Remark 2. In estimate (24) we used the fact that $q \ge 1$ to achieve the power $(q-1)$ for A. A careful inspection shows that this estimation coupled to the *explicit* appearance of $\nabla_x f_0$ (rather than simply f_0) results in the necessity of the C^3 norm.

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