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On analyticity of linear waves scattered by a layered medium

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Abstract

The scattering of linear waves by periodic structures is a crucial phenomena in many branches of applied physics and engineering. In this paper we establish rigorous analytic results necessary for the proper numerical analysis of a class of High-Order Perturbation of Surfaces methods for simulating such waves. More specifically, we prove a theorem on existence and uniqueness of solutions to a system of partial differential equations which model the interaction of linear waves with a multiply layered periodic structure in three dimensions. This result provides hypotheses under which a rigorous numerical analysis could be conducted for recent generalizations to the methods of Operator Expansions, Field Expansions, and Transformed Field Expansions.

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1. Introduction

The scattering of linear waves by periodic structures (both in two and three dimensions) is a crucial phenomena in many branches of applied physics and engineering. From acoustics (e.g., remote sensing [69], nondestructive testing [67], and underwater acoustics [14]), to electromag-

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netics (e.g., extraordinary optical transmission [31], surface enhanced spectroscopy [52], and surface plasmon resonance biosensing [38,44]), to elastodynamics (e.g., full waveform inversion [70,18] and hazard assessment [36,68]), examples abound. Obviously, the ability to rapidly simulate such configurations numerically with great accuracy and high fidelity is of the upmost importance to many disciplines.

The most popular approaches to these problems in the engineering literature are *volumetric* numerical methods. For instance, sampling from the seismic imaging community alone, formulations based upon Finite Differences [64], Finite Elements [73], and Spectral Elements [41] are common. However, these methods suffer from the requirement that they discretize the full volume of the problem domain which results in both a prohibitive number of degrees of freedom for the layered structures we consider here, and the difficult question of appropriately specifying far-field boundary conditions explicitly.

For these reasons, *surface* methods are an appealing alternative, particularly Boundary Integral Methods [23,43,66,3,12,13,10,45,19] and the High-Order Perturbation of Surfaces (HOPS) methods of Milder (Operator Expansions) [47,48] (see also [22]), Bruno and Reitich (Field Expansions) [15–17], and the author and Reitich (Transformed Field Expansions) [56–58]. These surface methods are greatly advantaged over the volumetric algorithms discussed above primarily in the greatly reduced number of degrees of freedom required to resolve a computation, in addition to the *exact* enforcement of far-field boundary conditions. Consequently, these approaches are an extremely important alternative and are becoming more widely used by practitioners.

Of course there has been a huge amount of rigorous analysis on the systems of partial differential equations which model these scattering phenomena, in addition to the design of computational schemes. Most of these results utilize either Integral Equations techniques or weak formulations of the volumetric problem (each of which naturally lead to numerical implementations). We find the Habilitationsschrift of T. Arens [5] a particularly readable and definitive reference for the periodic layered media problems we consider here. In particular, we point the interested reader to Chapter 1 which discusses in great detail the state-of-the-art in both two and three dimensions for solutions of the Helmholtz and Maxwell equations. To summarize, in two dimensions most of the questions of existence and uniqueness have been satisfactorily addressed and these results are summarized in surveys such as those of Petit [62] and Bao, Cowsar, and Masters [7]. For single layer configurations we point out the work of Alber [2], Wilcox [71], and Elschner and Schmidt [32]. In three dimensions, for the Helmholtz equation, most results are connected to variational formulations such as those of Abboud and Nedelec [4], Bao [6], Bao, Dobson, and Cox [9], and Dobson [30] (see also the work of Chen and Friedman [21] and Dobson and Friedman [28] in the context of Maxwell's equations). Arens summarizes these with the following sentence [5]: "There may exist at most a countable set of frequencies with infinity as the only possible accumulation point for which the problem is not uniquely solvable."

The purpose of this contribution is to establish rigorous analytic results necessary for the proper numerical analysis of HOPS methods. More specifically, we prove a result (Theorem 4.1) using boundary perturbations on existence and uniqueness of solutions to a system of Helmholtz equations which model the interaction of linear (acoustic) waves with a multiply layered periodic structure in three dimensions. The goal of this study is to provide hypotheses under which a rigorous numerical analysis could be conducted, and a solution to which our HOPS schemes can be shown to converge. More specifically, we seek a framework to study the generalizations to the Operator Expansions method of the author and Fang [54,35], to the Field Expansions approach by the author and Malcolm [50,49,51,55], and, based upon the recursions derived herein, to the Transformed Field Expansions approach. For the numerical analysis, we have in mind an investigation very much in the spirit of the author and Shen [60], which we will conduct in a future publication.

The organization of the paper is as follows: In § 2 we briefly discuss the well-known equations governing the scattering of linear (acoustic) radiation by a layered, three-dimensional periodic structure. In particular, we recall in § 2.1 how the far-field boundary conditions can be enforced transparently with the use of appropriate Dirichlet–Neumann Operators (DNOs), while the entire system of equations can be equivalently restated in terms of other DNOs in § 2.2. In § 3 we present a rather general and rigorously justifiable perturbative scheme for solving systems of linear systems of equations in Banach spaces. The appropriate analyticity theorems for a single perturbation parameter and one of multiple dimensions are presented and proven in § 3.1 and § 3.2, respectively. The application of these results to the governing equations presented in § 2 is made in § 4 where the novel result (Theorem 4.1) is established. This proof requires several rigorous analyses, and those of the invertibility of the linearized operator are given in § 4.1 (with a commentary on its relation to previously known results in § 4.2), and the analyticity of the inhomogeneity are presented in § 4.3. The final analysis is that the linear operator itself is analytic which proceeds from analyticity results for DNOs that are given in § 5. This treatment utilizes the Transformed Field Expansions methodology first derived by the author and Reitich [56–58], and we describe the relevant change of variables (§ 5.1), the transformed field equations (§ 5.2), the corresponding formulas for the DNO (§ 5.3), the formulas for the perturbation corrections (§ 5.4), and finally the rigorous demonstration (§ 5.5). We provide concluding remarks in § 6, while in Appendix A we describe details of the estimates of solutions to the fundamental elliptic boundary value problem at the core of our analyticity theorem on DNOs.

2. Governing equations

The Helmholtz equation governs the scattering of linear acoustic waves in a periodic layered structure, with insonification conditions at the upper interface, and upward and downward propagating wave conditions at positive and negative infinity [1,62,39,11]. For the latter of these we demand the "upward propagating Rayleigh expansion radiation condition" (URC) and its "downward propagating" analogue (DRC) as specified in [5] (which we make precise in § 2.1). In [54] we detailed a restatement of the classical governing equations in terms of Dirichlet–Neumann Operators (DNOs) which we reprise here for the reader's convenience.

We consider a multiply layered material with M interfaces at

$$z = a^{(m)} + g^{(m)}(x, y), \quad 1 \le m \le M,$$

which are $d_x \times d_y$ periodic

$$g^{(m)}(x + d_x, y + d_y) = g^{(m)}(x, y), \quad 1 \le m \le M,$$

separating $(M + 1)$-many layers which define the domains

$$\begin{aligned}
S^{(0)} &:= \{(x, y, z) \mid z > a^{(1)} + g^{(1)}(x, y)\}, \\
S^{(m)} &:= \{(x, y, z) \mid a^{(m+1)} + g^{(m+1)}(x, y) < z < a^{(m)} + g^{(m)}(x, y)\}, \qquad 1 \le m \le M - 1, \\
S^{(M)} &:= \{(x, y, z) \mid z < a^{(M)} + g^{(M)}(x, y)\},
\end{aligned}$$

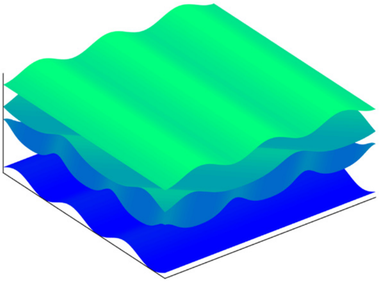

Fig. 1. Five-layer problem configuration with layer interfaces $z = a^{(m)} + g^{(m)}(x)$.

with (upward pointing) normals

$$N^{(m)} := (-\partial_x g^{(m)}(x, y), -\partial_y g^{(m)}(x, y), 1)^T;$$

see Fig. 1. The $(M + 1)$ domains are all lossless, constant-density acoustic media with velocities $c^{(m)}$ $(m = 0, \ldots, M)$ and we assume that plane-wave radiation is incident upon the structure from above

$$\underline{v}^{\text{inc}}(x, y, z, t) = e^{-i\omega t} e^{i(\alpha x + \beta y - \gamma^{(0)} z)} = e^{-i\omega t} v^{\text{inc}}(x, y, z).$$

In each layer the parameter $k^{(m)} = \omega / c^{(m)}$ characterizes both the properties of the material and the frequency of radiation in the structure. We denote the reduced scattered fields in $S^{(m)}$ by

$$v^{(m)}(x, y, z) = e^{i\omega t} \underline{v}^{(m)}(x, y, z, t),$$

(the full scattered fields with the periodic time dependence factored out) which, like the incident radiation, will be quasiperiodic [62]

$$v^{(m)}(x + d_x, y + d_y, z) = e^{i(\alpha d_x + \beta d_y)} v^{(m)}(x, y, z), \quad m = 0, \ldots, M.$$

These reduced fields satisfy the Helmholtz equations

$$\Delta v^{(m)} + \left(k^{(m)}\right)^2 v^{(m)} = 0, \quad \text{in } S^{(m)}, \quad 0 \leq m \leq M,$$

which are coupled through the Dirichlet and Neumann boundary conditions

$$\begin{aligned}
&v^{(m-1)} - v^{(m)} = \zeta^{(m)}, && z = a^{(m)} + g^{(m)}(x, y), \quad 1 \leq m \leq M, \\
&\partial_{N^{(m)}} \left[v^{(m-1)} - v^{(m)} \right] = \psi^{(m)}, && z = a^{(m)} + g^{(m)}(x, y), \quad 1 \leq m \leq M,
\end{aligned}$$

where

$$\zeta^{(1)}(x,y) := -v^{\text{inc}}(x,y,a^{(1)}+g^{(1)}(x,y))$$
$$= -e^{i(\alpha x+\beta y-\gamma^{(0)}(a^{(1)}+g^{(1)}(x,y)))}, \tag{2.1a}$$
$$\psi^{(1)}(x,y) := -\left[\partial_{N^{(1)}} v^{\text{inc}}(x,y,z)\right]_{z=a^{(1)}+g^{(1)}(x,y)}$$
$$= (i\gamma^{(0)} + i\alpha(\partial_x g^{(1)}) + i\beta(\partial_y g^{(1)}))e^{i(\alpha x+\beta y-\gamma^{(0)}(a^{(1)}+g^{(1)}(x,y)))}. \tag{2.1b}$$

If continuity is enforced inside the structure then $\zeta^{(m)} \equiv \psi^{(m)} \equiv 0$, $m = 2, \ldots, M$. However, as we shall see, it is no impediment to the method if we set these to any nonzero function.

2.1. Transparent boundary conditions

Regarding the upward/downward propagating wave conditions, we introduce the planes

$$z = \overline{a} > a^{(1)} + \left|g^{(1)}\right|_{L^\infty}, \quad z = \underline{a} < a^{(M)} - \left|g^{(M)}\right|_{L^\infty},$$

define the domains

$$\overline{S} := \{z > \overline{a}\}, \quad \underline{S} := \{z < \underline{a}\},$$

and note that we can find unique quasiperiodic solutions of the relevant Helmholtz problems on each of these domains given generic Dirichlet data, say $\xi(x,y)$ and $\mu(x,y)$. For this we use the Rayleigh expansions [65] which state that

$$v^{(0)}(x,y,z) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \hat{\xi}_{p,q} e^{i\alpha_p x + i\beta_q y + i\gamma_{p,q}^{(0)}(z-\overline{a})},$$
$$v^{(M)}(x,y,z) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \hat{\mu}_{p,q} e^{i\alpha_p x + i\beta_q y - i\gamma_{p,q}^{(M)}(z-\underline{a})},$$

where, for $p, q \in \mathbf{Z}$, $m \in \{0, \ldots, M\}$,

$$\alpha_p := \alpha + \left(\frac{2\pi}{d_x}\right)p, \quad \beta_q := \beta + \left(\frac{2\pi}{d_y}\right)q,$$
$$\gamma_{p,q}^{(m)} := \begin{cases} \sqrt{\left(k^{(m)}\right)^2 - \alpha_p^2 - \beta_q^2}, & (p,q) \in \mathcal{U}^{(m)}, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - \left(k^{(m)}\right)^2}, & (p,q) \notin \mathcal{U}^{(m)}, \end{cases} \tag{2.2}$$

and the set of propagating modes is

$$\mathcal{U}^{(m)} := \left\{(p,q) \in \mathbf{Z} \mid \alpha_p^2 + \beta_q^2 \le \left(k^{(m)}\right)^2\right\}.$$

We note that

$$v^{(0)}(x,y,\overline{a}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{\xi}_{p,q} e^{i\alpha_p x + i\beta_q y} = \xi(x,y),$$

$$v^{(M)}(x,y,\underline{a}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{\mu}_{p,q} e^{i\alpha_p x + i\beta_q y} = \mu(x,y).$$

With these formulas we can compute the *outward-pointing* Neumann data at the artificial boundaries

$$-\partial_z v^{(0)}(x,y,\overline{a}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} -(i\gamma_{p,q}^{(0)})\hat{\xi}_{p,q} e^{i\alpha_p x + i\beta_q y} =: T^{(0)}[\xi(x,y)],$$

$$\partial_z v^{(M)}(x,y,\underline{a}) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (-i\gamma_{p,q}^{(M)})\hat{\mu}_{p,q} e^{i\alpha_p x + i\beta_q y} =: T^{(M)}[\mu(x,y)],$$

which define the multipliers in Fourier space, $\{T^{(0)}, T^{(M)}\}$.

With these operators it is not difficult to see that quasiperiodic, upward propagating solutions to the Helmholtz equation

$$\Delta v^{(0)} + \left(k^{(0)}\right)^2 v^{(0)} = 0, \quad z > a^{(1)} + g^{(1)}(x,y),$$

equivalently solve

$$\Delta v^{(0)} + \left(k^{(0)}\right)^2 v^{(0)} = 0, \qquad a^{(1)} + g^{(1)}(x,y) < z < \overline{a}, \tag{2.3a}$$

$$\partial_z v^{(0)} + T^{(0)}\left[v^{(0)}\right] = 0, \qquad z = \overline{a}. \tag{2.3b}$$

Similarly, one can show that quasiperiodic, downward propagating solutions to the Helmholtz equation

$$\Delta v^{(M)} + \left(k^{(M)}\right)^2 v^{(M)} = 0, \quad z < a^{(M)} + g^{(M)}(x,y),$$

equivalently solve

$$\Delta v^{(M)} + \left(k^{(M)}\right)^2 v^{(M)} = 0, \qquad \underline{a} < z < a^{(M)} + g^{(M)}(x,y), \tag{2.4a}$$

$$\partial_z v^{(M)} - T^{(M)}\left[v^{(M)}\right] = 0, \qquad z = \underline{a}. \tag{2.4b}$$

Remark 2.1. We point out that the conditions (2.3b) and (2.4b) specify solutions which satisfy the UPC and DPC of Definition 2.6 in Arens [5]. It is these two conditions which guarantee the uniqueness of solutions on the unbounded domains $\{z > \overline{a}\}$ and $\{z < \underline{a}\}$.

2.2. Boundary formulation

We now follow [54] and reduce our set of unknowns to the following surface quantities, the (lower and upper) Dirichlet traces

$$V^{(m),\ell}(x,y) := v^{(m)}(x,y,a^{(m+1)} + g^{(m+1)}(x,y)), \qquad 0 \le m \le M-1,$$
$$V^{(m),u}(x,y) := v^{(m)}(x,y,a^{(m)} + g^{(m)}(x,y)), \qquad 1 \le m \le M,$$

and their (exterior, lower and upper) Neumann analogues

$$\tilde{V}^{(m),\ell}(x,y) := -(\partial_{N^{(m+1)}} v^{(m)})(x,y,a^{(m+1)} + g^{(m+1)}(x,y)), \qquad 0 \le m \le M-1,$$
$$\tilde{V}^{(m),u}(x,y) := (\partial_{N^{(m)}} v^{(m)})(x,y,a^{(m)} + g^{(m)}(x,y)), \qquad 1 \le m \le M.$$

Using the fact that, from these, one could recover the scattered field at any point with a suitable integral formula [33], we find that our governing equations reduce to the boundary conditions

$$V^{(m-1),\ell} - V^{(m),u} = \zeta^{(m)}, \qquad 1 \le m \le M, \tag{2.5a}$$
$$-\tilde{V}^{(m-1),\ell} - \tilde{V}^{(m),u} = \psi^{(m)}, \qquad 1 \le m \le M. \tag{2.5b}$$

We can further simplify by introducing Dirichlet–Neumann operators (DNOs). For this we make the following definitions.

Definition 2.2. Given a sufficiently smooth deformation $g^{(1)}(x,y)$, the unique quasiperiodic solution of

$$\Delta v^{(0)} + \left(k^{(0)}\right)^2 v^{(0)} = 0, \qquad a^{(1)} + g^{(1)}(x,y) < z < \overline{a},$$
$$v^{(0)} = V^{(0),\ell}, \qquad z = a^{(1)} + g^{(1)}(x,y),$$
$$\partial_z v^{(0)} + T^{(0)}\left[v^{(0)}\right] = 0, \qquad z = \overline{a},$$

defines the Dirichlet–Neumann Operator

$$G\left[V^{(0),\ell}\right] = G(\overline{a}, a^{(1)}, g^{(1)})\left[V^{(0),\ell}\right] := \tilde{V}^{(0),\ell}.$$

Definition 2.3. Given sufficiently smooth deformations $g^{(m)}(x,y)$ and $g^{(m+1)}(x,y)$, for $1 \le m \le M-1$, if a unique quasiperiodic solution exists to

$$\Delta v^{(m)} + \left(k^{(m)}\right)^2 v^{(m)} = 0, \qquad a^{(m+1)} + g^{(m+1)}(x,y) < z < a^{(m)} + g^{(m)}(x,y), \tag{2.6a}$$
$$v^{(m)} = V^{(m),\ell}, \qquad z = a^{(m+1)} + g^{(m+1)}(x,y), \tag{2.6b}$$
$$v^{(m)} = V^{(m),u}, \qquad z = a^{(m)} + g^{(m)}(x,y), \tag{2.6c}$$

it defines the Dirichlet–Neumann Operator

$$
\begin{aligned}
H(m)\left[\begin{pmatrix} V^{(m),u} \\ V^{(m),\ell} \end{pmatrix}\right] &= H(m; a^{(m)}, g^{(m)}, a^{(m+1)}, g^{(m+1)})\left[\begin{pmatrix} V^{(m),u} \\ V^{(m),\ell} \end{pmatrix}\right] \\
&= \begin{pmatrix} H^{uu}(m) & H^{u\ell}(m) \\ H^{\ell u}(m) & H^{\ell\ell}(m) \end{pmatrix}\left[\begin{pmatrix} V^{(m),u} \\ V^{(m),\ell} \end{pmatrix}\right] \\
&:= \begin{pmatrix} \tilde{V}^{(m),u} \\ \tilde{V}^{(m),\ell} \end{pmatrix}.
\end{aligned} \tag{2.7}
$$

Definition 2.4. Given a sufficiently smooth deformation $g^{(M)}(x, y)$, the unique quasiperiodic solution of

$$
\begin{aligned}
&\Delta v^{(M)} + \left(k^{(M)}\right)^2 v^{(M)} = 0, && \underline{a} < z < a^{(M)} + g^{(M)}(x, y), \\
&v^{(M)} = V^{(M),u}, && z = a^{(M)} + g^{(M)}(x, y), \\
&\partial_z v^{(M)} - T^{(M)}\left[v^{(M)}\right] = 0, && z = \underline{a},
\end{aligned}
$$

defines the Dirichlet–Neumann Operator

$$
J\left[V^{(M),u}\right] = J(\underline{a}, a^{(M)}, g^{(M)})\left[V^{(M),u}\right] := \tilde{V}^{(M),u}.
$$

Remark 2.5. In § 5 we will show that $g^{(m)} \in C^{s+3/2+\sigma}(d)$ for an integer $s \geq 0$, and any $\sigma > 0$, is smooth enough. However, Lipschitz smooth will also suffice [25,37].

In terms of this notation the boundary conditions, (2.5), become

$$
V^{(m-1),\ell} - V^{(m),u} = \zeta^{(m)}, \quad 1 \leq m \leq M,
$$

and

$$
\begin{aligned}
&- G[V^{(0),\ell}] - H^{uu}(1)[V^{(1),u}] - H^{u\ell}(1)[V^{(1),\ell}] = \psi^{(1)}, \\
&- H^{\ell u}(m-1)[V^{(m-1),u}] - H^{\ell\ell}(m-1)[V^{(m-1),\ell}] \\
&\quad - H^{uu}(m)[V^{(m),u}] - H^{u\ell}(m)[V^{(m),\ell}] = \psi^{(m)}, \qquad 2 \leq m \leq M-1, \\
&- H^{\ell u}(M-1)[V^{(M-1),u}] - H^{\ell\ell}(M-1)[V^{(M-1),\ell}] - J[V^{(M),u}] = \psi^{(M)}.
\end{aligned}
$$

The first of these can be used to eliminate $V^{(m),u}$

$$
V^{(m),u} = V^{(m-1),\ell} - \zeta^{(m)}, \quad 1 \leq m \leq M,
$$

so that the latter equations become

$$
\begin{aligned}
&- G[V^{(0),\ell}] - H^{uu}(1)[V^{(0),\ell} - \zeta^{(1)}] - H^{u\ell}(1)[V^{(1),\ell}] = \psi^{(1)}, \\
&- H^{\ell u}(m-1)[V^{(m-2),\ell} - \zeta^{(m-1)}] - H^{\ell\ell}(m-1)[V^{(m-1),\ell}] \\
&\quad - H^{uu}(m)[V^{(m-1),\ell} - \zeta^{(m)}] - H^{u\ell}(m)[V^{(m),\ell}] = \psi^{(m)}, \qquad 2 \leq m \leq M-1,
\end{aligned}
$$

$$
\begin{aligned}
&- H^{\ell u}(M-1)[V^{(M-2),\ell} - \zeta^{(M-1)}] - H^{\ell\ell}(M-1)[V^{(M-1),\ell}] \\
&\quad - J[V^{(M-1),\ell} - \zeta^{(M)}] = \psi^{(M)},
\end{aligned}
$$

or

$$
\begin{aligned}
&\left\{G + H^{uu}(1)\right\}[V^{(0),l}] + H^{u\ell}(1)[V^{(1),l}] \\
&\quad = -\psi^{(1)} + H^{uu}(1)[\zeta^{(1)}], \\
&H^{\ell u}(m-1)[V^{(m-2),l}] + \left\{H^{\ell\ell}(m-1) + H^{uu}(m)\right\}[V^{(m-1),l}] + H^{u\ell}(m)[V^{(m),l}] \\
&\quad = -\psi^{(m)} + H^{\ell u}(m-1)[\zeta^{(m-1)}] + H^{uu}(m)[\zeta^{(m)}], \qquad 2 \le m \le M-1, \\
&H^{\ell u}(M-1)[V^{(M-2),l}] + \left\{H^{\ell\ell}(M-1) + J\right\}[V^{(M-1),l}] \\
&\quad = -\psi^{(M)} + H^{\ell u}(M-1)[\zeta^{(M-1)}] + J[\zeta^{(M)}].
\end{aligned}
$$

Stated more compactly these read

$$
(\mathbf{L} + \mathbf{D} + \mathbf{U})\mathbf{V}^{\ell} = \mathbf{A}\mathbf{V}^{\ell} = \mathbf{R}, \tag{2.8}
$$

where

$$
\mathbf{V}^{\ell} := \begin{pmatrix} V^{(0),l} \\ V^{(1),l} \\ \vdots \\ V^{(M-2),l} \\ V^{(M-1),l} \end{pmatrix}, \tag{2.9}
$$

and

$$
\mathbf{R} := -\begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \\ \vdots \\ \psi^{(M-1)} \\ \psi^{(M)} \end{pmatrix} + \begin{pmatrix} H^{uu}(1)[\zeta^{(1)}] \\ H^{uu}(2)[\zeta^{(2)}] \\ \vdots \\ H^{uu}(M-1)[\zeta^{(M-1)}] \\ J[\zeta^{(M)}] \end{pmatrix} + \begin{pmatrix} 0 \\ H^{\ell u}(1)[\zeta^{(1)}] \\ \vdots \\ H^{\ell u}(M-2)[\zeta^{(M-2)}] \\ H^{\ell u}(M-1)[\zeta^{(M-1)}] \end{pmatrix}, \tag{2.10}
$$

and

$$
\mathbf{A} = \begin{pmatrix}
\mathbf{D}(1) & \mathbf{U}(1) & 0 & 0 & \cdots & 0 \\
\mathbf{L}(2) & \mathbf{D}(2) & \mathbf{U}(2) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mathbf{L}(M-1) & \mathbf{D}(M-1) & \mathbf{U}(M-1) \\
0 & \cdots & 0 & 0 & \mathbf{L}(M) & \mathbf{D}(M)
\end{pmatrix}, \tag{2.11a}
$$

where

$$\mathbf{U}(m) = H^{u\ell}(m), \qquad 1 \leq m \leq M-1, \tag{2.11b}$$

$$\mathbf{D}(m) = \begin{cases} G + H^{uu}(1), & m = 1, \\ H^{\ell\ell}(m-1) + H^{uu}(m), & 2 \leq m \leq M-1, \\ H^{\ell\ell}(M-1) + J, & m = M, \end{cases} \tag{2.11c}$$

$$\mathbf{L}(m) = H^{\ell u}(m-1), \qquad 2 \leq m \leq M. \tag{2.11d}$$

3. Analyticity of solutions of linear systems

In order to establish our desired results we consider quite general systems of linear equations of the form

$$\mathbf{A}(\varepsilon)\mathbf{V}(\varepsilon) = \mathbf{R}(\varepsilon), \tag{3.1}$$

and show how our equations, (2.8), which are clearly of this form, can be solved by regular perturbation theory. For this perturbative approach we provide two theorems which show that unique solutions exist, the first giving a sufficient result with a simpler methodology while the second gives a stronger result with greater complexity.

To begin, we consider $\varepsilon \in \mathbf{R}$ which is meant to model the situation where the geometry of the configuration is parameterized by *one* parameter (height/slope)

$$g^{(1)} = \varepsilon f^{(1)}, \quad g^{(2)} = \varepsilon f^{(2)}, \quad \dots, \quad g^{(M)} = \varepsilon f^{(M)}.$$

We then proceed to the more general case $\varepsilon \in \mathbf{R}^M$ in which each interface is perturbed by an *independent* parameter

$$g^{(1)} = \varepsilon_1 f^{(1)}, \quad g^{(2)} = \varepsilon_2 f^{(2)}, \quad \dots, \quad g^{(M)} = \varepsilon_M f^{(M)}.$$

3.1. A single parameter

In the single parameter case we assume

$$\mathbf{A}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{A}_n \varepsilon^n, \quad \mathbf{R}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{R}_n \varepsilon^n,$$

in (3.1) and seek a solution of the form

$$\mathbf{V}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{V}_n \varepsilon^n. \tag{3.2}$$

From (3.1) we find at order $\mathcal{O}(\varepsilon^n)$

$$\mathbf{A}_0 \mathbf{V}_n = \mathbf{R}_n - \sum_{\ell=0}^{n-1} \mathbf{A}_{n-\ell} \mathbf{V}_\ell,$$

or

$$\mathbf{V}_n = \mathbf{A}_0^{-1} \left[\mathbf{R}_n - \sum_{\ell=0}^{n-1} \mathbf{A}_{n-\ell} \mathbf{V}_\ell \right]. \tag{3.3}$$

With these we can establish the following theorem.

Theorem 3.1. *Given two Banach spaces X and Y, suppose that:*

1. $\mathbf{R}_n \in Y$ *for all* $n \geq 0$*, and there exist constants* $C_R > 0$*,* $B_R > 0$ *such that*

$$\|\mathbf{R}_n\|_Y \leq C_R B_R^n,$$

2. $\mathbf{A}_n : X \to Y$ *for all* $n \geq 0$*, and there exist constants* $C_A > 0$*,* $B_A > 0$ *such that*

$$\|\mathbf{A}_n\|_{X \to Y} \leq C_A B_A^n,$$

3. $\mathbf{A}_0^{-1} : Y \to X$*, and there exists a constant* $C_e > 0$ *such that*

$$\left\| \mathbf{A}_0^{-1} \right\|_{Y \to X} \leq C_e.$$

Then the equation (3.1) has a unique solution, (3.2), and there exist constants $C_V > 0$ and $B_V > 0$ such that

$$\|\mathbf{V}_n\|_X \leq C_V B_V^n, \tag{3.4}$$

for all $n \geq 0$ and any

$$C_V \geq 2 C_e C_R, \quad B_V \geq \max\left\{ B_R, 2B_A, 4 C_e C_A B_A \right\}.$$

This implies that, for any $0 \leq \rho < 1$, (3.2) converges for all ε such that $B\varepsilon < \rho$, i.e., $\varepsilon < \rho/B$.

Proof of Theorem 3.1. We work by induction, beginning with $n = 0$. At this order (3.3) yields

$$\mathbf{V}_0 = \mathbf{A}_0^{-1} \mathbf{R}_0,$$

and, from the properties of $\mathbf{A}_0^{-1}$, we have

$$\|\mathbf{V}_0\|_X = \left\| \mathbf{A}_0^{-1} \mathbf{R}_0 \right\|_X \leq C_e \|\mathbf{R}_0\|_Y =: C_V.$$

Now, assuming estimate (3.4) for all $n < \bar{n}$ we use (3.3) and the mapping properties of $\mathbf{A}_0^{-1}$ to give

$$\|\mathbf{V}_{\bar{n}}\|_X \leq C_e \left\{ \|\mathbf{R}_{\bar{n}}\|_Y + \sum_{\ell=0}^{\bar{n}-1} \|\mathbf{A}_{\bar{n}-\ell} \mathbf{V}_\ell\|_Y \right\}.$$

Now, using the estimates on $\mathbf{R}_n$ and $\mathbf{A}_n$ (for all n), and $\mathbf{V}_n$ ($n < \bar{n}$) we have

$$\begin{aligned}
\|\mathbf{V}_{\bar{n}}\|_X &\leq C_e \left\{ C_R B_R^{\bar{n}} + \sum_{\ell=0}^{\bar{n}-1} C_A B_A^{\bar{n}-\ell} C_V B_V^{\ell} \right\} \\
&\leq C_e C_R B_R^{\bar{n}} + C_e C_A C_V \left(\frac{B_A}{B_V}\right) B_V^{\bar{n}} \sum_{\ell=0}^{\bar{n}-1} \left(\frac{B_A}{B_V}\right)^{\bar{n}-\ell-1} \\
&\leq C_e C_R B_R^{\bar{n}} + C_e C_A C_V \left(\frac{B_A}{B_V}\right) B_V^{\bar{n}} \left(\frac{1}{1-1/2}\right),
\end{aligned}$$

if $B_A/B_V \leq 1/2$ (implying $B_V \geq 2B_A$). We are done if we demand that

$$B_V \geq B_R, \quad C_e C_R \leq C_V/2, \quad 2C_e C_A C_V (B_A/B_V) \leq C_V/2.$$

All of this can be achieved if

$$C_V \geq 2C_e C_R, \quad B_V \geq \max\{B_R, 2B_A, 4C_e C_A B_A\}. \quad \square$$

3.2. Multiple parameters

For the multiple parameter case of (3.1) we consider

$$\mathbf{A}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{A}_n \varepsilon^n, \quad \mathbf{R}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{R}_n \varepsilon^n,$$

using multi-index notation [33]

$$\varepsilon := \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_M \end{pmatrix}, \quad n := \begin{pmatrix} n_1 \\ \vdots \\ n_M \end{pmatrix},$$

and the convention

$$\sum_{n=0}^{\infty} \mathbf{A}_n \varepsilon^n := \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \mathbf{A}_{n_1,\ldots,n_M} \varepsilon_1^{n_1} \ldots \varepsilon_M^{n_M}.$$

We seek a solution of the form

$$\mathbf{V}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{V}_n \varepsilon^n, \tag{3.5}$$

and, from (3.1), we find at order $\mathcal{O}(\varepsilon^n)$

$$\mathbf{A}_0 \mathbf{V}_n = \mathbf{R}_n - \left(\sum_{\ell=0}^{n} \mathbf{A}_{n-\ell} \mathbf{V}_\ell - \mathbf{A}_0 \mathbf{V}_n \right),$$

or

$$\mathbf{V}_n = \mathbf{A}_0^{-1} \left[\mathbf{R}_n - \left(\sum_{\ell=0}^{n-1} \mathbf{A}_{n-\ell} \mathbf{V}_\ell - \mathbf{A}_0 \mathbf{V}_n \right) \right]. \tag{3.6}$$

In these we use the notation

$$\sum_{\ell=0}^{n} \mathbf{A}_{n-\ell} \mathbf{V}_\ell = \sum_{\ell_1=0}^{n_1} \cdots \sum_{\ell_M=0}^{n_M} \mathbf{A}_{n_1-\ell_1,\dots,n_M-\ell_M} \mathbf{V}_{\ell_1,\dots,\ell_M},$$

for $n = (n_1, \dots, n_M)$. Remembering the multi-index notation $0 = (0, \dots, 0)$ and the convention

$$n \geq 0 \quad \Longleftrightarrow \quad n_1 \geq 0, \dots, n_M \geq 0,$$

we can establish the following theorem.

Theorem 3.2. *Given two Banach spaces X and Y, suppose that:*

1. $\mathbf{R}_n \in Y$ *for all* $n \geq 0$, *and there exist multi-indexed constants* $C_R > 0$, $B_R > 0$ *such that*

$$\|\mathbf{R}_n\|_Y \leq C_R B_R^n,$$

2. $\mathbf{A}_n : X \to Y$ *for all* $n \geq 0$, *and there exist multi-indexed constants* $C_A > 0$, $B_A > 0$ *such that*

$$\|\mathbf{A}_n\|_{X \to Y} \leq C_A B_A^n,$$

3. $\mathbf{A}_0^{-1} : Y \to X$, *and there exists a constant* $C_e > 0$ *such that*

$$\left\| \mathbf{A}_0^{-1} \right\|_{Y \to X} \leq C_e.$$

Then the equation (3.1) has a unique solution, (3.5), and there exist multi-indexed constants $C_V > 0$ and $B_V > 0$ such that

$$\|\mathbf{V}_n\|_X \leq C_V B_V^n, \tag{3.7}$$

for all $n \geq 0$ and any

$$C_V \geq 2C_e C_R, \quad B \geq \max\{B_R, 2B_A, 4C_e C_A B_A\},$$

enforced componentwise. This implies that, for any $0 \leq \rho < 1$, (3.5) converges for all ε such that $B\varepsilon < \rho$, i.e., $\varepsilon < \rho/B$.

Proof of Theorem 3.2. We will focus upon the case $M = 2$; $M > 2$ (but finite) is a straightforward, though tedious, generalization. Thus, we seek to establish

$$\left\| \mathbf{V}_{n_1,n_2} \right\|_X \leq C_{V,1} C_{V,2} B_{V,1}^{n_1} B_{V,2}^{n_2}, \quad \forall n_1, n_2 \geq 0.$$

We prove this via an induction on n_2. The base case $n_2 = 0$:

$$\left\|\mathbf{V}_{n_1,0}\right\|_X \le C_{V,1} B_{V,1}^{n_1}, \quad \forall n_1 \ge 0,$$

is a special case of Theorem 3.1 with $\varepsilon = \varepsilon_1$. We now assume

$$\left\|\mathbf{V}_{n_1,n_2}\right\|_X \le C_{V,1} C_{V,2} B_{V,1}^{n_1} B_{V,2}^{n_2}, \quad \forall n_1 \ge 0, \quad \forall n_2 < \bar{n}_2,$$

and seek

$$\left\|\mathbf{V}_{n_1,\bar{n}_2}\right\|_X \le C_{V,1} C_{V,2} B_{V,1}^{n_1} B_{V,2}^{\bar{n}_2}, \quad \forall n_1 \ge 0.$$

This we establish by a second induction, this time on n_1. The base case $n_1 = 0$:

$$\left\|\mathbf{V}_{0,\bar{n}_2}\right\|_X \le C_{V,2} B_{V,2}^{\bar{n}_2}, \quad \forall \bar{n}_2 \ge 0,$$

is a special case of Theorem 3.1 with $\varepsilon = \varepsilon_2$. Finally, we assume

$$\left\|\mathbf{V}_{n_1,\bar{n}_2}\right\|_X \le C_{V,1} C_{V,2} B_{V,1}^{n_1} B_{V,2}^{\bar{n}_2}, \quad \forall n_1 < \bar{n}_1,$$

and seek

$$\left\|\mathbf{V}_{\bar{n}_1,\bar{n}_2}\right\|_X \le C_{V,1} C_{V,2} B_{V,1}^{\bar{n}_1} B_{V,2}^{\bar{n}_2}.$$

We now use (3.6) and the mapping properties of $\mathbf{A}_0^{-1}$ to realize

$$\left\|\mathbf{V}_{\bar{n}_1,\bar{n}_2}\right\|_X \le C_e \left\{ \left\|\mathbf{R}_{\bar{n}_1,\bar{n}_2}\right\|_Y + \sum_{|\ell|=0}^{|\bar{n}|-1} \left\|\mathbf{A}_{|\bar{n}|-\ell} \mathbf{V}_\ell\right\|_Y \right\}.$$

Using the estimates on $\mathbf{R}_{n_1,n_2}$ and $\mathbf{A}_{n_1,n_2}$ (for all n_1, n_2), and $\mathbf{V}_{n_1,n_2}$ ($n_1 < \bar{n}_1$, $n_2 < \bar{n}_2$) we have

$$\begin{aligned}
\left\|\mathbf{V}_{\bar{n}_1,\bar{n}_2}\right\|_X &\le C_e \left\{ C_{R,1} C_{R,2} B_{R,1}^{\bar{n}_1} B_{R,2}^{\bar{n}_2} + \sum_{|\ell|=0}^{|\bar{n}|-1} C_{A,1} C_{A,2} B_{A,1}^{\bar{n}_1-\ell_1} B_{A,2}^{\bar{n}_2-\ell_2} C_{V,1} C_{V,2} B_{V,1}^{\ell_1} B_{V,2}^{\ell_2} \right\} \\
&\le C_e C_{R,1} C_{R,2} B_{R,1}^{\bar{n}_1} B_{R,2}^{\bar{n}_2} + C_e C_{A,1} C_{A,2} C_{V,1} C_{V,2} \\
&\quad \times \left(\frac{B_{A,1}}{B_{V,1}}\right) B_{V,1}^{\bar{n}_1} \left(\frac{B_{A,2}}{B_{V,2}}\right) B_{V,2}^{\bar{n}_2} \sum_{|\ell|=0}^{|\bar{n}|-1} \left(\frac{B_{A,1}}{B_{V,1}}\right)^{\bar{n}_1-\ell_1-1} \left(\frac{B_{A,2}}{B_{V,2}}\right)^{\bar{n}_2-\ell_2-1} \\
&\le C_e C_{R,1} C_{R,2} B_{V,1}^{\bar{n}_1} B_{V,2}^{\bar{n}_2} + C_e C_{A,1} C_{A,2} C_{V,1} C_{V,2} \\
&\quad \times \left(\frac{B_{A,1}}{B_{V,1}}\right) B_{V,1}^{\bar{n}_1} \left(\frac{B_{A,2}}{B_{V,2}}\right) B_{V,2}^{\bar{n}_2} \left(\frac{1}{1-1/2}\right)^2,
\end{aligned}$$

if $B_{A,j}/B_{V,j} \leq 1/2$, $j = 1, 2$ (implying $B_{V,j} \geq 2B_{A,j}$). We are done if we demand that

$$B_{V,j} \geq B_{R,j}, \quad C_e C_{R,j} \leq C_{V,j}/2, \quad 2C_e C_{A,j} C_{V,j} (B_{A,j}/B_{V,j}) \leq C_{V,j}/2.$$

This can be realized if

$$C_{V,j} \geq 2C_e C_{R,j}, \quad B_{V,j} \geq \max\left\{B_{R,j}, 2B_{A,j}, 4C_e C_{A,j} B_{A,j}\right\}. \quad \square$$

4. Analyticity of solutions

We recall the surface formulation of our scattering problem,

$$(\mathbf{L} + \mathbf{D} + \mathbf{U})\mathbf{V}^\ell = \mathbf{A}\mathbf{V}^\ell = \mathbf{R},$$

cf. (2.8), where the known function $\mathbf{R}$ and operator $\mathbf{A}$ are specified in (2.10) and (2.11), respectively, and the vector of *unknowns* $\mathbf{V}^\ell$ is defined in (2.9). As we mentioned in the Introduction, our solution procedure is perturbative in nature and we will simply invoke Theorem 3.2 from § 3 to realize our desired result (which will deliver the conclusions of Theorem 3.1 as a special case). For this we formally expand

$$\mathbf{A}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{A}_n \varepsilon^n, \quad \mathbf{R}(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{R}_n \varepsilon^n,$$

which we presently justify rigorously, and seek a solution to (2.8) in the form

$$\mathbf{V}^\ell(\varepsilon) = \sum_{n=0}^{\infty} \mathbf{V}_n^\ell \varepsilon^n, \tag{4.1}$$

where $\varepsilon \in \mathbf{R}^M$.

To make our theorem precise we recall the classical L^2 based Sobolev spaces for $(d_x \times d_y)$-periodic *surface* functions with s-many derivatives [33] as

$$H^s(d) := \left\{\xi(x, y) \mid \xi \text{ is bounded, measurable, } \|\xi\|_{H^s} < \infty\right\},$$

where $d := [0, d_x] \times [0, d_y]$ and

$$\|\xi\|_{H^s}^2 := \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left|\hat{\xi}_{p,q}\right|^2 \langle (p, q)\rangle^2, \quad \langle (p, q)\rangle^2 := 1 + |p|^2 + |q|^2.$$

In addition, we require their vector-valued analogues

$$X^s(d) := \left\{ \boldsymbol{\xi} = \begin{pmatrix} \xi^{(1)}(x, y) \\ \vdots \\ \xi^{(M)}(x, y) \end{pmatrix} \middle| \xi^{(m)}(x, y) \in H^s(d) \right\},$$

with norm

$$\|\boldsymbol{\xi}\|_{X^s}^2 := \sum_{m=1}^{M} \left\| \xi^{(m)} \right\|_{H^s}^2 ,$$

with which we can now establish the following result. For this we will require the somewhat technical definitions of τ-allowable layer configuration, interface configuration, and medium configuration, which are found in Definitions 4.5, 4.9, and 4.11, respectively.

Theorem 4.1. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $f^{(m)} \in C^{s+3/2+\sigma}(d)$, $1 \leq m \leq M$, then, for a τ-allowable medium configuration, (4.7), the equation (2.8) has a unique solution, (4.1), and there exist constants $C > 0$ and $B > 0$ such that*

$$\left\| \mathbf{V}_n^{\ell} \right\|_{X^s} \leq C B^n ,$$

for all $n \geq 0$. However, $C = C(\tau) \to \infty$ as $\tau \to 0$. This implies that, for any $0 \leq \rho < 1$, (4.1) converges for all ε such that $B\varepsilon < \rho$, i.e., $\varepsilon < \rho/B$.

Proof of Theorem 4.1. As we mentioned above, our method of proof is to simply invoke Theorem 3.2, thus we must verify the relevant hypotheses. To begin, we consider the spaces

$$X = X^{s+1}, \quad Y = X^s .$$

In § 4.3 we will show that $\zeta^{(1)}$ and $\psi^{(1)}$ can be expanded in Taylor series which converge strongly in the spaces H^s for any $s \geq 0$ provided that $f^{(1)} \in C^{s+1}(d) \subset C^{s+3/2+\sigma}(d)$. This clearly implies that the $\mathbf{R}_n$ satisfy the estimates of Item 1 in Theorem 3.2.

In § 5 we show that the DNOs G, J, and $H(m)$ are analytic in boundary perturbations $f^{(m)} \in C^{s+3/2+\sigma}(d)$ provided that each layer is a τ-allowable layer configuration, (4.3). With these, it is clear that the $\mathbf{A}_n$ satisfy the estimates of Item 2 in Theorem 3.2.

Finally, in § 4.1 we show that the estimates and mapping properties of $\mathbf{A}_0^{-1}$ for Item 3 in Theorem 3.2 hold true provided that we are in a τ-allowable medium configuration, (4.7). □

Remark 4.2. The smoothness requirements, that $f^{(m)} \in C^{s+3/2+\sigma}(d)$, can be relaxed in exchange for a significantly more complicated demonstration and less convenient set of function spaces. The approach we have in mind was pursued by the author and Hu [37], and will deliver results which permit $f^{(m)}$ in the Schauder space $C^{1,\alpha}$ for any $\alpha > 0$, and even in the Lipschitz class.

Remark 4.3. The result above requires that ε be sufficiently small and we certainly cannot improve upon this. However, using the analysis of the author and Reitich [58] we can postulate about the *distribution* of poles in the complex plane. In particular, we believe that one should be able to demonstrate that the only singularities on the *real* axis are due to topological obstruction, e.g. choices of ε which produce cavitation of a layer.

4.1. The flat interface problem

As we just outlined, the key to our developments (as with all regular perturbation arguments) is the flat-interface version of (2.8)

$$\mathbf{A}_0 \mathbf{V}_0^\ell = \mathbf{R}_0,$$

where $\mathbf{A}_0 = \mathbf{A}(0)$, $\mathbf{V}_0^\ell = \mathbf{V}^\ell(0)$, and $\mathbf{R}_0 = \mathbf{R}(0)$, in particular the invertibility of $\mathbf{A}_0$ and the mapping properties of $\mathbf{A}_0^{-1}$. As we recall from (2.10) and (2.11), $\mathbf{R}_0$ and $\mathbf{A}_0$ consist of the operators G_0, J_0, and $H_0(m)$; the functions ζ_0 and ψ_0; and combinations of these.

It is not difficult to show, for plane-wave incidence from above, that

$$\zeta_0^{(1)} = -e^{i\alpha x + i\beta y}, \quad \psi_0^{(1)} = \left(i\gamma^{(0)}\right) e^{i\alpha x + i\beta y},$$

and $\zeta_0^{(m)} \equiv \psi_0^{(m)} \equiv 0$ for $m = 2, \dots, M$. (Again, nothing essential changes if all of the $\zeta_0^{(m)}$ and $\psi_0^{(m)}$ are non-zero so we do not eliminate them from our equations.) We demonstrated in [54] that

$$G_0 = -i\gamma_D^{(0)}, \quad J_0 = -i\gamma_D^{(M)},$$

where we have again used Fourier multiplier notation, i.e.,

$$m(D)\xi(x, y) = m(D_x, D_y)\xi(x, y) := \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} m(p, q)\hat{\xi}_{p,q} e^{i\alpha_p x + i\beta_q y},$$

where

$$\hat{\xi}_{p,q} := \frac{1}{d_x d_y} \int_0^{d_x} \int_0^{d_y} \xi(x, y) e^{-i\alpha_p x - i\beta_q y} \, dx \, dy.$$

Additionally, in [54] we derived that

$$H_0(m) = (i\gamma_D^{(m)}) \begin{pmatrix} \coth(i\gamma_D^{(m)} h^{(m)}) & -\operatorname{csch}(i\gamma_D^{(m)} h^{(m)}) \\ -\operatorname{csch}(i\gamma_D^{(m)} h^{(m)}) & \coth(i\gamma_D^{(m)} h^{(m)}) \end{pmatrix},$$

where $h^{(m)} := a^{(m)} - a^{(m+1)} > 0$.

Remark 4.4. To clarify this formula we note that (2.2) specifies not only that

$$\gamma_{p,q}^{(m)} \geq 0, \quad (p, q) \in \mathcal{U}^{(m)},$$

but also

$$\gamma_{p,q}^{(m)} = i\tilde{\gamma}_{p,q}^{(m)}, \quad \tilde{\gamma}_{p,q}^{(m)} > 0, \quad (p, q) \notin \mathcal{U}^{(m)},$$

which defines the positive real parameter $\tilde{\gamma}_{p,q}^{(m)}$. Seeking the *symbol* of $H_0(m)$, namely $\widehat{H_0(m)}_{p,q}$, we consider three cases for the quantity $\gamma_{p,q}^{(m)}$ separately

1. For $(p,q) \in \mathcal{U}^{(m)}$ and $\gamma_{p,q}^{(m)} > 0$,

$$\widehat{H_0(m)}_{p,q} = \gamma_{p,q}^{(m)} \begin{pmatrix} \cot\left(\gamma_{p,q}^{(m)} h^{(m)}\right) & -\csc\left(\gamma_{p,q}^{(m)} h^{(m)}\right) \\ -\csc\left(\gamma_{p,q}^{(m)} h^{(m)}\right) & \cot\left(\gamma_{p,q}^{(m)} h^{(m)}\right) \end{pmatrix}, \tag{4.2a}$$

   which is well-defined for

$$\gamma_{p,q}^{(m)} h^{(m)} \neq n\pi, \quad n \in \mathbf{Z}\backslash\{0\};$$

   we exclude $n = 0$ as this is handled in the next case.
2. For $\gamma_{p,q}^{(m)} = 0$ we have

$$\widehat{H_0(m)}_{p,q} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{4.2b}$$

   which is clearly free of singularities.
3. Finally, for $(p,q) \notin \mathcal{U}^{(m)}$,

$$\widehat{H_0(m)}_{p,q} = \tilde{\gamma}_{p,q}^{(m)} \begin{pmatrix} \coth\left(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}\right) & -\operatorname{csch}\left(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}\right) \\ -\operatorname{csch}\left(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}\right) & \coth\left(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}\right) \end{pmatrix}, \tag{4.2c}$$

   which is *always* well-defined.

Turning to the conditions which will ensure that $H_0(m)$ is well-defined, we now specifically exclude the "Dirichlet eigenvalues" [23], the $(p,q) \in \mathcal{U}^{(m)}$ where $\gamma_{p,q}^{(m)} h^{(m)}$ is a non-zero integer multiple of π. More precisely, we make the following definition to prohibit these.

Definition 4.5. A *layer configuration*, $\mathcal{C}_\ell^{(m)}$, of the m-th layer is a sextuple

$$\mathcal{C}_\ell^{(m)} := (d_x, d_y, h^{(m)}, \alpha, \beta, \gamma^{(m)}).$$

For any $\tau > 0$, the set of *τ-allowable layer configurations* is defined by

$$\mathcal{L}_\tau^{(m)} := \left\{ \mathcal{C}_\ell^{(m)} \middle| \ \left| \sin(\gamma_{p,q}^{(m)} h^{(m)}) \right| > \tau, \quad \forall (p,q) \in \mathcal{U}^{(m)} \right\}. \tag{4.3}$$

Remark 4.6. We shall only consider layer configurations $\mathcal{C}_\ell^{(m)} \in \mathcal{L}_\tau^{(m)}$, $m = 1, \ldots, M-1$, so that all of the operators $H_0(m)$ are well defined. We note that the operators G_0 and J_0 are well-defined for *all* configurations of the upper and lower layers.

Remark 4.7. We note that these resonances could be removed with the choice of operators different from the DNOs. One possible choice are the Impedance-to-Impedance operators first suggested by Monk and Wang [53] which were designed specifically for this issue.

With this we can prove the following lemma.

Lemma 4.8. *Given any integer $s \geq 0$ and $\tau > 0$ we have, for a τ-allowable layer configuration $\mathcal{C}_\ell^{(m)} \in \mathcal{L}_\tau^{(m)}$,*

1. $\zeta_0^{(m)}, \psi_0^{(m)} \in H^s(d)$,
2. $\{G_0, J_0\} : H^{s+1}(d) \to H^s(d)$,
3. $\{H_0^{uu}(m), H_0^{u\ell}(m), H_0^{\ell u}(m), H^{\ell\ell}(m)\} : H^{s+1}(d) \to H^s(d)$ *for all* $1 \leq m \leq M-1$,
4. $H_0^{u\ell}(m), H_0^{\ell u}(m)$ *are compact for all* $1 \leq m \leq M-1$.

Proof of Lemma 4.8. The proof of (1) is trivial upon consideration of the fact that $\zeta_0^{(1)}$ and $\psi_0^{(1)}$ consist of a single Fourier coefficient each, while $\zeta_0^{(m)}$ and $\psi_0^{(m)}$ are identically zero for $m = 2, \ldots, M$.

The proof of (2) comes easily from the symbols

$$\widehat{(G_0)}_{p,q} = -i\gamma_{p,q}^{(0)}, \quad \widehat{(J_0)}_{p,q} = -i\gamma_{p,q}^{(M)},$$

and the asymptotic behavior

$$(-i\gamma_{p,q}^{(0)}) \sim \sqrt{p^2 + q^2}, \qquad \langle (p,q) \rangle \to \infty,$$
$$(-i\gamma_{p,q}^{(M)}) \sim \sqrt{p^2 + q^2}, \qquad \langle (p,q) \rangle \to \infty.$$

The proof of (3) is not much more difficult given the forms for $H_0(m)$, (4.2), and their behavior

$$(i\gamma_{p,q}^{(m)}) \coth(i\gamma_{p,q}^{(m)} h^{(m)}) \sim \sqrt{p^2 + q^2}, \qquad \langle (p,q) \rangle \to \infty,$$
$$(-i\gamma_{p,q}^{(m)}) \operatorname{csch}(i\gamma_{p,q}^{(m)} h^{(m)}) \sim 0, \qquad \langle (p,q) \rangle \to \infty.$$

The layers require a little more attention (due to the Dirichlet eigenvalues) but one can estimate

$$\left| (i\gamma_{p,q}^{(m)}) \coth(i\gamma_{p,q}^{(m)} h^{(m)}) \right|^2 \leq C \langle (p,q) \rangle^2 \begin{cases} \tau^{-2}, & (p,q) \in \mathcal{U}^{(m)}, \\ C_0, & (p,q) \notin \mathcal{U}^{(m)}, \end{cases}$$

$$\left| (i\gamma_{p,q}^{(m)}) \operatorname{csch}(i\gamma_{p,q}^{(m)} h^{(m)}) \right|^2 \leq C \langle (p,q) \rangle^2 \begin{cases} \tau^{-2}, & (p,q) \in \mathcal{U}^{(m)}, \\ 1, & (p,q) \notin \mathcal{U}^{(m)}, \end{cases}$$

for some $C > 0$ provided we are in a τ*-allowable layer configuration*. In these we have used $\operatorname{csch}(z) \leq 1$ for $z \in \mathbf{R}$, and, for $z_0 > 0$,

$$\coth(z) \leq \coth(z_0), \quad z \in \mathbf{R}, \quad z \geq z_0 > 0.$$

The latter is helpful by setting

$$z_0 := \min_{(p,q)\notin\mathcal{U}^{(m)}} \left\{ \tilde{\gamma}_{p,q}^{(m)} h^{(m)} \right\} > 0, \quad C_0 := \coth(z_0).$$

With these we can show, for instance,

$$\begin{aligned}
\left\| H_0^{uu}(m)\xi \right\|_{H^s}^2 &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left| (i\gamma_{p,q}^{(m)}) \coth(i\gamma_{p,q}^{(m)} h^{(m)}) \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s} \\
&= \sum_{(p,q)\in\mathcal{U}^{(m)}} \left| \gamma_{p,q}^{(m)} \cot(\gamma_{p,q}^{(m)} h^{(m)}) \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s} \\
&\quad + \sum_{(p,q)\notin\mathcal{U}^{(m)}} \left| \tilde{\gamma}_{p,q}^{(m)} \coth(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}) \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s} \\
&\leq \sum_{(p,q)\in\mathcal{U}^{(m)}} \frac{C}{\tau^2} \left| \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s} \\
&\quad + \sum_{(p,q)\notin\mathcal{U}^{(m)}} C C_0 \langle (p,q) \rangle^2 \left| \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s} \\
&\leq C(\tau) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left| \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2(s+1)} \\
&\leq C(\tau) \left\| \xi \right\|_{H^{s+1}}^2,
\end{aligned}$$

where C is a constant (which may change from one line to the next), that behaves like τ^{-2} as $\tau \to 0$.

To show compactness of the operators $H_0^{u\ell} = H_0^{\ell u}$, (4), we follow the approach of Kress [42] and approximate them by finite dimensional range operators (which are compact, see Theorem 2.23 of [42]). We then show that these are norm convergent to $H_0^{u\ell} = H_0^{\ell u}$ which demonstrates that these are compact as well (Theorem 2.22 of [42]). We choose the natural approximating operator of finite dimensional range: The truncation after (P, Q) Fourier modes,

$$H_{0,P,Q}^{u\ell}[\xi] := \sum_{|p|\leq P} \sum_{|q|\leq Q} -(i\gamma_{p,q}^{(m)}) \operatorname{csch}(i\gamma_{p,q}^{(m)} h^{(m)}) \hat{\xi}_{p,q} e^{i\alpha_p x + i\beta_q y}. \tag{4.4}$$

Now, to show norm convergence we assume that P, Q are large enough so that

$$\gamma_{p,q} \notin \mathcal{U}^{(m)}, \quad p \geq P, \quad q \geq Q,$$

which allows us to compute

$$\left\| (H_0^{u\ell} - H_{0,P,Q}^{u\ell})\xi \right\|_{H^s}^2 = \sum_{|p|>P} \sum_{|q|>Q} \left| -\tilde{\gamma}_{p,q}^{(m)} \operatorname{csch}(\tilde{\gamma}_{p,q}^{(m)} h^{(m)}) \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s}$$

$$= \sum_{|p|>P} \sum_{|q|>Q} \left| \tilde{\gamma}_{p,q}^{(m)} \right|^2 C \left| \tilde{\gamma}_{p,q}^{(m)} h^{(m)} \right|^{-2r} \left| \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2s},$$

for any $r > 0$ (since $\operatorname{csch}(x) \leq C x^{-r}$ for any $r > 0$). Continuing,

$$\left\| (H_0^{u\ell} - H_{0,P,Q}^{u\ell})\xi \right\|_{H^s}^2 \leq \langle (P,Q) \rangle^{-2r} \sum_{|p|>P} \sum_{|q|>Q} \left| \hat{\xi}_{p,q} \right|^2 \langle (p,q) \rangle^{2(s+1)}$$

$$\leq \langle (P,Q) \rangle^{-2r} \left\| \xi \right\|_{H^{s+1}}^2,$$

which clearly tends to zero as $P^2 + Q^2 \to 0$ showing norm convergence. $\square$

Now, to deal with the fully layered medium problem we must consider other potential resonances (in addition to the Dirichlet eigenvalues) which may arise from the interaction of waves between different layers. To quantify these we make the following extensions to Definition 4.5.

Definition 4.9. An *interface configuration*, $\mathcal{C}_i^{(m)}$, of the m-th interface is an octuple

$$\mathcal{C}_i^{(m)} := (d_x, d_y, h^{(m-1)}, h^{(m)}, \alpha, \beta, \gamma^{(m-1)}, \gamma^{(m)}).$$

For any $\tau > 0$, the set of τ-*allowable interface configurations* is defined by

$$\mathcal{I}_\tau^{(m)} := \left\{ \mathcal{C}_i^{(m)} \,\middle|\, \left| \widehat{\mathbf{D}_0(m)}_{p,q} \right| > \tau, \quad \forall (p,q) \in \mathcal{U}^{(m)} \right\}, \tag{4.5}$$

where we recall that

$$\widehat{\mathbf{D}_0(m)}_{p,q} = \begin{cases} (-i\gamma_{p,q}^{(0)}) + (i\gamma_{p,q}^{(1)}) \coth(i\gamma_{p,q}^{(1)} h^{(1)}), & m = 1, \\ (i\gamma_{p,q}^{(m-1)}) \coth(i\gamma_{p,q}^{(m-1)} h^{(m-1)}) + (i\gamma_{p,q}^{(m)}) \coth(i\gamma_{p,q}^{(m)} h^{(m)}), & 2 \leq m \leq M-1, \\ (i\gamma_{p,q}^{(M-1)}) \coth(i\gamma_{p,q}^{(M-1)} h^{(M-1)}) + (-i\gamma_{p,q}^{(M)}), & m = M. \end{cases} \tag{4.6}$$

Remark 4.10. As before, we shall only consider interface configurations $\mathcal{C}_i^{(m)} \in \mathcal{I}_\tau^{(m)}$, $m = 1, \ldots, M$, so that all of the operators $\mathbf{D}_0(m)$ are well defined and invertible.

Definition 4.11. A *medium configuration*, $\mathcal{C}$, of a M-many layered medium is a $(2M+4)$-tuple

$$\mathcal{C} := (d_x, d_y, h^{(1)}, \ldots, h^{(M-1)}, \alpha, \beta, \gamma^{(0)}, \ldots, \gamma^{(M)}).$$

For any $\tau > 0$, the set of τ-*allowable structure configurations* is defined by

$$\mathcal{S}_\tau := \left\{ \mathcal{C} \,\middle|\, \left| \det\left\{ (\widehat{\mathbf{A}_0})_{p,q} \right\} \right| > \tau, \quad \forall\, (p,q) \in \mathbf{Z}^2 \right\},$$

where the determinant function $\det\left\{\widehat{(\mathbf{A}_0)}_{p,q}\right\}$ is theoretically straightforward to resolve, though laborious to compute. Finally, for any $\tau > 0$, the set of *τ-allowable medium configurations* is defined (with a slight abuse of notation) by

$$\mathcal{A}_\tau := \left(\bigcap_{m=1}^{M-1} \mathcal{L}_\tau^{(m)}\right) \bigcap \left(\bigcap_{m=1}^{M} \mathcal{I}_\tau^{(m)}\right) \bigcap \mathcal{S}_\tau. \tag{4.7}$$

Remark 4.12. At this point one can reasonably ask, "How severe are these constraints on a candidate configuration?" The answer is that they are not as onerous as they first appear.

- **Existence of DNOs:** To begin, the forbidden *layer* configurations correspond to the Dirichlet eigenvalues: Those for which

$$\sin(\gamma_{p,q}^{(m)} h^{(m)}) = 0 \quad \Longleftrightarrow \quad \gamma_{p,q}^{(m)} h^{(m)} = n\pi.$$

- **Invertibility of $\mathbf{D}_0(m)$:** The unacceptable set of *interface* configurations are those which support localized trapped modes. The only problematic wavenumbers must be in the *finite* set $\mathcal{U}^{(m-1)} \bigcup \mathcal{U}^{(m)}$, however, we can specify a prohibited interface. For instance, for $2 \leq m \leq M-1$, we have

$$\widehat{\mathbf{D}_0(m)}_{p,q} = \gamma_{p,q}^{(m-1)} \cot(\gamma_{p,q}^{(m-1)} h^{(m-1)}) + \gamma_{p,q}^{(m)} \cot(\gamma_{p,q}^{(m)} h^{(m)}),$$

  so that we can realize a zero value if

$$\gamma_{p,q}^{(m-1)} h^{(m-1)} = n\pi + \pi/2, \quad \text{and} \quad \gamma_{p,q}^{(m)} h^{(m-1)} = \ell\pi + \pi/2,$$

  for some integers n and ℓ. (Note that these are *not* Dirichlet eigenvalues.)
- **Invertibility of $\mathbf{A}_0$:** Finally, the set of acceptable *structures* are those for which the overall boundary value problem has condition number bounded below by $1/\tau$. As in the previous case, this determinant will be non-zero unless (p, q) is in a *finite* set

$$\mathcal{U} := \bigcup_{m=0}^{M} \mathcal{U}^{(m)},$$

  as the alternative guarantees that the tridiagonal matrix $\widehat{(\mathbf{A}_0)}_{p,q}$ is diagonally dominant.

With these we can prove the following Theorem.

Theorem 4.13. *Given any integer $s \geq 0$ and $\tau > 0$ we have, for a τ-allowable medium configuration $\mathcal{C} \in \mathcal{A}_\tau$,*

1. *The operator $\mathbf{D}_0^{-1}$ exists and maps $X^s(d)$ to $X^{s+1}(d)$,*
2. *The operator $\mathbf{L}_0 + \mathbf{U}_0$ is compact,*

3. *The operator* $\mathbf{A}_0^{-1}$ *exists and maps* $X^s(d)$ *to* $X^{s+1}(d)$*. More precisely, there exists a constant* $C_e = C_e(\tau)$ *such that*

$$\left\| \mathbf{A}_0^{-1} \boldsymbol{\xi} \right\|_{X^{s+1}} \leq C_e \left\| \boldsymbol{\xi} \right\|_{X^s} .$$

Proof of Theorem 4.13. The invertibility of $\mathbf{D}_0$ follows from the definition of the set $\mathcal{A}_\tau$ which demands membership in $\mathcal{I}_\tau^{(m)}$ for all $m = 1, \ldots, M$. The mapping property claimed in (1) follows readily from the asymptotic properties of $\widehat{\mathbf{D}_0(m)}_{p,q}$, cf. (4.6). More specifically, for $\langle (p,q) \rangle$ sufficiently large

$$\left| \widehat{\mathbf{D}_0(1)}_{p,q} \right| \sim \tilde{\gamma}_{p,q}^{(0)} + \tilde{\gamma}_{p,q}^{(1)} \coth\left(\tilde{\gamma}_{p,q}^{(1)} h^{(1)} \right) \sim \left| \gamma_{p,q}^{(0)} \right| + \left| \gamma_{p,q}^{(1)} \right|,$$

and, for $2 \leq m \leq M-1$,

$$\left| \widehat{\mathbf{D}_0(m)}_{p,q} \right| \sim \tilde{\gamma}_{p,q}^{(m-1)} \coth\left(\tilde{\gamma}_{p,q}^{(m-1)} h^{(m-1)} \right) + \tilde{\gamma}_{p,q}^{(m)} \coth\left(\tilde{\gamma}_{p,q}^{(m)} h^{(m)} \right) \sim \left| \gamma_{p,q}^{(m-1)} \right| + \left| \gamma_{p,q}^{(m)} \right|,$$

and

$$\left| \widehat{\mathbf{D}_0(M)}_{p,q} \right| \sim \tilde{\gamma}_{p,q}^{(M-1)} \coth\left(\tilde{\gamma}_{p,q}^{(M-1)} h^{(M-1)} \right) + \tilde{\gamma}_{p,q}^{(M)} \sim \left| \gamma_{p,q}^{(M-1)} \right| + \left| \gamma_{p,q}^{(M)} \right|.$$

Each of these delivers, for some $C > 0$,

$$\left| \widehat{\mathbf{D}_0(m)}_{p,q} \right|^2 \geq C \langle (p,q) \rangle^2.$$

The compactness result, (2), follows from the same reasoning used in the proof of Lemma 4.8, part (4). That is, approximation by the natural finite dimensional range operators (truncation after (P, Q) Fourier modes, cf. (4.4)) which are norm convergent to $\mathbf{L}_0 + \mathbf{U}_0$.

Finally, the invertibility of $\mathbf{A}_0$ and the estimate claimed in (3) follow from the classical Riesz Theory (see, e.g., Chapter 3 of Kress [42]). For instance, we may invoke Theorem 3.6 of Kress [42] which states the alternative that either

$$\mathbf{A}_0 \boldsymbol{\xi} = (\mathbf{D}_0 + \mathbf{L}_0 + \mathbf{U}_0)\, \boldsymbol{\xi} = 0 \tag{4.8}$$

has a non-trivial solution, or else, if $\mathbf{R}_0 \in X^s$ then

$$\mathbf{A}_0 \boldsymbol{\xi} = (\mathbf{D}_0 + \mathbf{L}_0 + \mathbf{U}_0)\, \boldsymbol{\xi} = \mathbf{R}_0,$$

has a unique solution in X^{s+1}. The former possibility is rendered impossible by the demand that the determinant of $\widehat{(\mathbf{A}_0)}_{p,q}$ is uniformly bounded away from zero. □

Remark 4.14. As we outlined above, our boundary perturbation approach to solving (2.8) requires a "base case" from which to perturb. The natural candidate is the flat interface configuration, however, in this instance the notion of a periodic grating loses its meaning. The *actual* flat interface problem is significantly more difficult as the wavenumbers (p, q) cannot be restricted to the (two-dimensional) lattice. However, in our approach we consider the "infinitesimally small" perturbed configuration as our base case which *can* be parameterized by the lattice (p, q) and is amenable to our analysis.

4.2. Previous results

Of course, rather than going to all of this work we could have simply applied to the theorems of [32] which were generalized to the three-dimensional acoustic wave setting in Chapter 2 of the beautifully clear and complete Habilitationsschrift of Arens [5]. More specifically, the reader will be rewarded by a careful study of Section 2.4 which presents the transmission problem we discuss here. In short, Arens considers the variational formulation of Dobson [29,30] and Bao [8] (see [5] for a much more extensive list of citations, particularly for the electromagnetics problem) and appeals to the Fredholm theory (see, e.g., Evans [33] and Kress [42]) for its solution. In brief, a "choice" arises which reduces the problem to one of ensuring uniqueness of solutions as we pursued above.

Defining V to be the space of (x, y)-quasiperiodic functions in $L^2(d \times [\underline{a}, \overline{a}])$ with first (weak) derivatives in $L^2(d \times [\underline{a}, \overline{a}])$, Arens [5] states the following result.

Corollary 4.15. *The scalar Transmission Problem is uniquely solvable except possibly for a sequence, $\{k_j\}$, of wavenumbers such that $k_j \to \infty$ as $j \to \infty$.*

With the existence of this result one can wonder why we expended so much effort to prove the theorems above, in particular Theorem 4.13. The answer is that since our method is perturbative in nature, the flat-interface (base) case determines the applicability of our result (Theorem 4.1). As a consequence we can study the unique solvability of the simple *trivial interface* problem to decide upon the utility of our theorem. This is characterized by the definition of the τ-allowable medium configuration, cf. Definition 4.11, so that *any* given configuration can be tested and assigned a value of τ. Arens' theorem is more general as it considers interfaces without reference to any "base configuration," however, it is impossible to test whether any given configuration features one of the prohibited k_j mentioned above. By contrast, our result lists three tests, (4.7), for membership in the set of allowable configurations which are readily computed.

4.3. Analyticity of the Dirichlet and Neumann data

We now study the analyticity properties of the Dirichlet, (2.1a), and Neumann data, (2.1b). For this we recall that $g = \varepsilon f$ (but $f^{(m)} \equiv 0$ for $2 \le m \le M$) and investigate the convergence of the Taylor series

$$\zeta^{(1)}(x, y; \varepsilon_1) = \sum_{n=0}^{\infty} \zeta_n^{(1)}(x, y)\varepsilon_1^n, \quad \psi^{(1)}(x, y; \varepsilon_1) = \sum_{n=0}^{\infty} \psi_n^{(1)}(x, y)\varepsilon_1^n.$$

It is not difficult to see that the terms in the Taylor series are given by

$$\zeta_n^{(1)}(x, y) = -\phi^{(1)}(x, y)\left(-i\gamma^{(0)}\right)^n F_n^{(1)}(x, y),$$

where

$$\phi^{(1)}(x,y) := e^{i(\alpha x+\beta y-\gamma^{(0)}a^{(1)})}, \quad F_n^{(1)}(x,y) := \frac{f^{(1)}(x,y)^n}{n!}.$$

Furthermore,

$$\begin{aligned}\psi_n^{(1)}(x,y) &= \phi^{(1)}(x,y)\left\{(i\alpha)(\partial_x f^{(1)}) + (i\beta)(\partial_y f^{(1)})\right\}\left(-i\gamma^{(0)}\right)^{n-1} F_{n-1}(x,y)\\ &\quad + \phi^{(1)}(x,y)(i\gamma^{(0)})\left(-i\gamma^{(0)}\right)^n F_n(x,y).\\ &= \left\{(i\alpha)(\partial_x f^{(1)}) + (i\beta)(\partial_y f^{(1)})\right\}\zeta_{n-1}^{(1)}(x,y) + (i\gamma^{(0)})\zeta_n^{(1)}(x,y).\end{aligned} \tag{4.9}$$

In order to establish our analyticity result we require the following "Algebra Property" of Sobolev spaces which permits us to estimate products of functions in these spaces. We refer the interested reader to Evans [33] or [56], for instance, for more details.

Lemma 4.16. *Given an integer $s \geq 0$ and any $\sigma > 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s(d)$, $w \in H^s(d)$ then $fw \in H^s(d)$ and*

$$\|fw\|_{H^s} \leq \mathcal{M}\,|f|_{C^s}\,\|w\|_{H^s},$$

and if $\tilde{f} \in C^{s+1/2+\sigma}(d)$, $\tilde{w} \in H^{s+1/2}(d)$ then $\tilde{f}\tilde{w} \in H^{s+1/2}(d)$ and

$$\left\|\tilde{f}\tilde{w}\right\|_{H^{s+1/2}} \leq \mathcal{M}\,|f|_{C^{s+1/2+\sigma}}\,\|w\|_{H^{s+1/2}}.$$

With this we can prove the following.

Lemma 4.17. *Given any integer $s \geq 0$ and any $\sigma > 0$, if $f^{(1)} \in C^{s+1}(d)$ then*

$$\left\|\zeta_n^{(1)}\right\|_{H^{s+1}} \leq K_\zeta^{(1)} B_\zeta^n, \tag{4.10a}$$

$$\left\|\psi_n^{(1)}\right\|_{H^s} \leq K_\psi^{(1)} B_\psi^n, \tag{4.10b}$$

for all $n \geq 0$ and constants $K_\zeta^{(1)}, K_\psi^{(1)}, B_\zeta, B_\psi > 0$.

Proof of Lemma 4.17. The proof of (4.10a) proceeds by induction and begins with $n = 0$ where we set

$$K_\zeta^{(1)} := \left\|\zeta_0^{(1)}\right\|_{H^{s+1}} = \left\|e^{i(\alpha x+\beta y-\gamma^{(0)}a^{(1)})}\right\|_{H^{s+1}}.$$

We now assume that (4.10a) holds for all $n < \bar{n}$ and note that

$$\zeta_{\bar{n}}^{(1)} = (-i\gamma^{(0)})\left(\frac{f^{(1)}}{\bar{n}}\right)\zeta_{\bar{n}-1}^{(1)}.$$

From this we find (using Lemma 4.16)

$$\left\| \zeta_{\bar{n}}^{(1)} \right\|_{H^{s+1}} \leq \left| \gamma^{(0)} \right| \frac{\mathcal{M}}{\bar{n}} \left| f^{(1)} \right|_{C^{s+1}} \left\| \zeta_{\bar{n}-1}^{(1)} \right\|_{H^{s+1}} \leq \left| \gamma^{(0)} \right| \mathcal{M} \left| f^{(1)} \right|_{C^{s+1}} K_\zeta B_\zeta^{\bar{n}-1},$$

and we are done provided

$$B_\zeta > \left| \gamma^{(0)} \right| \mathcal{M} \left| f^{(1)} \right|_{C^{s+1}}.$$

From (4.9) it is clear that the proof of (4.10b) follows in essentially the same manner. The only modification that one needs to make is to set

$$K_\psi^{(1)} := \left\| \psi_0^{(1)} \right\|_{H^s} = \left\| \left(i\gamma^{(0)} \right) e^{i(\alpha x + \beta y - \gamma^{(0)} a^{(1)})} \right\|_{H^s}. \quad \square$$

5. Analyticity of Dirichlet–Neumann operators

At this point we return to the analyticity of the DNOs G, J, and $H(m)$, which are fundamental to our analyticity proof presented above. The analyticity results for G and J have already been presented in [59], and we state them for completeness.

Theorem 5.1. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $\tilde{u}, \tilde{\ell} \in C^{s+3/2+\sigma}(d)$ then the series*

$$G(\varepsilon\tilde{\ell}) = \sum_{n=0}^{\infty} G_n(\tilde{\ell})\varepsilon^n, \quad J(\varepsilon\tilde{u}) = \sum_{n=0}^{\infty} J_n(\tilde{u})\varepsilon^n,$$

converge strongly as operators from $H^{s+1}(d)$ to $H^s(d)$. More precisely

$$\|G_n\|_{H^{s+1}\to H^s} \leq \tilde{K}_G^0 B_G^n, \quad \|J_n\|_{H^{s+1}\to H^s} \leq \tilde{K}_J^0 B_J^n,$$

for universal constants $\tilde{K}_G^0, \tilde{K}_J^0, B_G, B_J > 0$.

The corresponding result for $H(m)$ is novel and more interesting as the possibility for Dirichlet eigenvalues means that a DNO cannot be well-defined for all layer configuration choices, $\mathcal{C}_\ell^{(m)}$.

5.1. Change of variables

To begin, we recall the defining boundary value problem, (2.6), for the DNO, and the definition of the DNO itself (2.7). To streamline the presentation we simplify the notation slightly,

$$\Delta v + k^2 v = 0, \qquad \bar{\ell} + \ell(x, y) < z < \bar{u} + u(x, y), \tag{5.1a}$$

$$v = L, \qquad z = \bar{\ell} + \ell(x, y), \tag{5.1b}$$

$$v = U, \qquad z = \bar{u} + u(x, y), \tag{5.1c}$$

where $v = v^{(m)}$, $k = k^{(m)}$, $\bar{u} = a^{(m)}$, $\bar{\ell} = a^{(m+1)}$, $u = g^{(m)}$, $\ell = g^{(m+1)}$, and the DNO is given by

$$H[U, L] = \begin{pmatrix} H^{(u)}[U, L] \\ H^{(\ell)}[U, L] \end{pmatrix} = \begin{pmatrix} \left[\partial_z v - (\partial_x u)\partial_x v - (\partial_y u)\partial_y v\right]_{z=\bar{u}+u} \\ \left[-\partial_z v + (\partial_x \ell)\partial_x v + (\partial_y \ell)\partial_y v\right]_{z=\bar{\ell}+\ell} \end{pmatrix}.$$

We follow the lead of the author and Reitich [56,58,59] and introduce the following changes of variables (known as σ-coordinates [63] and the C-method [26,20])

$$x' = x, \quad y' = y, \quad z' = \bar{\ell}\left(\frac{\bar{u} + u(x, y) - z}{\bar{u} + u(x, y) - \bar{\ell} - \ell(x, y)}\right) + \bar{u}\left(\frac{z - \bar{\ell} - \ell(x, y)}{\bar{u} + u(x, y) - \bar{\ell} - \ell(x, y)}\right),$$

which maps the perturbed domain

$$S_{\bar{\ell}+\ell,\bar{u}+u} = \left\{\bar{\ell} + \ell(x, y) < z < \bar{u} + u(x, y)\right\},$$

to the flat-interface domain $S_{\bar{\ell},\bar{u}}$ which has height $\bar{h} := \bar{u} - \bar{\ell}$. We rewrite the vertical change of variables as

$$C(x, y)z' = z - D(x, y), \;\; C(x, y) := 1 + \frac{u(x, y) - \ell(x, y)}{\bar{h}}, \;\; D(x, y) := \frac{\bar{u}\ell(x, y) - \bar{\ell}u(x, y)}{\bar{h}}.$$

The function $v = v(x, y, z)$ transforms to

$$w = w(x', y', z') = v(x(x', y', z'), y(x', y', z'), z(x', y', z')).$$

It is not difficult to show that, via the chain rule, the derivative formulas are

$$C\partial_x = C\partial_{x'} - E^x\partial_{z'}, \quad C\partial_y = C\partial_{y'} - E^y\partial_{z'}, \quad C\partial_z = \partial_{z'},$$

where, given

$$Z_L := (z' - \bar{\ell})/\bar{h}, \quad Z_U := (\bar{u} - z')/\bar{h},$$

we have

$$E^x = E^x(x, y, z) = \frac{(\partial_{x'}u - \partial_{x'}\ell)z' + \bar{u}(\partial_{x'}\ell) - \bar{\ell}(\partial_{x'}u)}{\bar{h}} = (\partial_{x'}u)Z_L + (\partial_{x'}\ell)Z_U,$$

and

$$E^y = E^y(x, y, z) = \frac{(\partial_{y'}u - \partial_{y'}\ell)z' + \bar{u}(\partial_{y'}\ell) - \bar{\ell}(\partial_{y'}u)}{\bar{h}} = (\partial_{y'}u)Z_L + (\partial_{y'}\ell)Z_U.$$

5.2. The Helmholtz equation and boundary conditions

From these, upon multiplication by C^2, it is not difficult to show that the Helmholtz equation, (5.1a),

$$\Delta v + k^2 v = 0, \quad \bar{\ell} + \ell(x,y) < z < \bar{u} + u(x,y),$$

transforms to

$$\operatorname{div}' \left[A \nabla' w \right] + B \cdot \nabla' w + k^2 C^2 w = 0, \quad \bar{\ell} < z' < \bar{u}, \tag{5.2}$$

where

$$A = \begin{pmatrix} C^2 & 0 & -E^x C \\ 0 & C^2 & -E^y C \\ -E^x C & -E^y C & 1 + (E^x)^2 + (E^y)^2 \end{pmatrix}, \quad B = \begin{pmatrix} -(\partial_{x'} C) C \\ -(\partial_{y'} C) C \\ (\partial_{x'} C) E^x + (\partial_{y'} C) E^y \end{pmatrix}.$$

From here, for clarity of presentation, we drop the primed notation.

We note that, if $u = \varepsilon \tilde{u}$ and $\ell = \delta \tilde{\ell}$ then

$$\begin{aligned}
A &= A(\varepsilon,\delta) = A_{0,0} + A_{1,0}\varepsilon + A_{0,1}\delta + A_{2,0}\varepsilon^2 + A_{1,1}\varepsilon\delta + A_{0,2}\delta^2, \\
B &= B(\varepsilon,\delta) = B_{1,0}\varepsilon + B_{0,1}\delta + B_{2,0}\varepsilon^2 + B_{1,1}\varepsilon\delta + B_{0,2}\delta^2, \\
C^2 &= C^2(\varepsilon,\delta) = C^2_{0,0} + C^2_{1,0}\varepsilon + C^2_{0,1}\delta + C^2_{2,0}\varepsilon^2 + C^2_{1,1}\varepsilon\delta + C^2_{0,2}\delta^2,
\end{aligned}$$

where

$$A_{0,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_{1,0} = \frac{1}{\bar{h}} \begin{pmatrix} 2\tilde{u} & 0 & -\bar{h}(\partial_x \tilde{u}) Z_L \\ 0 & 2\tilde{u} & -\bar{h}(\partial_y \tilde{u}) Z_L \\ -\bar{h}(\partial_x \tilde{u}) Z_L & -\bar{h}(\partial_y \tilde{u}) Z_L & 0 \end{pmatrix},$$

$$A_{0,1} = \frac{1}{\bar{h}} \begin{pmatrix} -2\tilde{\ell} & 0 & -\bar{h}(\partial_x \tilde{\ell}) Z_U \\ 0 & -2\tilde{\ell} & -\bar{h}(\partial_y \tilde{\ell}) Z_U \\ -\bar{h}(\partial_x \tilde{\ell}) Z_U & -\bar{h}(\partial_y \tilde{\ell}) Z_U & 0 \end{pmatrix},$$

and

$$A_{2,0} = \frac{1}{\bar{h}^2} \begin{pmatrix} \tilde{u}^2 & 0 & -\bar{h}\tilde{u}(\partial_x \tilde{u}) Z_L \\ 0 & \tilde{u}^2 & -\bar{h}\tilde{u}(\partial_y \tilde{u}) Z_L \\ -\bar{h}\tilde{u}(\partial_x \tilde{u}) Z_L & -\bar{h}\tilde{u}(\partial_y \tilde{u}) Z_L & \bar{h}^2 \left\{ (\partial_x \tilde{u})^2 + (\partial_y \tilde{u})^2 \right\} Z_L^2 \end{pmatrix},$$

$$A_{1,1} = \frac{1}{\bar{h}^2} \begin{pmatrix} -2\tilde{u}\tilde{\ell} & 0 & \bar{h}\tilde{E}^x \\ 0 & -2\tilde{u}\tilde{\ell} & \bar{h}\tilde{E}^y \\ \bar{h}\tilde{E}^x & \bar{h}\tilde{E}^y & \bar{h}\tilde{E}^{x,y} \end{pmatrix},$$

$$A_{0,2} = \frac{1}{\bar{h}^2} \begin{pmatrix} \tilde{\ell}^2 & 0 & \bar{h}\tilde{\ell}(\partial_x \tilde{\ell}) Z_U \\ 0 & \tilde{\ell}^2 & \bar{h}\tilde{\ell}(\partial_y \tilde{\ell}) Z_U \\ \bar{h}\tilde{\ell}(\partial_x \tilde{\ell}) Z_U & \bar{h}\tilde{\ell}(\partial_y \tilde{\ell}) Z_U & \bar{h}^2 \left\{ (\partial_x \tilde{\ell})^2 + (\partial_y \tilde{\ell})^2 \right\} Z_U^2 \end{pmatrix},$$

where

$$\tilde{E^x} := \bar{h}\tilde{\ell}(\partial_x \tilde{u}) Z_L - \bar{h}\tilde{u}(\partial_x \tilde{\ell}) Z_U, \quad \tilde{E^y} := \bar{h}\tilde{\ell}(\partial_y \tilde{u}) Z_L - \bar{h}\tilde{u}(\partial_y \tilde{\ell}) Z_U,$$
$$\tilde{E}^{x,y} := 2\bar{h} \left\{ (\partial_x \tilde{u})(\partial_x \tilde{\ell}) + (\partial_y \tilde{u})(\partial_y \tilde{\ell}) \right\} Z_L Z_U.$$

In addition

$$B_{1,0} = \frac{1}{\bar{h}} \begin{pmatrix} -(\partial_x \tilde{u}) \\ -(\partial_y \tilde{u}) \\ 0 \end{pmatrix}, \quad B_{0,1} = \frac{1}{\bar{h}} \begin{pmatrix} (\partial_x \tilde{\ell}) \\ (\partial_y \tilde{\ell}) \\ 0 \end{pmatrix},$$

and

$$B_{2,0} = \frac{1}{\bar{h}^2} \begin{pmatrix} -\tilde{u}(\partial_x \tilde{u}) \\ -\tilde{u}(\partial_y \tilde{u}) \\ \bar{h} \left\{ (\partial_x \tilde{u})^2 + (\partial_y \tilde{u})^2 \right\} Z_L \end{pmatrix},$$
$$B_{1,1} = \frac{1}{\bar{h}^2} \begin{pmatrix} \tilde{u}(\partial_x \tilde{\ell}) + \tilde{\ell}(\partial_x \tilde{u}) \\ \tilde{u}(\partial_y \tilde{\ell}) + \tilde{\ell}(\partial_y \tilde{u}) \\ \bar{h} \left\{ (\partial_x \tilde{u})(\partial_x \tilde{\ell}) + (\partial_y \tilde{u})(\partial_y \tilde{\ell}) \right\} (Z_U - Z_L) \end{pmatrix},$$
$$B_{0,2} = \frac{1}{\bar{h}^2} \begin{pmatrix} -\tilde{\ell}(\partial_x \tilde{\ell}) \\ -\tilde{\ell}(\partial_y \tilde{\ell}) \\ -\bar{h} \left\{ (\partial_x \tilde{\ell})^2 + (\partial_y \tilde{\ell})^2 \right\} Z_U \end{pmatrix}.$$

Finally, we have

$$C^2_{0,0} = 1, \quad C^2_{1,0} = \frac{2}{\bar{h}} \tilde{u}, \quad C^2_{0,1} = -\frac{2}{\bar{h}} \tilde{\ell}, \quad C^2_{2,0} = \frac{1}{\bar{h}^2} \tilde{u}^2, \quad C^2_{1,1} = -\frac{2}{\bar{h}^2} \tilde{u}\tilde{\ell}, \quad C^2_{0,2} = \frac{1}{\bar{h}^2} \tilde{\ell}^2.$$

With these we write (5.2) as

$$\Delta w + k^2 w = F, \quad \bar{\ell} < z < \bar{u}, \tag{5.3}$$

where

$$F = \operatorname{div}\left[(I - A) \nabla w \right] - B \cdot \nabla w + k^2 (1 - C^2) w,$$

so that if $u = \varepsilon \tilde{u}$ and $\ell = \delta \tilde{\ell}$ then $F = \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)$.

It is easy to see that the boundary conditions, (5.1b) and (5.1c), simply transform to

$$w = L, \qquad z = \bar{\ell}, \tag{5.4a}$$
$$w = U, \qquad z = \bar{u}. \tag{5.4b}$$

5.3. The Dirichlet–Neumann operators

We close by noting that the generic DNO

$$\begin{pmatrix} H^{(u)}[U,L] \\ H^{(\ell)}[U,L] \end{pmatrix} = \begin{pmatrix} \left[\partial_z v - (\partial_x u)\partial_x v - (\partial_y u)\partial_y v\right]_{z=\bar{u}+u} \\ \left[-\partial_z v + (\partial_x \ell)\partial_x v + (\partial_y \ell)\partial_y v\right]_{z=\bar{\ell}+\ell} \end{pmatrix},$$

transforms (upon dropping primes) to

$$\begin{pmatrix} H^{(u)}[U,L] \\ H^{(\ell)}[U,L] \end{pmatrix} = \begin{pmatrix} \partial_z w(x,y,\bar{u}) \\ -\partial_z w(x,y,\bar{\ell}) \end{pmatrix} + \begin{pmatrix} Q^{(u)} \\ Q^{(\ell)} \end{pmatrix}, \tag{5.5}$$

where

$$\begin{aligned} \bar{h} Q^{(u)} &= -u H^{(u)}[U,L] + \ell H^{(u)}[U,L] - \bar{h}(\partial_x u)\partial_x w - u(\partial_x u)\partial_x w + \ell(\partial_x u)\partial_x w \\ &\quad - \bar{h}(\partial_y u)\partial_y w - u(\partial_y u)\partial_y w + \ell(\partial_y u)\partial_y w + \bar{h}(\partial_x u)^2\partial_z w + \bar{h}(\partial_y u)^2 \partial_z w, \\ \bar{h} Q^{(\ell)} &= -u H^{(\ell)}[U,L] + \ell H^{(\ell)}[U,L] + \bar{h}(\partial_x \ell)\partial_x w + u(\partial_x \ell)\partial_x w - \ell(\partial_x \ell)\partial_x w \\ &\quad + \bar{h}(\partial_y \ell)\partial_y w + u(\partial_y \ell)\partial_y w - \ell(\partial_y \ell)\partial_y w - \bar{h}(\partial_x \ell)^2\partial_z w - \bar{h}(\partial_y \ell)^2 \partial_z w, \end{aligned}$$

and again, if $u = \varepsilon\tilde{u}$ and $\ell = \delta\tilde{\ell}$, then $\{Q^{(u)}, Q^{(\ell)}\} = \mathcal{O}(\delta) + \mathcal{O}(\varepsilon)$.

5.4. Taylor expansion

We now gather our field equations in transformed coordinates

$$\begin{aligned} &\Delta w + k^2 w = F, && \bar{\ell} < z < \bar{u}, \\ &w = L, && z = \bar{\ell}, \\ &w = U, && z = \bar{u}, \end{aligned}$$

cf. (5.3) and (5.4), together with the transformed equation for the DNO

$$\begin{pmatrix} H^{(u)}[U,L] \\ H^{(\ell)}[U,L] \end{pmatrix} = \begin{pmatrix} \partial_z w(x,y,\bar{u}) \\ -\partial_z w(x,y,\bar{\ell}) \end{pmatrix} + \begin{pmatrix} Q^{(u)} \\ Q^{(\ell)} \end{pmatrix},$$

cf. (5.5). At this point we make the specification that, for $\varepsilon, \delta \in \mathbf{R}$,

$$u = \varepsilon\tilde{u}, \quad \ell = \delta\tilde{\ell},$$

where the (implicit) smallness assumptions on ε and δ can be removed (up to topological obstruction) [58,61]. With this we can formally expand

$$w = w(x, y, z; \varepsilon, \delta) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} w_{n,r}(x, y, z)\varepsilon^n \delta^r, \tag{5.6a}$$

$$H = \begin{pmatrix} H^{(u)} \\ H^{(\ell)} \end{pmatrix} = \begin{pmatrix} H^{(u)}(x, y; \varepsilon, \delta) \\ H^{(\ell)}(x, y; \varepsilon, \delta) \end{pmatrix} = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \begin{pmatrix} H^{(u)}_{n,r}(x, y) \\ H^{(\ell)}_{n,r}(x, y) \end{pmatrix} \varepsilon^n \delta^r, \tag{5.6b}$$

and find that, at each perturbation order $\mathcal{O}(\varepsilon^n \delta^r)$, we must solve

$$\Delta w_{n,r} + k^2 w_{n,r} = F_{n,r}, \qquad \bar{\ell} < z < \bar{u}, \tag{5.7a}$$

$$w_{n,r} = (1 - \delta_{n,r})L, \qquad z = \bar{\ell}, \tag{5.7b}$$

$$w_{n,r} = (1 - \delta_{n,r})U, \qquad z = \bar{u}, \tag{5.7c}$$

where $\delta_{n,r}$ is the Kronecker delta, and

$$\begin{pmatrix} H^{(u)}_{n,r}[U, L] \\ H^{(\ell)}_{n,r}[U, L] \end{pmatrix} = \begin{pmatrix} \partial_z w_{n,r}(x, y, \bar{u}) \\ -\partial_z w_{n,r}(x, y, \bar{\ell}) \end{pmatrix} + \begin{pmatrix} Q^{(u)}_{n,r} \\ Q^{(\ell)}_{n,r} \end{pmatrix}. \tag{5.8}$$

In these

$$F_{n,r} = -\sum_{\nu+\rho=1}^{2} \left\{ \operatorname{div}\left[A_{\nu,\rho} \nabla w_{n-\nu,r-\rho}\right] + B_{\nu,\rho} \cdot \nabla w_{n-\nu,r-\rho} + k^2 C^2_{\nu,\rho} w_{n-\nu,r-\rho} \right\},$$

and

$$\begin{aligned}
\bar{h} Q^{(u)}_{n,r} &= -\tilde{u} H^{(u)}_{n-1,r}[U, L] + \tilde{\ell} H^{(u)}_{n,r-1}[U, L] \\
&\quad - \bar{h}(\partial_x \tilde{u})\partial_x w_{n-1,r} - \tilde{u}(\partial_x \tilde{u})\partial_x w_{n-2,r} + \tilde{\ell}(\partial_x \tilde{u})\partial_x w_{n-1,r-1} \\
&\quad - \bar{h}(\partial_y \tilde{u})\partial_y w_{n-1,r} - \tilde{u}(\partial_y \tilde{u})\partial_y w_{n-2,r} + \tilde{\ell}(\partial_y \tilde{u})\partial_y w_{n-1,r-1} \\
&\quad + \bar{h}(\partial_x \tilde{u})^2 \partial_z w_{n-2,r} + \bar{h}(\partial_y \tilde{u})^2 \partial_z w_{n-2,r}, \\
\bar{h} Q^{(\ell)}_{n,r} &= -\tilde{u} H^{(\ell)}_{n-1,r}[U, L] + \tilde{\ell} H^{(\ell)}_{n,r-1}[U, L] \\
&\quad + \bar{h}(\partial_x \tilde{\ell})\partial_x w_{n,r-1} + \tilde{u}(\partial_x \tilde{\ell})\partial_x w_{n-1,r-1} - \tilde{\ell}(\partial_x \tilde{\ell})\partial_x w_{n,r-2} \\
&\quad + \bar{h}(\partial_y \tilde{\ell})\partial_y w_{n,r-1} + \tilde{u}(\partial_y \tilde{\ell})\partial_y w_{n-1,r-1} - \tilde{\ell}(\partial_y \tilde{\ell})\partial_y w_{n,r-2} \\
&\quad - \bar{h}(\partial_x \tilde{\ell})^2 \partial_z w_{n,r-2} - \bar{h}(\partial_y \tilde{\ell})^2 \partial_z w_{n,r-2}.
\end{aligned}$$

5.5. Analyticity Theorem and Proof

We now state the main theorems of this section together with the essential lemmas necessary to prove them. We close with the proofs of the fundamental results themselves. To begin we define a Sobolev norm for the field functions, $w = w(x, y, z)$ which helps us give a nearly optimal result in terms of boundary smoothness. We define, for laterally $(d_x \times d_y)$-periodic *volumetric* functions on the domain

$$V := d \times [\bar{\ell}, \bar{u}] = [0, d_x] \times [0, d_y] \times [\bar{\ell}, \bar{u}],$$

with s-many lateral derivatives, the Sobolev space,

$$Z^s(V) := \left\{ w(x, y, z) \mid w \text{ is bounded, measurable, } |||w|||_{H^s(V)} < \infty \right\},$$

where

$$|||w|||^2_{H^s(V)} := \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p, q) \rangle^{2s} \int_{\bar{\ell}}^{\bar{u}} \left| \hat{w}_{p,q}(z) \right|^2 \, dz.$$

With these we can prove the following useful results, the first a crucial elliptic estimate of the type used in previous results on analyticity of DNOs [56,58,61,34] which can easily be derived from classical theory found in, e.g., [46,27,33]. For completeness we outline the proof in Appendix A.

Theorem 5.2. *Given an integer $s \geq 0$, if $F \in H^{s-1/2}(V)$ and $U, L \in H^{s+1}(d)$ then there exists a solution of*

$$\Delta w + k^2 w = F, \qquad \bar{\ell} < z < \bar{u}, \tag{5.9a}$$

$$w = L, \qquad z = \bar{\ell}, \tag{5.9b}$$

$$w = U, \qquad z = \bar{u}, \tag{5.9c}$$

satisfying

$$\begin{aligned} \max \Big\{ & \|\partial_x w(x, y, \bar{u})\|_{H^s}, \left\|\partial_y w(x, y, \bar{u})\right\|_{H^s}, \|\partial_z w(x, y, \bar{u})\|_{H^s}, \|w(x, y, \bar{u})\|_{H^{s+1}}, \\ & \left\|\partial_x w(x, y, \bar{\ell})\right\|_{H^s}, \left\|\partial_y w(x, y, \bar{\ell})\right\|_{H^s}, \left\|\partial_z w(x, y, \bar{\ell})\right\|_{H^s}, \left\|w(x, y, \bar{\ell})\right\|_{H^{s+1}}, \\ & |||\partial_z w|||_{H^{s+1/2}(V)}, |||w|||_{H^{s+3/2}(V)} \Big\} \\ & \leq K_e \left\{ |||F|||_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \end{aligned} \tag{5.10}$$

where $K_e = K_e(\tau) > 0$ is a universal constant. The solution is unique in a τ-allowable configuration, but $K_e \to \infty$ as $\tau \to 0$.

In addition, it is not difficult to show the following lemma.

Lemma 5.3. *Given an integer $s \geq 0$, if $w \in Z^{s+1/2}(V)$ then*

$$\max \left\{ |||Z_L w|||_{H^{s+1/2}(V)}, |||Z_U w|||_{H^{s+1/2}(V)} \right\} \leq K_Z |||w|||_{H^{s+1/2}(V)},$$

where K_Z is a universal constant.

Also, one can generalize Lemma 4.16 in the following way.

Lemma 5.4. *Given an integer $s \geq 0$ and any $\sigma > 0$, there exists a constant $\mathcal{M} = \mathcal{M}(s)$ such that if $f \in C^s(d)$, $w \in Z^s(V)$ then $fw \in Z^s(V)$ and*

$$|||fw|||_{H^s(V)} \leq \mathcal{M}\,|f|_{C^s}\,|||w|||_{H^s(V)},$$

and if $\tilde{f} \in C^{s+1/2+\sigma}(d)$, $\tilde{w} \in Z^{s+1/2}(V)$ then $\tilde{f}\tilde{w} \in Z^{s+1/2}(V)$ and

$$\left|\left|\left| \tilde{f}\tilde{w} \right|\right|\right|_{H^{s+1/2}(V)} \leq \mathcal{M}\left|\tilde{f}\right|_{C^{s+1/2+\sigma}} |||\tilde{w}|||_{H^{s+1/2}(V)}.$$

We are now in a position to state our main results.

Theorem 5.5. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $\tilde{u}, \tilde{\ell} \in C^{s+3/2+\sigma}(d)$ and $U, L \in H^{s+1}(d)$, there exists a solution, (5.6a), of (5.1) and if the configuration is τ-allowable then it is unique. In any case, the solution satisfies*

$$\begin{aligned}
\max\Big\{&\left\|\partial_x w_{n,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_y w_{n,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_z w_{n,r}(x,y,\bar{u})\right\|_{H^s}, \left\|w_{n,r}(x,y,\bar{u})\right\|_{H^{s+1}},\\
&\left\|\partial_x w_{n,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w_{n,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w_{n,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|w_{n,r}(x,y,\bar{\ell})\right\|_{H^{s+1}},\\
&|||\partial_z w_{n,r}|||_{H^{s+1/2}(V)}, |||w_{n,r}|||_{H^{s+3/2}(V)}\Big\} \leq K^0_{u,\ell} B_u^n B_\ell^r, \qquad (5.11)
\end{aligned}$$

for

$$B_u > \max\left\{5K^1_{u,\ell}, \sqrt{5K^1_{u,\ell}}\right\} |\tilde{u}|_{C^{s+3/2+\sigma}}, \quad B_\ell > \max\left\{5K^1_{u,\ell}, \sqrt{5K^1_{u,\ell}}\right\} \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}},$$

where $K^1_{u,\ell}$ comes from Lemma 5.13 and $K^0_{u,\ell} > 0$ is a universal constant, but $K^0_{u,\ell} \to \infty$ as $\tau \to 0$.

From this we can easily prove the following result on analyticity of the DNO.

Theorem 5.6. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $\tilde{u}, \tilde{\ell} \in C^{s+3/2+\sigma}(d)$ then if the layer configuration is τ-allowable the series (5.6b) converge strongly as operators from $H^{s+1}(d)$ to $H^s(d)$. More precisely*

$$\max\left\{\left\|H^{(u)}_{n,r}\right\|_{H^{s+1}\to H^s}, \left\|H^{(\ell)}_{n,r}\right\|_{H^{s+1}\to H^s}\right\} \leq \tilde{K}^0_{u,\ell} B_u^n B_\ell^r, \qquad (5.12)$$

for universal constants $\tilde{K}^0_{u,\ell}, B_u, B_\ell > 0$, but $\tilde{K}^0_{u,\ell} \to \infty$ as $\tau \to 0$.

To justify these we establish four preliminary results which show that both the field and the DNO are analytic in upper (lower) boundary perturbation provided that the lower (upper) one is trivial (i.e., zero).

Theorem 5.7. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $\tilde{u} \in C^{s+3/2+\sigma}(d)$ and $\tilde{\ell} \equiv 0$, and $U, L \in H^{s+1}(d)$, there exists a solution, (5.6a), of (5.1) and if the configuration is τ-allowable then it is unique. In any case, the solution satisfies*

$$\max\left\{\left\|\partial_x w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_y w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_z w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|w_{n,0}(x,y,\bar{u})\right\|_{H^{s+1}},\right.$$
$$\left\|\partial_x w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|w_{n,0}(x,y,\bar{\ell})\right\|_{H^{s+1}},$$
$$\left.|||\partial_z w_{n,0}|||_{H^{s+1/2}(V)}, |||w_{n,0}|||_{H^{s+3/2}(V)}\right\} \le K^0_{u,0} B^n_u, \tag{5.13}$$

for

$$B_u > \max\left\{2K^1_{u,0}, \sqrt{2K^1_{u,0}}\right\} |\tilde{u}|_{C^{s+3/2+\sigma}},$$

where $K^1_{u,0}$ *comes from Lemma 5.11 and* $K^0_{u,0} > 0$ *is a universal constant, but* $K^0_{u,0} \to \infty$ *as* $\tau \to 0$.

From this one can show the following result on analyticity of DNOs.

Theorem 5.8. *Given an integer* $s \ge 0$ *and any* $\sigma > 0$, *if* $\tilde{u} \in C^{s+3/2+\sigma}(d)$ *and* $\tilde{\ell} \equiv 0$ *then if the layer configuration is* τ*-allowable the series (5.6b) converge strongly as operators from* $H^{s+1}(d)$ *to* $H^s(d)$. *More precisely*

$$\max\left\{\left\|H^{(u)}_{n,0}\right\|_{H^{s+1}\to H^s}, \left\|H^{(\ell)}_{n,0}\right\|_{H^{s+1}\to H^s}\right\} \le \tilde{K}^0_{u,0} B^n_u, \tag{5.14}$$

for universal constants $\tilde{K}^0_{u,0}$, $B_u > 0$, *but* $\tilde{K}^0_{u,0} \to \infty$ *as* $\tau \to 0$.

For variations of the bottom layer we have the following.

Theorem 5.9. *Given an integer* $s \ge 0$ *and any* $\sigma > 0$, *if* $\tilde{\ell} \in C^{s+3/2+\sigma}(d)$ *and* $\tilde{u} \equiv 0$, *and* $U, L \in H^{s+1}(d)$, *there exists a solution, (5.6a), of (5.1) and if the configuration is* τ*-allowable then it is unique. In any case, the solution satisfies*

$$\max\left\{\left\|\partial_x w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_y w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_z w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|w_{0,r}(x,y,\bar{u})\right\|_{H^{s+1}},\right.$$
$$\left\|\partial_x w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|w_{0,r}(x,y,\bar{\ell})\right\|_{H^{s+1}},$$
$$\left.|||\partial_z w_{0,r}|||_{H^{s+1/2}(V)}, |||w_{0,r}|||_{H^{s+3/2}(V)}\right\} \le K^0_{0,\ell} B^r_\ell,$$

for

$$B_\ell > \max\left\{2K^1_{0,\ell}, \sqrt{2K^1_{0,\ell}}\right\} \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}},$$

where $K^1_{0,\ell}$ *comes from Lemma 5.12 and* $K^0_{0,\ell} > 0$ *is a universal constant, but* $K^0_{0,\ell} \to \infty$ *as* $\tau \to 0$.

As before, the following can be readily established.

Theorem 5.10. *Given an integer $s \geq 0$ and any $\sigma > 0$, if $\tilde{\ell} \in C^{s+3/2+\sigma}(d)$ and $\tilde{u} \equiv 0$ then if the layer configuration is τ-allowable the series (5.6b) converge strongly as operators from $H^{s+1}(d)$ to $H^s(d)$. More precisely*

$$\max\left\{\left\|H_{0,r}^{(u)}\right\|_{H^{s+1}\to H^s}, \left\|H_{0,r}^{(\ell)}\right\|_{H^{s+1}\to H^s}\right\} \leq \tilde{K}_{0,\ell}^0 B_\ell^r,$$

for universal constants $\tilde{K}_{0,\ell}^0, B_\ell > 0$, but $\tilde{K}_{0,\ell}^0 \to \infty$ as $\tau \to 0$.

We now require two lemmas for the recursive estimation of the inhomogeneities in (5.7), the first in perturbation orders n and the second in orders r.

Lemma 5.11. *Let $s \geq 0$ be an integer and let $\tilde{u} \in C^{s+3/2+\sigma}(d)$ for some $\sigma > 0$. Assume*

$$\begin{aligned}\max\Big\{&\left\|\partial_x w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_y w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_z w_{n,0}(x,y,\bar{u})\right\|_{H^s}, \left\|w_{n,0}(x,y,\bar{u})\right\|_{H^{s+1}},\\ &\left\|\partial_x w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w_{n,0}(x,y,\bar{\ell})\right\|_{H^s}, \left\|w_{n,0}(x,y,\bar{\ell})\right\|_{H^{s+1}},\\ &\left|\left|\left|\partial_z w_{n,0}\right|\right|\right|_{H^{s+1/2}(V)}, \left|\left|\left|w_{n,0}\right|\right|\right|_{H^{s+3/2}(V)}\Big\} \leq K_{u,0}^0 B_u^n,\end{aligned}$$

for constants $K_{u,0}^0, B_u > 0$ and all $n < \bar{n}$. Then the function $F_{\bar{n},0}$ from (5.7) satisfies

$$\begin{aligned}&\max\left\{\left\|F_{\bar{n},0}(x,y,\bar{u})\right\|_{H^s}, \left\|F_{\bar{n},0}(x,y,\bar{\ell})\right\|_{H^s}, \left|\left|\left|F_{\bar{n},0}\right|\right|\right|_{H^{s-1/2}(V)}\right\}\\ &\qquad\leq K_{u,0}^1 K_{u,0}^0\left\{|\tilde{u}|_{C^{s+1}} B_u^{\bar{n}-1} + |\tilde{u}|_{C^{s+1}}^2 B_u^{\bar{n}-2}\right\},\end{aligned}$$

for a universal constant $K_{u,0}^1 > 0$.

In addition to this we have the following.

Lemma 5.12. *Let $s \geq 0$ be an integer and let $\tilde{\ell} \in C^{s+3/2+\sigma}(d)$ for some $\sigma > 0$. Assume*

$$\begin{aligned}\max\Big\{&\left\|\partial_x w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_y w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|\partial_z w_{0,r}(x,y,\bar{u})\right\|_{H^s}, \left\|w_{0,r}(x,y,\bar{u})\right\|_{H^{s+1}},\\ &\left\|\partial_x w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w_{0,r}(x,y,\bar{\ell})\right\|_{H^s}, \left\|w_{0,r}(x,y,\bar{\ell})\right\|_{H^{s+1}},\\ &\left|\left|\left|\partial_z w_{0,r}\right|\right|\right|_{H^{s+1/2}(V)}, \left|\left|\left|w_{0,r}\right|\right|\right|_{H^{s+3/2}(V)}\Big\} \leq K_{0,\ell}^0 B_\ell^r,\end{aligned}$$

for constants $K_{0,\ell}^0, B_\ell > 0$ and all $r < \bar{r}$. Then the function $F_{0,\bar{r}}$ from (5.7) satisfies

$$\begin{aligned}&\max\left\{\left\|F_{0,\bar{r}}(x,y,\bar{u})\right\|_{H^s}, \left\|F_{0,\bar{r}}(x,y,\bar{\ell})\right\|_{H^s}, \left|\left|\left|F_{0,\bar{r}}\right|\right|\right|_{H^{s-1/2}(V)}\right\}\\ &\qquad\leq K_{0,\ell}^1 K_{0,\ell}^0\left\{\left|\tilde{\ell}\right|_{C^{s+1}} B_\ell^{\bar{r}-1} + \left|\tilde{\ell}\right|_{C^{s+1}}^2 B_\ell^{\bar{r}-2}\right\},\end{aligned}$$

for a universal constant $K_{0,\ell}^1 > 0$.

The proof of each of these is quite similar and thus we present only that for Lemma 5.11.

Proof of Lemma 5.11. For brevity we consider only one representative term which must be estimated. We focus upon

$$\begin{aligned}\bar{h}^2 F_{\bar{n},0}^{(2,0)} &= \partial_x \left[(\tilde{u})^2 \partial_x w_{\bar{n}-2,0} - \bar{h}\tilde{u}(\partial_x \tilde{u}) Z_L \partial_z w_{\bar{n}-2,0} \right] \\ &+ \partial_y \left[(\tilde{u})^2 \partial_y w_{\bar{n}-2,0} - \bar{h}\tilde{u}(\partial_y \tilde{u}) Z_L \partial_z w_{\bar{n}-2,0} \right] \\ &+ \partial_z \left[-\bar{h}\tilde{u}(\partial_x \tilde{u}) Z_L \partial_x w_{\bar{n}-2,0} - \bar{h}\tilde{u}(\partial_y \tilde{u}) Z_L \partial_y w_{\bar{n}-2,0} \right. \\ &\left. + \bar{h}^2 \left\{ (\partial_x \tilde{u})^2 + (\partial_y \tilde{u})^2 \right\} Z_L^2 \partial_z w_{\bar{n}-2,0} \right].\end{aligned}$$

We now estimate

$$\begin{aligned}\bar{h}^2 \left\|\left| F_{\bar{n},0}^{(2,0)} \right|\right\|_{H^{s-1/2}(V)} \le & \left\|\left| (\tilde{u})^2 \partial_x w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} + \bar{h} \left\|\left| \tilde{u}(\partial_x \tilde{u}) Z_L \partial_z w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} \\ &+ \left\|\left| (\tilde{u})^2 \partial_y w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} + \bar{h} \left\|\left| \tilde{u}(\partial_y \tilde{u}) Z_L \partial_z w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} \\ &+ \bar{h} \left\|\left| \tilde{u}(\partial_x \tilde{u}) Z_L \partial_x w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} \\ &+ \bar{h} \left\|\left| \tilde{u}(\partial_y \tilde{u}) Z_L \partial_y w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} \\ &+ \bar{h}^2 \left\|\left| (\partial_x \tilde{u})^2 Z_L^2 \partial_z w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)} \\ &+ \bar{h}^2 \left\|\left| (\partial_y \tilde{u})^2 Z_L^2 \partial_z w_{\bar{n}-2,0} \right|\right\|_{H^{s+1/2}(V)}\end{aligned}$$

We can now use Lemma 5.4 to estimate

$$\begin{aligned}\bar{h}^2 \left\|\left| F_{\bar{n},0}^{(2,0)} \right|\right\|_{H^{s-1/2}(V)} \le & \mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}}^2 \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}} \left| \tilde{u} \right|_{C^{s+3/2+\sigma}} K_Z \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}}^2 \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}} \left| \tilde{u} \right|_{C^{s+3/2+\sigma}} K_Z \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}} \left| \tilde{u} \right|_{C^{s+3/2+\sigma}} K_Z \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+1/2+\sigma}} \left| \tilde{u} \right|_{C^{s+3/2+\sigma}} K_Z \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}^2\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+3/2+\sigma}}^2 K_Z^2 \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)} \\ &+ \bar{h}^2\mathcal{M}^2 \left| \tilde{u} \right|_{C^{s+3/2+\sigma}}^2 K_Z^2 \left\|\left| w_{\bar{n}-2,0} \right|\right\|_{H^{s+3/2}(V)},\end{aligned}$$

where $\mathcal{M} = \mathcal{M}(s + 3/2 + \sigma)$. Upon using

$$\left| \tilde{u} \right|_{C^{s+1/2+\sigma}} \le \left| \tilde{u} \right|_{C^{s+3/2+\sigma}},$$

we are finished by making the estimate

$$\bar{h}^2 \left\| F_{\bar{n},0}^{(2,0)} \right\|_{H^{s-1/2}(V)} \leq \mathcal{M}^2 \left\{ 2 + 4\bar{h} K_Z + 2\bar{h}^2 K_Z^2 \right\} |\tilde{u}|_{C^{s+3/2+\sigma}}^2 \, K_{u,0}^0 B_u^{\bar{n}-2},$$

and choosing

$$K^1 := \mathcal{M}^2 \left\{ 2 + 4\bar{h} K_Z + 2\bar{h}^2 K_Z^2 \right\} / \bar{h}^2. \quad \square$$

We are now in position to prove Theorem 5.7 and once again note that the corresponding proof of Theorem 5.9 is nearly identical so that we omit it for brevity.

Proof of Theorem 5.7. Our proof proceeds by induction in n; at order $n = 0$ we note that in (5.7) we have

$$F_{0,0} \equiv 0,$$

and it is a simple matter to write down the exact solution using the method of separation of variables. We have assumed that this solution is unique and set

$$K_{u,0}^0 = |||w_{0,0}|||_{H^{s+3/2}(V)},$$

thereby establishing the base case of our induction. We now assume our estimates (5.13) for all $n < \bar{n}$. As we have assumed that a solution of (5.7) exists, we can apply Theorem 5.2 to (5.7) to estimate

$$\begin{aligned} \max \Big\{ & \left\| \partial_x w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^{s+1}}, \\ & \left\| \partial_x w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^{s+1}}, \\ & |||\partial_z w_{\bar{n},0}|||_{H^{s+1/2}(V)}, |||w_{\bar{n},0}|||_{H^{s+3/2}(V)} \Big\} \leq K_e |||F_{\bar{n},0}|||_{H^{s-1/2}(V)}. \end{aligned}$$

We now invoke Lemma 5.11 to discover that

$$\begin{aligned} \max \Big\{ & \left\| \partial_x w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^s}, \left\| w_{\bar{n},0}(x,y,\bar{u}) \right\|_{H^{s+1}}, \\ & \left\| \partial_x w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| w_{\bar{n},0}(x,y,\bar{\ell}) \right\|_{H^{s+1}}, \\ & |||\partial_z w_{\bar{n},0}|||_{H^{s+1/2}(V)}, |||w_{\bar{n},0}|||_{H^{s+3/2}(V)} \Big\} \\ & \leq K_e K_{u,0}^0 K_{u,0}^1 \left\{ |\tilde{u}|_{C^{s+1}} B_u^{\bar{n}-1} + |\tilde{u}|_{C^{s+1}}^2 B_u^{\bar{n}-2} \right\}, \end{aligned}$$

and we are done provided

$$B_u > \max \left\{ 2K_{u,0}^1 K_e, \sqrt{2K_{u,0}^1 K_e} \right\} |\tilde{u}|_{C^{s+3/2+\sigma}}. \quad \square$$

With this we can establish Theorem 5.8 and, as before, note that the corresponding proof of Theorem 5.10 is nearly the same and thus omitted.

Proof of Theorem 5.8. Again, our proof is inductive in n; at order $n = 0$ we note that in (5.8) we have

$$Q^{(u)}_{0,0} \equiv Q^{(\ell)}_{0,0} \equiv 0,$$

so that

$$\begin{pmatrix} H^{(u)}_{0,0}[U, L] \\ H^{(\ell)}_{0,0}[U, L] \end{pmatrix} = \begin{pmatrix} \partial_z w_{0,0}(x, y, \bar{u}) \\ -\partial_z w_{0,0}(x, y, \bar{\ell}) \end{pmatrix}.$$

We have assumed configurations where solutions are unique, thus, from Theorem 5.5, we have that

$$\partial_z w_{0,0}(x, y, \bar{u}), \partial_z w_{0,0}(x, y, \bar{\ell}) \in H^s(d),$$

so that we may set

$$\tilde{K}^0_{u,0} = \max\left\{\left\|\partial_z w_{0,0}(x, y, \bar{u})\right\|_{H^s}, \left\|\partial_z w_{0,0}(x, y, \bar{\ell})\right\|_{H^s}\right\},$$

thereby establishing the base case of our induction. We now assume our estimates (5.14) for all $n < \bar{n}$. Equation (5.8) with $r = 0$ delivers,

$$\begin{pmatrix} H^{(u)}_{\bar{n},0}[U, L] \\ H^{(\ell)}_{\bar{n},0}[U, L] \end{pmatrix} = \begin{pmatrix} \partial_z w_{\bar{n},0}(x, y, \bar{u}) \\ -\partial_z w_{\bar{n},0}(x, y, \bar{\ell}) \end{pmatrix} + \begin{pmatrix} Q^{(u)}_{\bar{n},0} \\ Q^{(\ell)}_{\bar{n},0} \end{pmatrix},$$

and we focus upon one representative term in $Q^{(u)}_{n,0}$, namely

$$\bar{h} Q^{(1,0)}_{\bar{n},0} = -\tilde{u} H^{(u)}_{\bar{n}-1,0}[U, L] - \bar{h}(\partial_x \tilde{u})\partial_x w_{\bar{n}-1,0} - \bar{h}(\partial_y \tilde{u})\partial_y w_{\bar{n}-1,0},$$

(the others can be handled in a similar fashion). We now utilize Theorem 5.7 to deliver

$$\begin{aligned} \bar{h}\left\| Q^{(1,0)}_{\bar{n},0}\right\|_{H^s} &= \left\| -\tilde{u} H^{(u)}_{\bar{n}-1,0}[U, L] - \bar{h}(\partial_x \tilde{u})\partial_x w_{\bar{n}-1,0} - \bar{h}(\partial_y \tilde{u})\partial_y w_{\bar{n}-1,0}\right\|_{H^s} \\ &\leq \mathcal{M}(s)\left|\tilde{u}\right|_{C^{s+1}}\left\{\left\| H^{(u)}_{\bar{n}-1,0}[U, L]\right\|_{H^s} + \bar{h}\left\|\partial_x w_{\bar{n}-1,0}\right\|_{H^s} + \bar{h}\left\|\partial_y w_{\bar{n}-1,0}\right\|_{H^s}\right\} \\ &\leq \mathcal{M}(s)\left|\tilde{u}\right|_{C^{s+1}}\left\{\tilde{K}^0_{u,0} B_u^{\bar{n}-1} + 2\bar{h} K^0_{u,0} B_u^{\bar{n}-1}\right\}. \end{aligned}$$

We are done provided that we choose

$$B_u > 2\mathcal{M}\left|\tilde{u}\right|_{C^{s+1}} \max\{\frac{1}{\bar{h}}, 2K^0_{u,0}/\tilde{K}^0_{u,0}\}. \quad \square$$

We require one final recursive lemma whose proof is very similar to that of Lemma 5.11 so it is omitted.

Lemma 5.13. *Let $s \geq 0$ be an integer and let $\tilde{u}, \tilde{\ell} \in C^{s+3/2+\sigma}(d)$ for some $\sigma > 0$. Assume*

$$\max \Big\{ \left\| \partial_x w_{n,r}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_y w_{n,r}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_z w_{n,r}(x,y,\bar{u}) \right\|_{H^s}, \left\| w_{n,r}(x,y,\bar{u}) \right\|_{H^{s+1}}, \\ \left\| \partial_x w_{n,r}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_y w_{n,r}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_z w_{n,r}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| w_{n,r}(x,y,\bar{\ell}) \right\|_{H^{s+1}}, \\ |||\partial_z w_{n,r}|||_{H^{s+1/2}(V)}, |||w_{n,r}|||_{H^{s+3/2}(V)} \Big\} \leq K^0_{u,\ell} B^n_u B^r_\ell,$$

for constants $K^0_{u,\ell}, B_u, B_\ell > 0$, all n if $r < \bar{r}$, and $n < \bar{n}$ if $r = \bar{r}$. Then the function $F_{\bar{n},\bar{r}}$ from (5.7) satisfies

$$\max \Big\{ \left\| F_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| F_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, |||F_{\bar{n},\bar{r}}|||_{H^{s-1/2}(V)} \Big\} \\ \leq K^1_{u,\ell} K^0_{u,\ell} \Big\{ |\tilde{u}|_{C^{s+3/2+\sigma}} B^{\bar{n}-1}_u + \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}} B^{\bar{r}-1}_\ell \\ + |\tilde{u}|^2_{C^{s+3/2+\sigma}} B^{\bar{n}-2}_u + |\tilde{u}|_{C^{s+3/2+\sigma}} \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}} B^{\bar{n}-1}_u B^{\bar{r}-1}_\ell + \left|\tilde{\ell}\right|^2_{C^{s+3/2+\sigma}} B^{\bar{r}-2}_\ell \Big\}, \quad (5.15)$$

for a universal constant $K^1_{u,\ell} > 0$.

At last we can prove Theorem 5.5.

Proof of Theorem 5.5. We will now proceed by a double induction, beginning in r. The base case is (5.11) with $r = 0$ which we have established in Theorem 5.7. We now assume (5.11) for all n if $r < \bar{r}$, and seek to establish (5.11) for all n and $r = \bar{r}$. For this we perform a second induction on n. Here the base case is to show (5.11) for $n = 0$ and any $\bar{r}$, but this we verified in Theorem 5.9. Thus, we are finished provided that we can establish (5.11) for $(n = \bar{n}, r = \bar{r})$ provided that this estimate holds for all n if $r < \bar{r}$ and $n < \bar{n}$ if $r = \bar{r}$. As we have assumed that a solution of (5.7) exists, we can apply Theorem 5.2 to estimate

$$\max \Big\{ \left\| \partial_x w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^{s+1}}, \\ \left\| \partial_x w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^{s+1}}, \\ |||\partial_z w_{\bar{n},\bar{r}}|||_{H^{s+1/2}(V)}, |||w_{\bar{n},\bar{r}}|||_{H^{s+3/2}(V)} \Big\} \leq K_e |||F_{\bar{n},\bar{r}}|||_{H^{s-1/2}(V)}.$$

We now invoke Lemma 5.13 to discover that

$$\max \Big\{ \left\| \partial_x w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^s}, \left\| w_{\bar{n},\bar{r}}(x,y,\bar{u}) \right\|_{H^{s+1}}, \\ \left\| \partial_x w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_y w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| \partial_z w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^s}, \left\| w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \right\|_{H^{s+1}}, \\ |||\partial_z w_{\bar{n},\bar{r}}|||_{H^{s+1/2}(V)}, |||w_{\bar{n},\bar{r}}|||_{H^{s+3/2}(V)} \Big\} \\ \leq K_e K^1_{u,\ell} K^0_{u,\ell} \Big\{ |\tilde{u}|_{C^{s+3/2+\sigma}} B^{\bar{n}-1}_u + \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}} B^{\bar{r}-1}_\ell \\ + |\tilde{u}|^2_{C^{s+3/2+\sigma}} B^{\bar{n}-2}_u + |\tilde{u}|_{C^{s+3/2+\sigma}} \left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}} B^{\bar{n}-1}_u B^{\bar{r}-1}_\ell + \left|\tilde{\ell}\right|^2_{C^{s+3/2+\sigma}} B^{\bar{r}-2}_\ell \Big\},$$

and we are done provided

$$B_u > \max\left\{5K^1_{u,\ell}K_e, \sqrt{5K^1_{u,\ell}K_e}\right\}|\tilde{u}|_{C^{s+3/2+\sigma}},$$

$$B_\ell > \max\left\{5K^1_{u,\ell}K_e, \sqrt{5K^1_{u,\ell}K_e}\right\}\left|\tilde{\ell}\right|_{C^{s+3/2+\sigma}}. \quad \square$$

And now, after all of this work, we can establish Theorem 5.6

Proof of Theorem 5.6. As in the proof of Theorem 5.5 we conduct a double induction beginning in r. The base case is (5.12) in the case $r = 0$ which we verified in Theorem 5.8. We now assume (5.12) for all n if $r < \bar{r}$, and seek to establish (5.12) for all n if $r = \bar{r}$. We perform this by another induction in n. The base case is (5.12) for $r = \bar{r}$ if $n = 0$, but this was established in Theorem 5.10. We are finished if we can show (5.12) for $(n = \bar{n}, r = \bar{r})$ provided that this bound holds for all n if $r < \bar{r}$ and $n < \bar{n}$ if $r = \bar{r}$. Now (5.8) gives

$$\begin{pmatrix} H^{(u)}_{\bar{n},\bar{r}}[U,L] \\ H^{(\ell)}_{\bar{n},\bar{r}}[U,L] \end{pmatrix} = \begin{pmatrix} \partial_z w_{\bar{n},\bar{r}}(x,y,\bar{u}) \\ -\partial_z w_{\bar{n},\bar{r}}(x,y,\bar{\ell}) \end{pmatrix} + \begin{pmatrix} Q^{(u)}_{\bar{n},\bar{r}} \\ Q^{(\ell)}_{\bar{n},\bar{r}} \end{pmatrix},$$

and we focus on one term in $Q^{(\ell)}_{\bar{n},\bar{r}}$,

$$\bar{h}Q^{(1,0)}_{\bar{n},\bar{r}} = -\tilde{u}H^{(u)}_{\bar{n}-1,\bar{r}}[U,L] - \bar{h}(\partial_x\tilde{u})\partial_x w_{\bar{n}-1,\bar{r}} - \bar{h}(\partial_y\tilde{u})\partial_y w_{\bar{n}-1,\bar{r}},$$

(our analysis of the other terms is nearly identical). Theorem 5.5 gives

$$\begin{aligned}
\bar{h}\left\|Q^{(1,0)}_{\bar{n},\bar{r}}\right\|_{H^s} &\le \left\|\tilde{u}H^{(u)}_{\bar{n}-1,\bar{r}}[U,L]\right\|_{H^s} + \left\|\bar{h}(\partial_x\tilde{u})\partial_x w_{\bar{n}-1,\bar{r}}\right\|_{H^s} + \left\|\bar{h}(\partial_y\tilde{u})\partial_y w_{\bar{n}-1,\bar{r}}\right\|_{H^s} \\
&\le \mathcal{M}(s)\,|\tilde{u}|_{C^s}\left\|H^{(u)}_{\bar{n}-1,\bar{r}}[U,L]\right\|_{H^s} + 2\bar{h}\mathcal{M}(s)\,|\tilde{u}|_{C^{s+1}}\left\|w_{\bar{n}-1,\bar{r}}\right\|_{H^{s+1}} \\
&\le \mathcal{M}(s)\,|\tilde{u}|_{C^s}\,\tilde{K}^0_{u,\ell}B_u^{\bar{n}-1}B_\ell^{\bar{r}} + 2\bar{h}\mathcal{M}(s)\,|\tilde{u}|_{C^{s+1}}\,K^0_{u,\ell}B_u^{\bar{n}-1}B_\ell^{\bar{r}}.
\end{aligned}$$

We are done provided we choose

$$B_u > \max\{\frac{2}{\bar{h}}, 4\bar{h}K^0_{u,\ell}/\tilde{K}^0_{u,\ell}\}\mathcal{M}\,|\tilde{u}|_{C^{s+1}}. \quad \square$$

6. Conclusions

In this contribution we have established rigorous analytic results necessary for the proper numerical analysis of a class of High-Order Perturbation of Surfaces (HOPS) methods. In particular, we have proven a theorem (see Theorem 4.1) on existence and uniqueness of solutions to a system of partial differential equations which model the interaction of linear waves with a multiply layered periodic structure in three dimensions. With this we now have hypotheses under which a rigorous numerical analysis could be conducted, and a solution to which our HOPS schemes can be shown to converge. This will apply not only to the classical methods of Milder (Operator Expansions) and Bruno and Reitich (Field Expansions)—perhaps with more stringent further hypotheses—but also to the stabilized approach of the author and Reitich (Transformed Field Expansions).

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Appendix A. Elliptic estimate

The goal of this appendix is to give an idea of the proof of Theorem 5.2 in § 5.5. Recall that in the middle layer we solve (5.9)

$$\begin{aligned} &\Delta w + k^2 w = F, && \{0 < z < h\}, \\ &w = L, && z = 0, \\ &w = U, && z = h, \end{aligned}$$

after a simple linear change of variables in z from $[\bar{\ell}, \bar{u}]$ to $[0, h]$. For this we express the solution as

$$w(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{w}_{p,q}(z) e^{i\alpha_p x + i\beta_q y},$$

which delivers the following two point boundary value problems to solve

$$\partial_z^2 \hat{w}_{p,q} + \gamma_{p,q}^2 \hat{w}_{p,q} = \hat{F}_{p,q}, \qquad \{0 < z < h\}, \tag{A.1a}$$

$$\hat{w}_{p,q}(0) = \hat{L}_{p,q}, \tag{A.1b}$$

$$\hat{w}_{p,q}(h) = \hat{U}_{p,q}. \tag{A.1c}$$

We recall that,

$$\alpha_p := \alpha + \left(\frac{2\pi}{d_x}\right) p, \quad \beta_q := \beta + \left(\frac{2\pi}{d_y}\right) q,$$

$$\gamma_{p,q} := \begin{cases} \sqrt{k^2 - \alpha_p^2 - \beta_q^2}, & (p,q) \in \mathcal{U}, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - k^2}, & (p,q) \notin \mathcal{U}, \end{cases}$$

$$\mathcal{U} := \left\{ (p,q) \in \mathbf{Z} \mid \alpha_p^2 + \beta_q^2 \le k^2 \right\},$$

cf. (2.2).

A.1. Existence and uniqueness of solution

To begin, we establish the *existence* and *uniqueness* of solutions to (A.1) provided $\gamma_{p,q} \neq n\pi$ using the classical result of Keller [40], later extended in the "Integrated Solution Method" of Zhang [72] (which the author learned from [24]). In the notation of [24], this method considers problems of the form

$$\mathbf{u}'(z) + M(z)\mathbf{u}(z) = \mathbf{f}(z), \qquad 0 < z < h, \tag{A.2a}$$

$$A_0\mathbf{u} = \mathbf{r}_0, \qquad z = 0, \tag{A.2b}$$

$$B_1\mathbf{u} = \mathbf{s}_1, \qquad z = h, \tag{A.2c}$$

where

$$\mathbf{f}(z) \in \mathbf{C}^m, \quad \mathbf{r}_0(z) \in \mathbf{C}^{m_1}, \quad \mathbf{s}_1(z) \in \mathbf{C}^{m_2},$$

are vector fields ($m = m_1 + m_2$). Further,

$$M(z) \in \mathbf{C}^{m\times m}, \quad A_0 \in \mathbf{C}^{m_1\times m}, \quad B_1 \in \mathbf{C}^{m_2\times m},$$

are full rank matrices. Let $\Phi(z)$ be the fundamental matrix solution of the system

$$\Phi'(z) + M(z)\Phi(z) = 0, \quad \Phi(0) = I_m,$$

where I_m is the $m \times m$ identity matrix. With these Keller [40] proves the following.

Theorem A.1. *The two-point value problem (A.2) has a unique solution if and only if*

$$\det\begin{pmatrix} A_0 \\ B_1\Phi(h) \end{pmatrix} \neq 0.$$

For our system we have $m = 2$, $m_1 = m_2 = 1$, and

$$\mathbf{u} = \begin{pmatrix} \hat{w}_{p,q} \\ \partial_z \hat{w}_{p,q} \end{pmatrix}, \quad M(z) = \begin{pmatrix} 0 & -1 \\ \gamma_{p,q}^2 & 0 \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ \hat{F}_{p,q} \end{pmatrix},$$
$$A_0 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad r_0 = \hat{L}_{p,q}, \quad s_0 = \hat{U}_{p,q},$$

where, the behavior of solutions depend strongly upon the character of $\gamma_{p,q}$:

1. First, $\gamma_{p,q}$ may be real and positive so that

$$\Phi(z) = \begin{pmatrix} \cos(\gamma_{p,q}z) & \sin(\gamma_{p,q}z)/\gamma_{p,q} \\ -\gamma_{p,q}\sin(\gamma_{p,q}z) & \cos(\gamma_{p,q}z) \end{pmatrix},$$

and

$$\det\begin{pmatrix} A_0 \\ B_1\Phi(h) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ \cos(\gamma_{p,q}h) & \sin(\gamma_{p,q}h)/\gamma_{p,q} \end{pmatrix} = \sin(\gamma_{p,q}h)/\gamma_{p,q}.$$

Thus, a unique solution exists if and only if

$$\sin(\gamma_{p,q}h) \neq 0 \quad \Longleftrightarrow \quad \gamma_{p,q}h \neq n\pi, \quad n \neq 0.$$

2. Second, $\gamma_{p,q}$ may be zero so that

$$\Phi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

and

$$\det \begin{pmatrix} A_0 \\ B_1 \Phi(h) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 1 & h \end{pmatrix} = h.$$

Therefore, a unique solution exists in this case.

3. Finally, $\gamma_{p,q}$ may be purely complex with positive imaginary part, $\gamma_{p,q} = i\tilde{\gamma}_{p,q}$ ($\tilde{\gamma}_{p,q} > 0$),

$$\Phi(z) = \begin{pmatrix} \cosh(\tilde{\gamma}_{p,q} z) & \sinh(\tilde{\gamma}_{p,q} z)/\tilde{\gamma}_{p,q} \\ \tilde{\gamma}_{p,q} \sinh(\tilde{\gamma}_{p,q} z) & \cosh(\tilde{\gamma}_{p,q} z) \end{pmatrix},$$

and

$$\det \begin{pmatrix} A_0 \\ B_1 \Phi(h) \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ \cosh(\tilde{\gamma}_{p,q} h) & \sinh(\tilde{\gamma}_{p,q} h)/\tilde{\gamma}_{p,q} \end{pmatrix} = \sinh(\tilde{\gamma}_{p,q} h)/\tilde{\gamma}_{p,q}.$$

Since $\sinh(\tilde{\gamma}_{p,q} h) \neq 0$, for $\tilde{\gamma}_{p,q} h \neq 0$, a unique solution always exists.

A.2. Solution formula

We now exclude the possibility of non-uniqueness by considering layer configurations which are τ-allowable, Definition 4.5, so that $\gamma_{p,q} h \neq n\pi$ ($n \neq 0$). Under these circumstances, the unique solution can be written in terms of the following homogeneous solutions of (A.1a)

$$W_{p,q}(z) := \begin{cases} \sin(\gamma_{p,q} z)/\sin(\gamma_{p,q} h), & \gamma_{p,q} \in \mathbf{R}^+, \\ z/h, & \gamma_{p,q} = 0, \\ \sinh(\tilde{\gamma}_{p,q} z)/\sinh(\tilde{\gamma}_{p,q} h), & \gamma_{p,q} = i\tilde{\gamma}_{p,q} \in i\mathbf{R}^+, \end{cases}$$

which are well-defined (if $\gamma_{p,q} h \neq n\pi$). Using the facts that

$$W_{p,q}(0) = 0, \quad W_{p,q}(h) = 1,$$

we write the solution of (A.1) as

$$\hat{w}_{p,q}(z) = \hat{L}_{p,q} W_{p,q}(h-z) + \hat{U}_{p,q} W_{p,q}(z) \\ + C W_{p,q}(h-z) I_0[\hat{F}_{p,q}](z) + C W_{p,q}(z) I_h[\hat{F}_{p,q}](z), \quad \text{(A.3)}$$

for some constant C, where

$$I_0[\hat{F}_{p,q}](z) := \int_0^z W_{p,q}(s) \hat{F}_{p,q}(s)\, ds, \quad I_h[\hat{F}_{p,q}](z) := \int_z^h W_{p,q}(h-s) \hat{F}_{p,q}(s)\, ds.$$

To find C we note that

$$\partial_z W_{p,q}(z) := \begin{cases} \gamma_{p,q}\cos(\gamma_{p,q}z)/\sin(\gamma_{p,q}h), & \gamma_{p,q} \in \mathbf{R}^+, \\ 1/h, & \gamma_{p,q} = 0, \\ \tilde{\gamma}_{p,q}\cosh(\tilde{\gamma}_{p,q}z)/\sinh(\tilde{\gamma}_{p,q}h), & \gamma_{p,q} = i\tilde{\gamma}_{p,q} \in i\mathbf{R}^+, \end{cases}$$

and

$$\partial_z^2 W_{p,q}(z) := \begin{cases} -\gamma_{p,q}^2\sin(\gamma_{p,q}z)/\sin(\gamma_{p,q}h), & \gamma_{p,q} \in \mathbf{R}^+, \\ 0, & \gamma_{p,q} = 0, \\ \tilde{\gamma}_{p,q}^2\sinh(\tilde{\gamma}_{p,q}z)/\sinh(\tilde{\gamma}_{p,q}h), & \gamma_{p,q} = i\tilde{\gamma}_{p,q} \in i\mathbf{R}^+, \end{cases}$$

or $\partial_z^2 W_{p,q} = -\gamma_{p,q}^2 W_{p,q}(z)$. In order to satisfy (A.1a) we choose

$$C = \begin{cases} -\sin(\gamma_{p,q}h)/\gamma_{p,q}, & \gamma_{p,q} \in \mathbf{R}^+, \\ -h, & \gamma_{p,q} = 0, \\ -\sinh(\gamma_{p,q}h)/\gamma_{p,q}, & \gamma_{p,q} \in i\mathbf{R}^+. \end{cases}$$

A.3. Estimates

We recall the estimates we require to establish Theorem 5.2,

$$\left|\!\left|\!\left| w \right|\!\right|\!\right|_{H^{s+3/2}(V)} \le K_e \left\{ \left|\!\left|\!\left| F \right|\!\right|\!\right|_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \tag{A.4a}$$

$$\left|\!\left|\!\left| \partial_z w \right|\!\right|\!\right|_{H^{s+1/2}(V)} \le K_e \left\{ \left|\!\left|\!\left| F \right|\!\right|\!\right|_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \tag{A.4b}$$

$$\begin{aligned} &\max\left\{ \|w(x,y,\bar{u})\|_{H^{s+1}}, \left\|w(x,y,\bar{\ell})\right\|_{H^{s+1}} \right\} \\ &\quad \le K_e \left\{ \left|\!\left|\!\left| F \right|\!\right|\!\right|_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \end{aligned} \tag{A.4c}$$

$$\begin{aligned} &\max\left\{ \|\partial_x w(x,y,\bar{u})\|_{H^s}, \left\|\partial_y w(x,y,\bar{u})\right\|_{H^s}, \|\partial_z w(x,y,\bar{u})\|_{H^s} \right\} \\ &\quad \le K_e \left\{ \left|\!\left|\!\left| F \right|\!\right|\!\right|_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \end{aligned} \tag{A.4d}$$

$$\begin{aligned} &\max\left\{ \left\|\partial_x w(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_y w(x,y,\bar{\ell})\right\|_{H^s}, \left\|\partial_z w(x,y,\bar{\ell})\right\|_{H^s} \right\} \\ &\quad \le K_e \left\{ \left|\!\left|\!\left| F \right|\!\right|\!\right|_{H^{s-1/2}(V)} + \|U\|_{H^{s+1}} + \|L\|_{H^{s+1}} \right\}, \end{aligned} \tag{A.4e}$$

cf. (5.10). We now demonstrate that these follow from the following lemma whose proof comes from the exact solution formula (A.3) and arguments very much akin to those given in [56].

Lemma A.2. *It can be shown that*

$$\left\|\hat{w}_{p,q}(z)\right\|_{L^2}^2 \le K_e \left\{ \langle (p,q)\rangle^{-4} \left\|\hat{F}_{p,q}(x)\right\|_{L^2}^2 + \langle (p,q)\rangle^{-1} \left|\hat{U}_{p,q}\right|^2 + \langle (p,q)\rangle^{-1} \left|\hat{L}_{p,q}\right|^2 \right\}, \tag{A.5a}$$

$$\left\|\partial_z\hat{w}_{p,q}(z)\right\|_{L^2}^2 \le K_e \left\{ \langle (p,q)\rangle^{-2} \left\|\hat{F}_{p,q}(x)\right\|_{L^2}^2 + \langle (p,q)\rangle^{1} \left|\hat{U}_{p,q}\right|^2 + \langle (p,q)\rangle^{1} \left|\hat{L}_{p,q}\right|^2 \right\}, \tag{A.5b}$$

$$\left|\hat{w}_{p,q}(\bar{u})\right|^2 \leq K_e \left\{ \langle (p,q) \rangle^{-3} \left\| \hat{F}_{p,q}(x) \right\|_{L^2}^2 + \left|\hat{U}_{p,q}\right|^2 + \left|\hat{L}_{p,q}\right|^2 \right\}, \tag{A.5c}$$

$$\left|\partial_z \hat{w}_{p,q}(\bar{u})\right|^2 \leq K_e \left\{ \langle (p,q) \rangle^{-1} \left\| \hat{F}_{p,q}(x) \right\|_{L^2}^2 + \langle (p,q) \rangle^2 \left|\hat{U}_{p,q}\right|^2 + \langle (p,q) \rangle^2 \left|\hat{L}_{p,q}\right|^2 \right\}, \tag{A.5d}$$

for some universal constant $K_e > 0$.

Given this, to establish (A.4a) we estimate

$$\begin{aligned} \|w\|_{H^{s+3/2}(V)}^2 &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^{2s+3} \left\| \hat{w}_{p,q}(z) \right\|_{L^2}^2 \\ &\leq K_e \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left\{ \langle (p,q) \rangle^{2s-1} \left\| \hat{F}_{p,q}(z) \right\|_{L^2}^2 \right. \\ &\quad \left. + \langle (p,q) \rangle^{2s+2} \left|\hat{U}_{p,q}\right|^2 + \langle (p,q) \rangle^{2s+2} \left|\hat{L}_{p,q}\right|^2 \right\}, \end{aligned}$$

which follows from (A.5a) and we are done. Now,

$$\begin{aligned} \|\partial_z w\|_{H^{s+1/2}(V)}^2 &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^{2s+1} \left\| \partial_z \hat{w}_{p,q}(z) \right\|_{L^2}^2 \\ &\leq K_e \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left\{ \langle (p,q) \rangle^{2s-1} \left\| \hat{F}_{p,q}(z) \right\|_{L^2}^2 \right. \\ &\quad \left. + \langle (p,q) \rangle^{2s+2} \left|\hat{U}_{p,q}\right|^2 + \langle (p,q) \rangle^{2s+2} \left|\hat{L}_{p,q}\right|^2 \right\}, \end{aligned}$$

comes from (A.5b) and we are done with (A.4b). Continuing, to verify (A.4c) we begin with

$$\begin{aligned} \|w(x,y,\bar{u})\|_{H^{s+1}}^2 &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^{2s+2} \left| \hat{w}_{p,q}(\bar{u}) \right|^2 \\ &\leq K_e \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left\{ \langle (p,q) \rangle^{2s-1} \left\| \hat{F}_{p,q}(z) \right\|_{L^2}^2 \right. \\ &\quad \left. + \langle (p,q) \rangle^{2s+2} \left|\hat{U}_{p,q}\right|^2 + \langle (p,q) \rangle^{2s+2} \left|\hat{L}_{p,q}\right|^2 \right\}, \end{aligned}$$

which results from (A.5c) and we are done. Since

$$\widehat{\partial_x w}_{p,q}(\bar{u}) = (i\alpha_p)\hat{w}_{p,q}(\bar{u}), \quad \widehat{\partial_y w}_{p,q}(\bar{u}) = (i\beta_q)\hat{w}_{p,q}(\bar{u}),$$

estimate (A.5c) suffices to show the estimates for $(\partial_x w)(x,y,\bar{u})$ and $(\partial_y w)(x,y,\bar{u})$ in (A.4c). Now,

$$
\begin{aligned}
\|\partial_z w(x,y,\bar{u})\|_{H^s}^2 &= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^{2s} \left| \partial_z \hat{w}_{p,q}(\bar{u}) \right|^2 \\
&\leq K_e \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left\{ \langle (p,q) \rangle^{2s-1} \left\| \hat{F}_{p,q}(z) \right\|_{L^2}^2 \right. \\
&\qquad \left. + \langle (p,q) \rangle^{2s+2} \left| \hat{U}_{p,q} \right|^2 + \langle (p,q) \rangle^{2s+2} \left| \hat{L}_{p,q} \right|^2 \right\},
\end{aligned}
$$

from (A.5d) and we are done with (A.4d). The estimates in (A.4e) are validated in a similar fashion. This completes the proof.

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